The Lower Estimate for Bernstein Operator

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For functions belonging to the classes $C^2[0,1]$ and $C^3[0,1]$, we establish the lower estimate with an explicit constant in approximation by Bernstein polynomials in terms of the second order Ditzian-Totik modulus of smoothness. Several applications to some concrete examples of functions are presented.

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1 Introduction

For every function $f \in C[0,1]$ the Bernstein polynomial operator is given by

$$B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n}) \cdot \binom{n}{k} x^k (1-x)^{n-k}, \ x \in [0,1]. \qquad (1.1)$$

To handle this operator, it is useful to utilize the second order Ditzian-Totik modulus of smoothness $\omega_\varphi^2(f,t)$ which is equivalent to the $K-$ functional given by

$$K_\varphi^2(f,t^2) := \inf_{g \in W_\infty^2(\varphi)} \left\{ \|f-g\| + t^2 \|\varphi^2 g''\|_{L_\infty[0,1]} \right\}, \qquad (1.2)$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $t > 0, n \in N$ and the weighted Sobolev space $W_\infty^2(\varphi)$, is given by

$$W_\infty^2(\varphi) = \left\{ f \in AC[0,1], \ f' \in AC_{loc}, \ \varphi^2 f'' \in L_\infty[0,1] \right\}. \qquad (1.3)$$

Remark 1. For definition, properties and many applications of $\omega_\varphi^2(f, t)$ and $K_\varphi^2(f, t^2)$ see [3]. For example, it is known that if in the definition of $K_\varphi^2(f, t^2)$, instead of the space $W_\infty^2(\varphi)$ we consider the space

$$C^2[0,1] = \{g : [0,1] \to \mathbb{R}; g \text{ is twice continuously diferentiable on } [0,1]\},$$

where $C^2[0,1] \subset W_\infty^2(\varphi)$, this replacement does not have any effect on the equivalence between $\omega_\varphi^2(f, t)$ and $K_\varphi^2(f, t^2)$.

In a pioneering work of Ditzian and Ivanov [2], a general theory was developed to obtain strong converse inequalities for a broad class of operators. For the Bernstein operator, in [2] it was proved a strong converse inequality of type B. The first proof of strong converse inequality of type A for Bernstein operator was given in 1994 by Knoop and Zhou in [15] and Totik in [19], which we cite here as:

Theorem A. *There exist two absolute constants $C_1, C_2 > 0$ such that*

$$C_1 \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \leq \|f - B_n f\| \leq C_2 \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \qquad (1.4)$$

holds for all $f \in C[0,1]$ and all $n \in N$.

The proof of Theorem A is very complicated. Concerning the absolute constants, very recently using different methods, in [17] among others it was proved that $C_2 = 3$ could be placed in the right-hand side of (1.4). But as far as we know, nothing is known about the constant in the lower estimate in (1.4). It is the aim of this paper to establish for the first time concrete value of the constant in the left-hand side of (1.4), but only for the functions $f \in C^2[0,1]$ and $f \in C^3[0,1]$.

Our main results can be stated as follows.

Theorem 1. *For any $f \in C^2[0,1]$ and any $\mu_0 \in (0,1)$, there exists $n_1(f, \mu_0) \in \mathbb{N}$ (depending on f and μ_0), such that for all $n \geq n_1$, we have*

$$\frac{\mu_0}{32} \cdot \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \leq \|f - B_n f\| \leq 3 \cdot \omega_\varphi^2(f, \frac{1}{\sqrt{n}}). \qquad (1.5)$$

Corollary 1. *For $m > 0$, let us define the class of functions*

$$C_{M,m}^{3,2}[0,1] = \{f : [0,1] \to \mathbb{R}; f \in C^3[0,1], \|f'''\| \leq M, |f''(x)| \geq m, x \in [0,1]\}.$$

Then for all $f \in C_{M,m}^{3,2}[0,1]$ and $n \geq n_1$ with $n_1 = \left[\frac{1024M^2}{m^2}\right] + 1$, we have

$$\frac{1}{64} \cdot \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \leq \|f - B_n f\| \leq 3\omega_\varphi^2(f, \frac{1}{\sqrt{n}}).$$

