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On Majorization for Matrices

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In this paper, we give several results for majorized matrices by using continuous convex function and Green function. We obtain mean value theorems for majorized matrices and also give corresponding Cauchy means, as well as prove that these means are monotonic. We prove positive semi-definiteness of matrices generated by differences deduced from majorized matrices which implies exponential convexity and log-convexity of these differences and also obtain Lypunov's and Dresher's type inequalities for these differences.

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1 Introduction

For fixed $n \geq 2$, let

$$\mathbf{x} = (x_1, ..., x_n)\,, \;\; \mathbf{y} = (y_1, ..., y_n)$$

denote two n-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[n]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(n)}$$

be their ordered components.

Definition 1.1. (see [14, p.319]) $\mathbf{y}$ is said to majorize $\mathbf{x}$ (or $\mathbf{x}$ is said to be majorized by $\mathbf{y}$), in symbol, $\mathbf{y} \succ \mathbf{x}$, if

$$\sum_{i=1}^{m} x_{[i]} \leq \sum_{i=1}^{m} y_{[i]} \tag{1.1}$$

holds for $m = 1, 2, ..., n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Note that (1.1) is equivalent to

$$\sum_{i=n-m+1}^{n} x_{(i)} \leq \sum_{i=n-m+1}^{n} y_{(i)}$$

holds for $m = 1, 2, ..., n-1$.

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin [9, p.11] (see also [14, p.320]):

Theorem 1.1. *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}$, $\mathbf{y}$ be two n-tuples such that x_i, $y_i \in I$ $(i = 1, ..., n)$. Then*

$$\sum_{i=1}^{n} \phi(y_i) \leq \sum_{i=1}^{n} \phi(x_i) \tag{1.2}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ iff $\mathbf{x} \succ \mathbf{y}$ holds.

Remark 1.1. ([8]) If ϕ is a strictly convex function, then equality in is valid iff $x_{[i]} = y_{[i]}$, $i = 1, ..., n$.

The following theorem can be regarded as a weighted version of Theorem 1.1 and is proved by Fuchs in ([4], [14, p.323]):

Theorem 1.2. *Let $\mathbf{x}$, $\mathbf{y}$ be two decreasing real n-tuples, let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{k} w_i\, y_i \leq \sum_{i=1}^{k} w_i\, x_i \;\; \textit{for} \;\; k = 1, ..., n-1, \tag{1.3}$$

and

$$\sum_{i=1}^{n} w_i\, y_i = \sum_{i=1}^{n} w_i\, x_i. \tag{1.4}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{n} w_i\, \phi(y_i) \leq \sum_{i=1}^{n} w_i\, \phi(x_i). \tag{1.5}$$

The following theorem is valid ([11, p.32]):

Theorem 1.3. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on an interval I, $\mathbf{w}$ be a positive n-tuple and $\mathbf{x}$, $\mathbf{y} \in I^n$ satisfying (1.3) and (1.4).*

1. *If* $\mathbf{y}$ *is decreasing* n*-tuple, then* (1.5) *holds.*

2. *If* $\mathbf{x}$ *is increasing* n*-tuple, then reverse inequality in* (1.5) *holds.*

Theorem 1.4. ([6]) *Let* $\phi : I \to \mathbb{R}$ *be a continuous convex function on an interval* I, $x_i, y_i \in I$ $(i = 1, 2, ..., n)$, $w_i \geq 0$ $(i = 1, 2, ..., n)$ *with* $W_n = \sum_{i=1}^{n} w_i > 0$. *If* $(x_i - y_i)_{(i=\overline{1,n})}$ *is nondecreasing (nonincreasing),* $(y_i)_{(i=\overline{1,n})}$ *is nondecreasing (nonincreasing) and satisfying* (1.4), *then* (1.5) *holds.*

In the following result, inner product is defined in a usual way on $\mathbb{R}^m$. Furthermore, $e = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ is a basis in $\mathbb{R}^m$, and $d = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m\}$ is the dual basis of e, that is $\langle \mathbf{e}_i, \mathbf{d}_j \rangle = \delta_{ij}$ (Kronecker delta). One denotes $J = \{1, 2, \ldots, m\}$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$. Let $\mathbf{v} \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$. A vector $\mathbf{z} \in \mathbb{R}^m$ is said to be $\mu, \mathbf{v}$*-separable* on J_1 and J_2 (with respect to the basis e), if

$$\langle \mathbf{e}_i, \mathbf{z} - \mu\mathbf{v} \rangle \geq 0 \ \text{ for } i \in J_1, \ \text{ and } \ \langle \mathbf{e}_j, \mathbf{z} - \mu\mathbf{v} \rangle \leq 0 \ \text{ for } j \in J_2. \tag{1.6}$$