Remark 2. It is evident that Theorem 1 and Corollary 1 have the disadvantage that the constants $\frac{\mu_0}{32}$ and $\frac{1}{64}$ in the corresponding lower estimates and the index n_1 depend on the functions f, as being valid only for $n \geq n_1$ and for functions in $C^2[0,1]$ and in $C^3[0,1]$, respectively. However, their advantage is that it is for the first time when for a broad class of functions (including many particular elementary functions, like $\exp(x)$, $\sin(x)$, $\cos(x)$, $\arctg(x)$, etc.) these results allow to give concrete lower bounds in terms of the modulus $\omega_\varphi^2(f, \frac{1}{\sqrt{n}})$ for the norm of $\|B_n f - f\|$. Note that these constants are not possible to be deduced from all other well known results, like those from Knoop and Zhou [15], Totik [19], Ditzian and Ivanov [2], and many others.

In Section 2 we give some auxiliary results and establish in Theorems 3 and 4 norm estimates in Voronovskaja's theorem for Bernstein operator. In Section 3 we prove Theorem 1 and Corollary 1. Finally, in Section 4 several applications to some concrete examples of functions are presented.

2 Auxiliary results

As already mentioned, the moduli $\omega_\varphi^2(f, t)$ and $K_\varphi^2(f, t^2)$ are equivalent. For example, Theorem 6.2 in Chapter 6 in [1] (see also [3]) states that, there are constants $C_1, C_2 > 0$, such that for all $f \in L_\infty$

$$C_1 \omega_\varphi^2(f, t) \leq K_\varphi^2(f, t^2) \leq C_2 \omega_\varphi^2(f, t), \, 0 < t \leq \frac{1}{2}. \tag{2.1}$$

For our goals it is important to determine explicitly the values of C_1 and C_2 in (2.1). We recall the following result, established in [17] and in [9]:

Theorem B. *For all $f \in W_\infty^2(\varphi)$ and all $n \geq 1$ the following holds true:*

$$A_1 \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \leq K_\varphi^2(f, \frac{1}{n}) \leq A_2 \omega_\varphi^2(f, \frac{1}{\sqrt{n}}), \tag{2.2}$$

where the value of $A_1 = \frac{1}{16}$ was established in [17] and $A_2 = 10$ follows from [9].

The next estimate was proved in Theorem 6.1 in [1]:

Theorem C. *There is a constant $C > 0$, depending only on r, such that for each $f \in W_p^2(\varphi)$, $1 \leq p \leq \infty$ we have*

$$\omega_\varphi^r(f, t)_p \leq C t^r \|\varphi^r f^{(r)}\|_p, \quad 0 \leq t \leq \frac{1}{2r}. \tag{2.3}$$

Note that from the proof of Theorem C, it is not possible to determine the magnitude of C in (2.3). But using Theorem B, we may proceed as follows:

For $f \in C^2[0,1]$ we have

$$\omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \leq \frac{1}{A_1} \cdot K_\varphi^2(f, \frac{1}{n}) \leq \frac{1}{A_1} \cdot \frac{1}{n} \|\varphi^2 f''\|. \tag{2.4}$$

For the value of $A_1 = \frac{1}{16}$ see Theorem 2 in [17] and for the value of $A_2 = 10$ see Corollary 9 in [9].

Even more, it is possible to establish a lower bound for $\omega_\varphi^2(f, \frac{1}{\sqrt{n}})$, $f \in C^2[0,1]$, as follows.

Theorem 2. *For any $f \in C^2[0,1]$ and any $\lambda_0 \in (0,1)$, there exists n_0 (depending on f and λ_0) such that for all $n \geq n_0$ the following*

$$\lambda_0 \cdot \frac{1}{n} \|\varphi^2 f''\| \leq \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \tag{2.5}$$

holds true.

P r o o f. If f is a polynomial of degree ≤ 1 on $[0,1]$, then the inequality one reduces to the equality $0 = 0$.