A vector $\mathbf{z} \in \mathbb{R}^m$ is said to be $\mathbf{v}$*-separable* on J_1 and J_2 (w.r.t. e), if $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2 for some μ. One says that a function $\varphi : I \to \mathbb{R}$ *preserves* $\mathbf{v}$*-separability* on J_1 and J_2 w.r.t. e, if $(\varphi(z_1), \varphi(z_2), \ldots, \varphi(z_m))$ is $\mathbf{v}$-separable on J_1 and J_2 w.r.t. e for each $\mathbf{z} = (z_1, z_2, \ldots, z_m) \in I^m$ such that $\mathbf{z}$ is $\mathbf{v}$-separable on J_1 and J_2 w.r.t. e.

Theorem 1.5. ([12, *Theorem* 2.2]) *Let* $\phi : I \to \mathbb{R}$ *be a continuous convex function on an interval* I. *Assume* $\varphi \in \partial\phi$, *where* $\partial\phi$ *is the subdifferential of* ϕ. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\mathbf{y} = (y_1, y_1, \ldots, y_m)$ *and* $\mathbf{w} = (w_1, w_1, \ldots, w_m)$, *where* $x_i, y_i \in I$, $w_i > 0$ *for* $i \in J = \{1, 2, \ldots, m\}$, *and let* $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^m$ *with* $\langle \mathbf{u}, \mathbf{v} \rangle > 0$. *If there exist index sets* J_1 *and* J_2 *with* $J_1 \cup J_2 = J$ *such that*

(i) $\mathbf{y}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *w.r.t.* e,

(ii) $\mathbf{x}$ - $\mathbf{y}$ *is* λ, $\mathbf{u}$*-separable on* J_1 *and* J_2 *w.r.t.* d, *where* $\lambda = \langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle / \langle \mathbf{u}, \mathbf{v} \rangle$,

(iii) $\langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle = 0$, *or* $\langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle \langle \mathbf{z}, \mathbf{u} \rangle \geq 0$, *where* $\mathbf{z} = (\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_m))$,

(iv) φ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *w.r.t.* e,

then (1.5) *holds.*

Matrix majorization: The notion of majorization concerns a partial ordering of the diversity of the components of two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. A natural problem of interest is the extension of this notion from m-tuples (vectors) to $n \times m$ matrices. For example, let

$$X = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)' \ \text{ and } \ Y = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n)'$$

be two $n \times m$ real matrices, where $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$; $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$ are the corresponding row vectors.

Definition 1.2. Let X, Y be two $n \times m$ real matrices for $n \geq 2$, $m \geq 2$. X is said to row-wise majorize Y ($X \succ^r Y$) if $\mathbf{x}_i \succ \mathbf{y}_i$ holds for $i = 1, 2, ..., n$.

To define Cauchy type means for majorized matrices, the following families of functions will be useful.

Lemma 1.1. *Let us define the functions* $\eta_t : [0, \infty) \to \mathbb{R}$

$$\eta_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 1; \\ x \log x, & t = 1. \end{cases} \tag{1.7}$$

Then $\eta_t''(x) = x^{t-2}$, *that is* η_t *is convex for* $x \geq 0, t > 0$, *with the convention that* $0 \log 0 = 0$

Lemma 1.2. *Let us define the functions* $\psi_t : (0, \infty) \to \mathbb{R}$

$$\psi_t(x) = \begin{cases} \frac{x^t}{t(t-1)} \ , & t \neq 0, 1; \\ -\log x \ , & t = 0; \\ x \log x \ , & t = 1. \end{cases} \tag{1.8}$$

Then $\varphi_t''(x) = x^{t-2}$, *that is* φ_t *is convex for* $x > 0, t \in \mathbb{R}$.

Lemma 1.3. *Let us define the functions* $\delta_t : \mathbb{R} \to \mathbb{R}$

$$\delta_t(x) = \begin{cases} \frac{1}{t^2} e^{tx} \ , & t \neq 0; \\ \frac{1}{2} x^2, & t = 0. \end{cases} \tag{1.9}$$

Then $\phi_t''(x) = e^{tx}$, *that is* ϕ_t *is convex for* $x \in \mathbb{R}, t \in \mathbb{R}$.

The following lemma is equivalent to definition of convex function([14, p.2]).

Lemma 1.4. *If* f *is convex on an interval* $I \subseteq \mathbb{R}$, *then*

$$f(s_1)(s_3 - s_2) + f(s_2)(s_1 - s_3) + f(s_3)(s_2 - s_1) \geq 0. \tag{1.10}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

The following important subclass, i.e. the class of exponentially convex functions, introduced by Bernstein [2], will be crucial importance in studying the properties of Cauchy type means for majorized matrices (for example monotonicity). Also our method can give a method of producing families of exponentially convex functions.

Definition 1.3. A function $\phi : I \to \mathbb{R}$ is exponentially convex on an interval $I \subseteq \mathbb{R}$ if it is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(x_k + x_l) \geq 0$$

for all $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, .., n$ such that $x_k + x_l \in I, 1 \leq k, l \leq n$.

Proposition 1.1. *Let $\phi : I \to \mathbb{R}$. Then the following propositions are equivalent:*

(i) ϕ is exponentially convex.

(ii) ϕ is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(\frac{x_k + x_l}{2}) \geq 0$$

for every $n \in \mathbb{N}$, for every $a_k \in \mathbb{R}$ and $x_k \in I$, $k, l = 1, 2, .., n$, $1 \leq k \leq n$.

The following corollary is given in ([2], [10]):

Corollary 1.1. *If ϕ is exponentially convex function then*

$$\det\left[\phi(\frac{x_k + x_l}{2})\right]_{k,l=1}^{n} \geq 0$$

for every $n \in \mathbb{N}$ $x_k \in I$, $k = 1, 2, .., n$.

Corollary 1.2. *If $\phi : I \to (0, \infty)$ is exponentially convex function, then ϕ is a* log*-convex function.*

This paper is organized in this manner: in Section 2, we give analogues of Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5 in matrix form. We also introduce majorization result for matrices by using Green function. In Section 3, we give mean value theorems for majorized matrices and prove positive semi-definiteness of matrices generated by differences deduced from majorized matrices which implies exponential convexity and log-convexity of these differences and also obtain Lypunov's and Dresher's type inequalities for these differences. We introduce Cauchy type means and prove that these are monotonic.

2 Main results

Theorem 2.1. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on an interval I and $X = [x_{ij}]$, $Y = [y_{ij}]$ and $W = [w_{ij}]$ be matrices, where $x_{ij}, y_{ij} \in I$ and $w_{ij} \in \mathbb{R}$ $(i = 1, 2, ..., n,\ j = 1, 2, ..., m)$.*

(a) *If $X \succ^r Y$, the following inequality holds*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(x_{ij}) \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(y_{ij}). \tag{2.11}$$

If ϕ is strictly convex on I, then the strict inequality holds in (2.11) *if and only if $X \neq Y$.*

(b) *If $(x_{ij})_{j=\overline{1,m}}$, $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, .., n)$ are decreasing and satisfy the following conditions,*

$$\sum_{j=1}^{k} w_{ij}x_{ij} \geq \sum_{j=1}^{k} w_{ij}y_{ij}, \quad k = 1, 2, ..., m-1, \tag{2.12}$$

for $i = 1, 2, ..., n$ and

$$\sum_{j=1}^{m} w_{ij}x_{ij} = \sum_{j=1}^{m} w_{ij}y_{ij} \tag{2.13}$$

for $i = 1, 2, ..., n$.
Then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}). \tag{2.14}$$

(c) (c_1) *If $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, .., n)$ is decreasing with $w_{ij} > 0$ $(i = 1, 2, ..., n$, $j = 1, 2, ..., m)$ and satisfying conditions* (2.12) *and* (2.13)*, then* (2.14) *holds. If ϕ is strictly convex on I, then the strict inequality holds in* (2.14) *if and only if $X \neq Y$.*

(c_2) *If $(x_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, .., n)$ is increasing with $w_{ij} > 0$ $(i = 1, 2, ..., n$, $j = 1, 2, ..., m)$ and satisfying conditions* (2.12) *and* (2.13)*, then reverse inequality in* (2.14) *holds. If ϕ is strictly convex on I, then the reverse strict inequality holds in* (2.14) *if and only if $X \neq Y$.*

(d) *If $(x_{ij} - y_{ij})_{j=\overline{1,m}}$ and $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, .., n)$ are nondecreasing (nonincreasing) with $w_{ij} \geq 0$ $(i = 1, 2, ..., n,\ j = 1, 2, ..., m)$ and satisfying condition* (2.13)*, then* (2.14) *holds. If ϕ is strictly convex on I and $w_{ij} > 0$, then the strict inequality holds in* (2.14) *if and only if $X \neq Y$.*