Therefore, suppose that f is not a polynomial of degree ≤ 1. By the definition of $\omega_\varphi^2(f, \delta)$, for all $0 \leq \delta < 1$ we can write (see e.g. [1])

$$\omega_\varphi^2(f, \delta) = \sup\{\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|;$$

$$x \in [h^2/(1 + h^2), 1/(1 + h^2)]\}; 0 \leq h \leq \delta\}.$$

Because f is not a polynomial of degree ≤ 1, it follows that $f''(x)$ is not identical equal to zero on $[0,1]$ and that there exists a point $x_0 \in [0,1]$ such that $\|\varphi^2 f''\|_\infty = |\varphi^2(x_0)f''(x_0)| > 0$ (contrariwise would easily follow that f is a polynomial of degree ≤ 1, a contradiction).

Since evidently that $x_0 \in (0,1)$, this implies that there exists $h_0 \in (0,1)$ such that for all $h \in [0, h_0]$ we have $x_0 \in [h^2/(1 + h^2), 1/(1 + h^2)]$, and

$$\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in [h^2/(1 + h^2), 1/(1 + h^2)]\}$$

$$\geq |f(x_0 + h\varphi(x_0) - 2f(x_0) + f(x - h\varphi(x_0)| = h^2\varphi^2(x_0) \cdot |f''(\xi_{h,x_0})|,$$

for all $h \in [0, h_0]$, where from the mean value theorem

$$\xi_{h,x_0} \in [x_0 - h\varphi(x_0), x + h\varphi(x_0)].$$

For $h \to 0$, evidently that $\xi_{h,x_0} \to x_0$ and from the continuity of f'' on $[0,1]$, it follows that $\lim_{h \to 0} |f''(\xi_{h,x_0})| = |f''(x_0| > 0$. Therefore, for $\lambda_0 \in (0,1)$, there

exists $0 \leq h_1 < h_0$, such that for all $0 \leq h < h_1$, we have $|f''(\xi_{h,x_0})| \geq \lambda_0 |f''(x_0)|$ and combined with the above lower estimate, implies

$$\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in [h^2/(1 + h^2), 1/(1 + h^2)]\}$$

$$\geq \lambda_0 h^2 \|\varphi^2 f''\|.$$

Now, let $n_0 \in \mathbb{N}$ be the smallest natural number such that $\frac{1}{\sqrt{n}} \leq h_1$, for all $n \geq n_0$.

Then, for fixed arbitrary $n \geq n_0$, passing above to supremum after $h \in [0, 1/\sqrt{n})$, we immediately get

$$\omega_\varphi^2(f; 1/\sqrt{n}) \geq \lambda_0 \cdot \frac{1}{n} \|\varphi^2 f''\|,$$

which completes the proof.   ∎

The crucial step in the proof of Theorem 1 is to establish norm estimate in the theorem of Voronovskaja. This theorem was first proved in [20] and is given in the book of DeVore and Lorentz [1] as follows:

Theorem D. *If f is bounded on $[0, 1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ for some $x \in [0, 1]$, then*

$$\lim_{n \to \infty} n \cdot [B_n(f, x) - f(x)] = \frac{x(1 - x)}{2} \cdot f''(x). \tag{2.6}$$

If $f \in C^2[0, 1]$, the convergence is uniform.

This result has attracted the attention of many authors in the last 80 years. Very recently some quantitative estimates in pointwise variant of Voronovskaja's theorem are obtained in [11,12,13,14,18] and for the complex-valued functions of complex variable in [4,5,6]. Concerning norm estimates we cite Lemma 8.3 in [2] as:

Theorem E. *For $f \in W_\infty^3(\varphi)$ and $n \geq 12$ we have*

$$\|B_n f - f - \frac{1}{2n} \cdot \varphi^2 f''\|_{L_\infty} \leq n^{-\frac{3}{2}} \cdot \|\varphi^3 f'''\|_{L_\infty}. \tag{2.7}$$

Our next two statements extend this result to $f \in C^2[0, 1]$.

Theorem 3. *For any $f \in C^2[0, 1]$, there exists $n_0 := n_0(f)$, such that for all $n \geq n_0$ we have*

$$\left\| B_n f - f - \frac{1}{2n} \cdot \varphi^2 f'' \right\| \leq 4\omega_\varphi^2(f, \frac{1}{\sqrt{n}}). \tag{2.8}$$

P r o o f. Obviously,

$$\left\| B_n f - f - \frac{1}{2n} \cdot \varphi^2 f'' \right\| \le \| B_n f - f \| + \frac{1}{2n} \| \varphi^2 f'' \|$$

$$\le 3\omega_\varphi^2(f, \frac{1}{\sqrt{n}}) + \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) = 4\omega_\varphi^2(f, \frac{1}{\sqrt{n}}),$$

where we have applied the upper estimate in [10] and (2.5), respectively. The proof is completed.   ∎

Moreover, in terms of the usual moduli of continuity, we can obtain the following better estimate.

Theorem 4. *If $f \in C^2[0, 1]$, then for all $n \in \mathbb{N}$, $n \ge 2$, it holds*

$$\| B_n f - f - \frac{1}{2n} \cdot \varphi^2 f'' \| \le \frac{5}{8n}\omega_1(f'', \frac{1}{\sqrt{n}}) + \frac{13}{64n}\omega_2(f'', \frac{1}{\sqrt{n}}). \qquad (2.9)$$

P r o o f. By Theorem 4 in Gonska-Raşa [13], for $f \in C^2[0, 1]$, $n \ge 2$, $x \in [0, 1]$ and $X = x(1 - x)$, we have

$$\left| B_n(f, x) - f(x) - \frac{x(1 - x)}{2n} f''(x) \right|$$

$$\le \frac{X}{n} \left\{ \frac{X'}{\sqrt{3(n - 2)X + 1}} \cdot \frac{5}{6}\omega_1\left(f'', \sqrt{\frac{3(n - 2)X + 1}{n^2}}\right) \right.$$

$$\left. + \frac{13}{16}\omega_2\left(f'', \sqrt{\frac{3(n - 2)X + 1}{n^2}}\right) \right\}.$$

Passing to uniform norm and taking into account the inequalities $0 \le X \le 1/4$, $|X'| = |1 - 2x| \le 3$, for all $x \in [0, 1]$ we immediately obtain

$$\| B_n f - f - \frac{1}{2n} \cdot \varphi^2 f'' \| \le \frac{1}{4n} \left\{ 3 \cdot \frac{5}{6}\omega_1\left(f'', \sqrt{\frac{[3(n - 2)/4] + 1}{n^2}}\right) \right.$$

$$\left. + \frac{13}{16}\omega_2\left(f'', \sqrt{\frac{[3(n - 2)/4] + 1}{n^2}}\right) \right\}$$

$$\le \frac{5}{8n}\omega_1(f'', \frac{1}{\sqrt{n}}) + \frac{13}{64n}\omega_2(f'', \frac{1}{\sqrt{n}}),$$

which proves the theorem.   ∎

3   Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1.

P r o o f. First, note that in the statement of Theorem 1, we may suppose that f is not a polynomial of degree ≤ 1, because if f is a polynomial of degree ≤ 1 then Theorem 1 holds trivially. This supposition obviously implies that in what follows we have $\|\varphi^2 f''\| > 0$.

We apply the ideas in the case of Bernstein polynomials of complex variable in the proof of Theorem 2.1 in [8]. The following identity is valid for all f, which are not polynomials of degree ≤ 1

$$B_n(f, x) - f(x) = \omega_\varphi^2(f, \frac{1}{\sqrt{n}})\Big\{ \frac{\varphi^2(x) f''(x)}{2} \cdot \frac{1}{n\omega_\varphi^2(f, \frac{1}{\sqrt{n}})}$$
$$+ \left[\frac{1}{\omega_\varphi^2(f, \frac{1}{\sqrt{n}})} \left(B_n(f, x) - f(x) - \frac{\varphi^2(x) f''(x)}{2n} \right) \right] \Big\}. \tag{3.1}$$