(e) *Let $w_{ij} > 0$ $(i = 1, 2, ..., n, j = 1, 2, .., m)$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ with $\langle \mathbf{u}, \mathbf{v} \rangle > 0$. If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that for each $i = 1, 2, .., n$*

(i) *$(y_{ij})_{j=\overline{1,m}}$ is $\mathbf{v}$-separable on J_1 and J_2 w.r.t. e,*

(ii) $(x_{ij} - y_{ij})_{j=\overline{1,m}}$ *is* $\lambda, \mathbf{u}$*-separable on* J_1 *and* J_2 *w.r.t.* d*, where* $\lambda = \langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle / \langle \mathbf{u}, \mathbf{v}\rangle$,

(iii) $\langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle = 0$, *or* $\langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle \langle (z_{ij})_{j=\overline{1,m}}, \mathbf{u}\rangle \geq 0$, *where* $(z_{ij})_{j=\overline{1,m}} = (\varphi(y_{i1}), \dots, \varphi(y_{im}))$,

(iv) φ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *w.r.t.* e,

then (2.14) *holds.*

Proof. (a) By using Theorem 1.1, we can write

$$\sum_{j=1}^{m} \phi\left(x_{ij}\right) \geq \sum_{j=1}^{m} \phi\left(y_{ij}\right), \quad \text{for } i = 1, 2, ..., n. \tag{2.15}$$

Summing (2.15) over i from 1 to n, we get (2.11).
In a similar way, we can prove (b), (c), (d) and (e). ■

Now, we give majorization type result for matrices by using the Green function.

Consider the Green function G defined on $[\alpha, \beta] \times [\alpha, \beta]$ by

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{2.16}$$

The function G is convex under s, it is symmetric, so it is also convex under t. The function G is continuous under s and continuous under t.

For any function $\phi : [\alpha, \beta] \to \mathbb{R}$, $\phi \in C^2([\alpha, \beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_{\alpha}^{\beta} G(x, s)\phi''(s)ds, \tag{2.17}$$

where the function G is defined as above in (2.16) ([16]).

Theorem 2.2. *Let* $\phi : I \to \mathbb{R}$ *be a continuous convex function on an interval* I *and* $X = [x_{ij}]$, $Y = [y_{ij}]$ *and* $W = [w_{ij}]$ *be matrices, where* $x_{ij}, y_{ij} \in I$ *and* $w_{ij} \in \mathbb{R}$ $(i = 1, 2, ..., n,\ j = 1, 2, ..., m)$ *such that satisfy condition* (2.13).

Then the following two statements are equivalent:

(i) For every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, (2.14) *holds.*

(ii) For all $s \in [\alpha, \beta]$ *holds*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} G(x_{ij}, s) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} G\left(y_{ij}, s\right). \tag{2.18}$$

Moreover, the statements (i) *and* (ii) *are also equivalent if we change the sign of inequality in both inequalities, in* (2.14) *and in* (2.18).

Proof. (i)⇒(ii): Let (i) holds. As the function $G(\cdot, s)$ ($s \in [\alpha, \beta]$) is also continuous and convex, it follows that also for this function (2.14) holds, i.e., it holds (2.18).

(ii)⇒(i): Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a convex function, $\phi \in C^2([\alpha, \beta])$ and (ii) holds. Then, we can represent function ϕ in the form (2.17), where the function G is defined in (2.16). By easy calculation, using (2.17), we can easily get that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij})$$

$$= \int_{\alpha}^{\beta} \left[\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}G(x_{ij}, s) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}G(y_{ij}, s) \right] \phi''(s)\, ds.$$

Since ϕ is a convex function, then $\phi''(s) \geq 0$ for all $s \in [\alpha, \beta]$. So, if for every $s \in [\alpha, \beta]$ holds (2.18), then it follows that for every convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, with $\phi \in C^2([\alpha, \beta])$, inequality (2.14) holds.

At the end, note that it is not necessary to demand the existence of the second derivative of the function ϕ ([14, p.172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials. ∎

3 Mean value theorems and generalized Cauchy means

Theorem 3.1. *Let X, Y and W be matrices as in Theorem* 2 *such that satisfy condition* (2.13). *Let also $\phi \in C^2([\alpha, \beta])$. If for all $s \in [\alpha, \beta]$, the inequality* (2.18) *holds or if for all $s \in [\alpha, \beta]$, the reverse inequality in* (2.18) *holds, then there exists $\xi \in [\alpha, \beta]$ such that*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}) = \frac{\phi''(\xi)}{2} \left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2 \right). \tag{3.19}$$

Proof. Since $\phi''(x)$ is continuous on $[\alpha, \beta]$, let $m = \min_{x \in [\alpha, \beta]} \phi''(x)$ and $M = \max_{x \in [\alpha, \beta]} \phi''(x)$, so $m \leq \phi''(x) \leq M$ for $x \in [\alpha, \beta]$.

Consider the functions ϕ_1, ϕ_2 defined as

$$\phi_1(x) = \frac{Mx^2}{2} - \phi(x),$$

and

$$\phi_2(x) = \phi(x) - \frac{mx^2}{2}.$$

Since

$$\phi_1''(x) = M - \phi''(x) \geq 0,$$

and

$$\phi_2''(x) = \phi''(x) - m \geq 0,$$

it follows that $\phi_i(x)$ for $i = 1, 2$ are convex.
Now by applying ϕ_1 for ϕ in (2.14), we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\left[\frac{Mx_{ij}^2}{2} - \phi(x_{ij})\right] \geq \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\left[\frac{My_{ij}^2}{2} - \phi(y_{ij})\right]. \tag{3.20}$$

From (3.20) we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}) \leq \frac{1}{2}M\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2\right) \tag{3.21}$$

and similarly by applying ϕ_2 for ϕ in (2.14), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}) \geq \frac{1}{2}m\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2\right). \tag{3.22}$$

If $\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2 = 0$, then from (3.21) and (3.22) follows that for any $\xi \in I$ (3.19) holds.

If $\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2 > 0$, it follows by combining (3.21) and (3.22) that

$$m \leq \frac{2\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij})\right)}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2} \leq M.$$

Now using the fact that for $m \leq \rho \leq M$ there exists $\xi \in I$ such that $f''(\xi) = \rho$ we get (3.19). ■

Theorem 3.2. *Let X, Y and W be matrices as in Theorem 2 such that satisfy condition (2.13). Let also $\phi, \psi \in C^2([\alpha, \beta])$. If for all $s \in [\alpha, \beta]$, the inequality (2.18) holds or if for all $s \in [\alpha, \beta]$, the reverse inequality in (2.18) holds, then there exists $\xi \in [\alpha, \beta]$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(x_{ij}) \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(y_{ij})}. \tag{3.23}$$

Provided that the denominators are non zero.

Proof. Let the function $k \in C^2([\alpha, \beta])$ be defined by

$$k = c_1\phi - c_2\psi,$$

where c_1 and c_2 are defined as

$$c_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(x_{ij}) \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(y_{ij}),$$

$$c_2 = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}).$$

Then, using Theorem 3.1 with $\phi = k$, we have

$$0 = \left(\frac{c_1\phi''(\xi)}{2} - \frac{c_2\psi''(\xi)}{2}\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2\right). \tag{3.24}$$

By using (3.19) for ψ, left hand side of (3.19) is non-zero by our assumption, it follows that $\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 \; - \; \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2 \neq 0$.
Therefore, (3.24) gives us

$$\frac{c_2}{c_1} = \frac{\phi''(\xi)}{\psi''(\xi)}.$$

After putting the values of c_1 and c_2 , we get (3.23). ■

Corollary 3.1. *Let X, Y and W be matrices as in Theorem 2 such that satisfy condition (2.13). If for all $s \in [\alpha, \beta]$, the inequality (2.18) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (2.18) holds and $[\alpha, \beta]$ is closed interval in $\mathbb{R}^+$, then it exists $\xi \in [\alpha, \beta]$ such that*

$$\xi^{u-v} = \frac{v(v-1)}{u(u-1)} \cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^u - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^u}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^v - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^v}. \tag{3.25}$$

P r o o f. Set $\phi(x) = x^u$ and $\psi(x) = x^v$ in Theorem 3, we get (3.25). ■

Now we are able to introduce generalized Cauchy means from (3.23). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function, then from (3.23) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(y_{ij})}\right). \tag{3.26}$$

R e m a r k 3.1. Note that we can consider the interval $[\alpha, \beta] = [m_{x,y}, M_{x,y}]$, where $m_{x,y} = \min\{\min_{ij} x_{ij}, \min_{ij} y_{ij}\}$, $M_{x,y} = \max\{\max_{ij} x_{ij}, \max_{ij} y_{ij}\}$.

Since the function $\xi \to \xi^{u-v}$, $u \neq v$ is invertible, then from (3.25) we have

$$m_{x,y} \leq \left\{\frac{v(v-1)}{u(u-1)} \cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^{u} - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^{u}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^{v} - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^{v}}\right\}^{\frac{1}{u-v}} \leq M_{x,y}. \tag{3.27}$$

We shall say that the expression in the middle is a mean of x_{ij} and y_{ij}.