This immediately implies

$$\|B_n f - f\| \geq \omega_\varphi^2(f, \frac{1}{\sqrt{n}})\Big\{ \frac{\|\varphi^2 f''\|}{2} \cdot \frac{1}{n\omega_\varphi^2(f, \frac{1}{\sqrt{n}})}$$
$$- \left[\frac{1}{\omega_\varphi^2(f, \frac{1}{\sqrt{n}})} \cdot \|B_n f - f - \frac{\varphi^2 f''}{2n}\| \right] \Big\}. \tag{3.2}$$

Now from (2.4) we obtain

$$\frac{1}{n\omega_\varphi^2(f, \frac{1}{\sqrt{n}})} \geq \frac{A_1}{\|\varphi^2 f''\|}, \quad \text{for all } n \in \mathbb{N}, \tag{3.3}$$

which implies

$$\|B_n f - f\| \geq \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \cdot \left\{ \frac{A_1}{2} - A_n \right\}, \tag{3.4}$$

where

$$A_n = \frac{\frac{5}{8n}\omega_1(f'', 1/\sqrt{n}) + \frac{13}{64n}\omega_2(f'', 1/\sqrt{n})}{\omega_\varphi^2(f, \frac{1}{\sqrt{n}})} \to 0, \quad \text{as } n \to \infty.$$

More precisely, using (2.4) and (2.5), we immediately obtain the double inequality

$$\frac{1}{16\|\varphi^2 f''\|} \left[\frac{5}{8}\omega_1(f''; 1/\sqrt{n}) + \frac{13}{64}\omega_2(f''; 1/\sqrt{n}) \right] \leq A_n$$

$$\leq \frac{1}{\lambda_0 \|\varphi^2 f''\|} \left[\frac{5}{8} \omega_1(f''; 1/\sqrt{n}) + \frac{13}{64} \omega_2(f''; 1/\sqrt{n}) \right], \tag{3.5}$$

where the right-hand side holds for all $n \geq n_0(f, \lambda_0)$ (that comes from (2.5)), while the left-hand side holds for all $n \geq 2$.

Therefore, from the right-hand side of (3.5) it follows: $\lim_{n \to \infty} A_n = 0$.

From (3.4) and (3.5), it is evident that there exists $n_1(f, \lambda_0, \mu_0) > n_0(f, \lambda_0)$, such that for all $n \geq n_1(f, \lambda_0, \mu_0)$

$$\frac{A_1}{2} - A_n \geq \frac{\mu_0 A_1}{2} = \frac{\mu_0}{32}. \tag{3.6}$$

Therefore, from (3.4) we have

$$\|B_n f - f\| \geq \frac{\mu_0}{32} \cdot \omega_\varphi^2(f, \frac{1}{\sqrt{n}}), \tag{3.7}$$

for all $n \geq n_1$, which proves Theorem 1.   ∎

Proof of Corollary 1.

P r o o f. We have to prove just the left-hand side inequality. Following the lines in the proof of Theorem 2, since $\frac{1}{2} \in [h^2/(1 + h^2), 1/(1 + h^2)]$ for all $0 \leq h < 1$, we easily get

$$\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \in [h^2/(1 + h^2), 1/(1 + h^2)]\}$$

$$\geq |f(1/2 + h\varphi(1/2) - 2f(1/2) + f(1/2 - h\varphi(1/2)| = h^2\varphi^2(1/2) \cdot |f''(\xi_{h,1/2})|$$

$$\geq \frac{mh^2}{4}, \text{ for all } 0 \leq h < 1,$$

where passing to supremum with $0 \leq h \leq \delta$ implies $\omega_\varphi^2(f, \delta) \geq \frac{m}{4} \cdot \delta^2$.

Therefore, by taking $\delta = \frac{1}{\sqrt{n}}$, it follows $\omega_\varphi^2(f, 1/\sqrt{n}) \geq \frac{m}{4} \cdot \frac{1}{n}$.