4 Exponential convexity and monotonicity of Cauchy means related to majorized matrices

In this section, we want to give some very important applications of generalized Cauchy means i.e., monotonicity of these means. Also we prove positive semi-definiteness of matrices generated by differences deduced from the majorized matrices which implies exponential convexity and log-convexity of these differences and also obtain Lypunov's and Dresher's type inequalities for these differences.

Let X, Y, W and ϕ be stated as in Theorem 2.2. We define the functional $\tilde{A}(X, Y, W; \phi)$ by

$$\tilde{A}(X, Y, W; \phi) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}).$$

We begin with defining Cauchy type means for the family of functions η_t.

Theorem 4.1. *Let X, Y and W be matrices as in Theorem* 2.1 *with $x_{ij}, y_{ij} \geq 0$ $(i = 1, 2, ..., n, j = 1, 2, ..., m)$ such that satisfy condition* (2.13). *Consider $Q_t^1 = \tilde{A}(X, Y, W; \eta_t)$, if (2.18) holds for every $\tau \in [\alpha, \beta]$ and $Q_t^2 = -\tilde{A}(X, Y, W; \eta_t)$, if (2.18) holds in the opposite direction for every $\tau \in [\alpha, \beta]$.*

Then the following statements are valid for $Q_t^i (i = 1, 2)$:

(a) for every $n \in \mathbb{N}$ and for every $p_k \in \mathbb{R}^+$, $k \in \{1, 2, ..., n\}$, the matrix $[Q_{\frac{p_k+p_l}{2}}]_{k,l=1}^n$ is a positive semi-definite matrix. Particularly

$$\det[Q^i_{\frac{p_k+p_l}{2}}]_{k,l=1}^n \geq 0; \tag{4.28}$$

(b) the function $t \mapsto Q^i_t$ is exponentially convex,

(c) if $Q^i_t > 0$, then the function $t \mapsto Q^i_t$ is log*-convex, i.e. for $0 < r < s < t < \infty$, we have*

$$(Q^i_s)^{t-r} \leq (Q^i_r)^{t-s}(Q^i_t)^{s-r}. \tag{4.29}$$

Proof. (a) Let us consider the function defined by

$$\omega(x) = \sum_{k,l=1}^{n} a_k a_l \eta_{p_{kl}}(x),$$

where $p_{kl} = \frac{p_k+p_l}{2} > 0$ and $a_k \in \mathbb{R}$ for all $k \in \{1, 2, ..., n\}$, $x \geq 0$.
We have

$$\omega''(x) = \sum_{k,l=1}^{n} a_k a_l x^{p_{kl}-2} = \left(\sum_{k=1}^{n} a_k x^{\frac{p_k-2}{2}}\right)^2 \geq 0.$$

Therefore, $\omega(x)$ is convex for $x \geq 0$. Using (2.14) we get

$$\sum_{k,l=1}^{n} a_k a_l Q^i_{p_{kl}} \geq 0,$$

so the matrix $[Q^i_{\frac{p_k+p_l}{2}}]_{k,l=1}^n$ is a positive semi-definite.

(b) Since $\lim_{t\to 1} Q^i_t = Q^i_1$ and $0 \log 0 = 0$, so Q^i_t is continuous for all $t > 0, x \geq 0$ and $[Q^i_{\frac{p_k+p_l}{2}}]_{k,l=1}^n$ is positive semi-definite matrix, so using Proposition 1.1 we have exponentially convexity of the function $t \to Q^i_t$.

(c) Let $Q^i_t > 0$, then by Corollary 1.2 we have Q^i_t is log-convex i.e $t \to \log Q^i_t$ is convex, by Lemma 1.4 for $0 < r < s < t < \infty$ and taking $f(t) = \log Q^i_t$, we get

$$(t-s)\log Q^i_r + (r-t)\log Q^i_s + (s-r)\log Q^i_t \geq 0.$$

Which is equivalent to (4.29). ∎

Let X, Y and W be matrices as in Theorem 2.1 with $x_{ij}, y_{ij} \geq 0$ $(i = 1, 2, ..., n, j = 1, 2, ..., m)$ such that satisfy condition (2.13). If for all $s \in [\alpha, \beta]$, the inequality (2.18) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (2.18) holds. Also let $Q_t^i > 0$ for $t > 0$,

$$S_{u,v} = \left(\frac{Q_u^i}{Q_v^i}\right)^{\frac{1}{u-v}}, \quad i = 1, 2, \tag{4.30}$$

for $0 < u \neq v < +\infty$ are means of x_{ij} and y_{ij}. Moreover we can extend these means in other cases.