Now, following the lines in the proof of Theorem 1, where instead of Theorem 4 we use Theorem E, we get the same relationship (3.4), where because of the hypothesis, the estimate in Theorem E becomes

$$\|B_n f - f - \frac{1}{2n} \cdot \varphi^2 f''\| \leq \frac{M}{8n^{3/2}},$$

with A_n upper bounded as follows :

$$A_n \leq \frac{Mn^{-3/2}}{8\omega_\varphi^2(f, 1/\sqrt{n})} \leq \frac{M}{8n^{3/2}} \cdot \frac{4n}{m} = \frac{M}{2m\sqrt{n}}.$$

Therefore, from the lines in the proof of Theorem 1 we obtain

$$\|B_n f - f\| \geq \omega_\varphi^2(f, \frac{1}{\sqrt{n}}) \cdot \left\{ \frac{A_1}{2} - \frac{M}{2m\sqrt{n}} \right\},$$

which by (3.6) is valid for all n satisfying

$$\frac{M}{2m\sqrt{n}} \leq \frac{1}{64},$$

that is for all $n \geq n_1$ with $n_1 = \left[\frac{1024M^2}{m^2} \right] + 1$, which proves the corollary. $\blacksquare$

Remark 3. If we suppose, in addition, that for example $f \in C^4[0,1]$, then from the above proof of Theorem 1, relationship (3.6), it easily follows that $A_n \leq \frac{1-\mu_0}{32}$, and therefore that the index n_1 in the statement, necessarily must be chosen greater than the smallest number $n_2 \in \mathbb{N}$ that satisfies the inequality

$$\frac{1}{\lambda_0 \|\varphi^2 f''\|} \left[\frac{5}{8\sqrt{n}} \cdot \|f'''\| + \frac{13}{64n} \cdot \|f^{(4)}\| \right] \leq \frac{1-\mu_0}{32}.$$

Indeed, this immediately follows from the inequality

$$\frac{1}{\lambda_0 \|\varphi^2 f''\|} \left[\frac{5}{8} \omega_1(f''; 1/\sqrt{n}) + \frac{13}{64} \omega_2(f''; 1/\sqrt{n}) \right]$$

$$\leq \frac{1}{\lambda_0 \|\varphi^2 f''\|} \left[\frac{5}{8\sqrt{n}} \cdot \|f'''\| + \frac{13}{64n} \cdot \|f^{(4)}\| \right].$$

If instead of $f \in C^4[0,1]$, we suppose that $f \in W_\infty^3(\varphi)$ and in the proof of Theorem 1, instead of Theorem 4 we use Theorem E, then we easily get

$$A_n = \frac{n^{-3/2} \|\varphi^3 f'''\|}{[\lambda_0 \|\varphi^2 f''\|/(n)]} = \frac{1}{\sqrt{n}} \cdot \frac{\|\varphi^3 f'''\|}{\lambda_0 \|\varphi^2 f''\|},$$

for all $n \geq n_0(f, \lambda_0)$ and that the index $n_1(f, \mu_0, \lambda_0)$ in the statement of Theorem 1, necessarily must be greater than the smallest n_2 that satisfies the inequality $A_n \leq \frac{1-\mu_0}{32}$. Simple calculation shows that we may take

$$n_2 = \left[\left(\frac{32 \|\varphi^3 f'''\|}{\lambda_0 (1-\mu_0) \|\varphi^2 f''\|} \right)^2 \right] + 1,$$

where $[a]$ means the integer part of a.

Remark 4. Note that for the limit case of Bernstein operator, $U_n(f, x)$, it was proved by Parvanov and Popov the following strong converse inequality in [16]:

$$\frac{1}{2} \|U_n f - f\| \leq K_\varphi^2(f, \frac{1}{n}) \leq (6 + \sqrt{8}) \|U_n f - f\|. \tag{3.8}$$

The proof relies on the commutativity of U_n-a property, which is not available for B_n.

4   Particular Examples

For some particular classes of functions, the index n_1 in Theorem 1 and Corollary 1 can be explicitly obtained, as follows.