So by limit we have, for $u \neq 1$,

$$S_{u,u} = \exp\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij} x_{ij}^u \log x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij} y_{ij}^u \log y_{ij}}{\sum_{i=1}^n \sum_{j=1}^m w_{ij} x_{ij}^u - \sum_{i=1}^n \sum_{j=1}^m w_{ij} y_{ij}^u} - \frac{2u-1}{u(u-1)}\right),$$

$$S_{1,1} = \exp\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij} x_{ij} \log^2 x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij} y_{ij} \log^2 y_{ij}}{2\left(\sum_{i=1}^n \sum_{j=1}^m w_{ij} x_{ij} \log x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij} y_{ij} \log y_{ij}\right)} - 1\right).$$

Theorem 4.2. *Let $t, s, u, v \in \mathbb{R}^+$ such that $t \leq u, s \leq v$, then the following inequality is valid.*

$$S_{t,s} \leq S_{u,v}. \tag{4.31}$$

Proof. For convex function ϕ it holds ([14, p.2])

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1} \tag{4.32}$$

with $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Since by Theorem 4.1, Q_t^i is log convex, we can set in (4.32): $\phi(x) = \log Q_x^i$, $x_1 = t$, $x_2 = s$, $y_1 = u$, $y_2 = v$, we get

$$\frac{\log Q_s^i - \log Q_t^i}{s - t} \leq \frac{\log Q_v^i - \log Q_u^i}{v - u}. \tag{4.33}$$

From (4.33), we get (4.31) for $s \neq t$ and $u \neq v$.
For $s = t$ and $u = v$ we have limiting cases. ∎

Now, we define Cauchy type means for the family of functions ψ_t.

Theorem 4.3. *Let X, Y and W be matrices as in Theorem 2.1 with $x_{ij}, y_{ij} > 0$ $(i = 1, 2, ..., n, j = 1, 2, ..., m)$ and such that satisfy condition (2.13). Consider $\tilde{Q}_t^1 = \tilde{A}(X, Y, W; \psi_t)$, if (2.18) holds for every $\tau \in [\alpha, \beta]$ and $\tilde{Q}_t^2 = -\tilde{A}(X, Y, W; \psi_t)$, if (2.18) holds in the opposite direction for every $\tau \in [\alpha, \beta]$. Then the following statements are valid for $\tilde{Q}_t^i$(i=1,2):*

(a) for every $n \in \mathbb{N}$ and for every $p_k \in \mathbb{R}$, $k \in \{1, 2, ..., n\}$, the matrix $[\tilde{Q^i}_{\frac{p_k+p_l}{2}}]_{k,l=1}^n$ is a positive semi-definite matrix. Particularly

$$\det[\tilde{Q^i}_{\frac{p_k+p_l}{2}}]_{k,l=1}^n \geq 0; \tag{4.34}$$

(b) the function $t \to \tilde{Q^i}_t$ is exponentially convex,

(c) if $\tilde{Q^i}_t > 0$, then the function $t \to \tilde{Q^i}_t$ is log*-convex, i.e for $-\infty < r < s < t < \infty$, we have*

$$(\tilde{Q^i}_s)^{t-r} \leq (\tilde{Q^i}_r)^{t-s}(\tilde{Q^i}_t)^{s-r}. \tag{4.35}$$

Proof. The proof is similar to the proof of Theorem 4.1. ■

Let X, Y and W be matrices as in Theorem 2.1 with $x_{ij}, y_{ij} > 0$ ($i = 1, 2, ..., n, j = 1, 2, ..., m$) such that satisfy condition (2.13). If for all $s \in [\alpha, \beta]$, the inequality (2.18) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (2.18) holds and let $\tilde{Q^i}_t > 0$ for $t \in \mathbb{R}$,

$$\tilde{S}_{u,v} = \left(\frac{\tilde{Q^i}_u}{\tilde{Q^i}_v}\right)^{\frac{1}{u-v}} \quad i = 1, 2, \tag{4.36}$$

for $-\infty < u \neq v < +\infty$ are means of x_{ij} and y_{ij}. Moreover we can extend these means in other cases.

So by limit we have, for $u \neq 0, 1$,

$$\tilde{S}_{u,u} = \exp\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij}^u \log x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij}^u \log y_{ij}}{\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij}^u - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij}^u} - \frac{2u-1}{u(u-1)}\right),$$

$$\tilde{S}_{0,0} = \exp\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij} \log^2 x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij} \log^2 y_{ij}}{2\left(\sum_{i=1}^n \sum_{j=1}^m w_{ij} \log x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij} \log y_{ij}\right)} + 1\right),$$

$$\tilde{S}_{1,1} = \exp\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij} \log^2 x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij} \log^2 y_{ij}}{2\left(\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij} \log x_{ij} - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij} \log y_{ij}\right)} - 1\right).$$

Theorem 4.4 *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u, s \leq v$, then the following inequality is valid.*

$$\tilde{S}_{t,s} \leq \tilde{S}_{u,v}. \tag{4.37}$$

P r o o f. The proof is similar to the proof of Theorem 4.2. ∎

Finally, we define Cauchy type means for the family of functions δ_t.

Theorem 4.5 *Let X, Y and W be matrices as in Theorem 2.1 such that satisfy condition (2.13). Consider $\overline{Q}_t^1 = \tilde{A}(X,Y,W;\delta_t)$, if (2.18) holds for every $\tau \in [\alpha,\beta]$ and $\overline{Q}_t^2 = -\tilde{A}(X,Y,W;\delta_t)$, if (2.18) holds in the opposite direction for every $\tau \in [\alpha,\beta]$.*

Then the following statements are valid for $\overline{Q}_t^i$(i=1,2):

(a) for every $n \in \mathbb{N}$ and for every $p_k \in \mathbb{R}$, $k \in \{1,2,..,n\}$, the matrix $[\overline{Q^i}_{\frac{p_k+p_l}{2}}]_{k,l=1}^n$ is a positive semi-definite matrix. Particularly

$$\det[\overline{Q^i}_{\frac{p_k+p_l}{2}}]_{k,l=1}^n \geq 0; \tag{4.38}$$

(b) the function $t \to \overline{Q^i}_t$ is exponentially convex,

(c) if $\overline{Q^i}_t > 0$, then the function $t \to \overline{Q^i}_t$ is log-*convex, i.e for $-\infty < r < s < t < \infty$, we have*

$$(\overline{Q^i}_s)^{t-r} \leq (\overline{Q^i}_r)^{t-s}(\overline{Q^i}_t)^{s-r}. \tag{4.39}$$

P r o o f. The proof is similar to the proof of Theorem 4.1. ∎

Let X, Y and W be matrices as in Theorem 2.1 such that satisfy condition (2.13). If for all $s \in [\alpha,\beta]$, the inequality (2.18) holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in (2.18) holds and let $\overline{Q^i}_t > 0$ for $t \in \mathbb{R}$,

$$\overline{S}_{u,v} = \frac{1}{u-v}\log\left(\frac{v^2}{u^2}.\frac{\sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{ux_{ij}} - \sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{uy_{ij}}}{\sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{vx_{ij}} - \sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{vy_{ij}}}\right) \tag{4.40}$$

for $-\infty < u \neq v < +\infty$ are means of x_{ij} and y_{ij}. Moreover we can extend these means in other cases.

So by limit we have

$$\overline{S}_{u,u} = \frac{\sum_{i=1}^n\sum_{j=1}^m w_{ij}x_{ij}e^{ux_{ij}} - \sum_{i=1}^n\sum_{j=1}^m w_{ij}y_{ij}e^{uy_{ij}}}{\sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{ux_{ij}} - \sum_{i=1}^n\sum_{j=1}^m w_{ij}e^{uy_{ij}}} - \frac{2}{u}, \quad u \neq 0,$$

$$\overline{S}_{0,0} = \frac{\sum_{i=1}^n\sum_{j=1}^m w_{ij}x_{ij}^3 - \sum_{i=1}^n\sum_{j=1}^m w_{ij}y_{ij}^3}{3\left(\sum_{i=1}^n\sum_{j=1}^m w_{ij}x_{ij}^2 - \sum_{i=1}^n\sum_{j=1}^m w_{ij}y_{ij}^2\right)}.$$

Theorem 4.6 *Let $t,s,u,v \in \mathbb{R}$ such that $t \leq u, s \leq v$, then the following inequality is valid.*

$$\overline{S}_{t,s} \leq \overline{S}_{u,v}. \tag{4.41}$$

P r o o f. The proof is similar to the proof of Theorem 4.2. ■

R e m a r k 4.1. We can prove Theorem 3.1, Theorem 3.2, Corollary 3.1, Remark 3.1, Theorem 4.1, Cauchy type means (4.30), Theorem 4.2, Theorem 4.3, Cauchy type means (4.36), Theorem 4.4, Theorem 4.5, Cauchy type means (4.40) and Theorem 4.6 in a similar fashion for Theorem 2.1 (a), (b), (c), (d) and (e).

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