Example 1. First, consider $f(x) = \exp(x)$, $x \in [0, 1]$. Since we have $\|f'''\| \leq e$ and $|f''(x)| \geq 1$, for all $x \in [0, 1]$, it follows that we can apply Corollary 1, obtaining that we have

$$\|B_n(\exp, \cdot) - \exp\| \geq \frac{1}{64}\omega_\varphi^2(\exp, 1/\sqrt{n}), \quad \text{for all } n > 1024e^2.$$

Example 2. Secondly, consider $f(x) = \cos(x)$, $x \in [0, 1]$. Since have $\|f'''\| \leq 1$ and $|f''(x)| \geq \cos(1) \approx 0.540302306$, for all $x \in [0, 1]$, it follows that again we can apply Corollary 1, obtaining that we have

$$\|B_n(\cos, \cdot) - \cos\| \geq \frac{1}{64}\omega_\varphi^2(\cos, 1/\sqrt{n}), \quad \text{for all } n \geq \left[\frac{1024}{\cos(1)}\right] + 1.$$

Example 3. For third example, by taking $f(x) = \sin(x)$, $x \in [0, 1]$, it is clear that we cannot apply Corollary 1, but we can apply Theorem 1, or more exactly its variant expressed by Remark 3, first part.

First, let us find the index n_0 in the Theorem 2 (as Theorem 2 is used in the proof of Theorem 1). For this purpose, note that simple calculation leads us to

$$|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))| = 2|\sin(x)[1 - \cos(h\varphi(x))]|$$

$$= 4\sin(x) \cdot \sin^2(h\varphi(x)/2) \geq 4 \cdot \frac{2}{\pi}x \cdot \frac{4}{\pi^2} \cdot \frac{h^2\varphi^2(x)}{4} = \frac{8}{\pi^3}xh^2\varphi^2(x),$$

for all $x \in \left[\frac{h^2}{1+h^2}, \frac{1}{1+h^2}\right]$. Here we used the inequality $\sin(x) \geq \frac{2}{\pi}x$, for all $x \in [0, \pi/2]$.

Suppose in what follows that $0 \leq h \leq \delta \leq \frac{1}{\sqrt{2}}$. It follows that $\frac{2}{3} \in \left[\frac{h^2}{1+h^2}, \frac{1}{1+h^2}\right]$. Also, it is clear that $x\varphi^2(x)$ attains its maximum value $\frac{4}{27}$ on $[0, 1]$ at $x = 2/3$.

Therefore, passing above to supremum firstly with respect to x, and then with respect to $h \in [0, \delta]$, with $\delta \leq \frac{1}{\sqrt{2}}$, it is immediate that

$$\omega_\varphi^2(\sin, \delta) \geq \frac{8}{\pi^3} \cdot \frac{4}{27} \cdot \delta^2 = \frac{32}{27\pi^3}\delta^2,$$

that is for $\delta = 1/\sqrt{n}$ with $n \geq 2$, we obtain

$$\omega_\varphi^2(\sin, 1/\sqrt{n}) \geq \frac{32}{27\pi^3} \cdot \frac{1}{n} \geq \frac{32}{27\pi^3} \cdot \frac{\|\varphi^2 f''\|}{n}, \quad \text{for all } n \geq 2.$$

It follows that Theorem 2 holds with $\lambda_0 = \frac{32}{27\pi^3}$ and $n_0 = 2$.

Now, because

$$\|\varphi^2 f''\| \geq \varphi^2(1/2)\sin(1/2) = \frac{1}{4}\sin(1/2) \approx 0.25 \times 0.4794 = 0.11985,$$

by the first part of Remark 3 we obtain that n_2 there, can be chosen as the smallest index n satisfying the last inequality below

$$\frac{1}{\lambda_0\|\varphi^2 f''\|}\left[\frac{5}{8\sqrt{n}}\cdot\|f'''\| + \frac{13}{64n}\cdot\|f^{(4)}\|\right] \leq \frac{4}{\lambda_0\sin(1/2)}\left[\frac{5}{8\sqrt{n}} + \frac{13}{64n}\right]$$

$$\leq \frac{1-\mu_0}{32}, \tag{4.1}$$

with $\lambda_0, \mu_0 \in (0,1)$, arbitrary fixed.

From here, it is immediate that n_2 can be chosen the smallest index n satisfying

$$\frac{4}{\lambda_0\sin(1/2)}\left[\frac{5}{8} + \frac{13}{64}\right]\cdot\frac{1}{\sqrt{n}} \leq \frac{1-\mu_0}{32},$$

which by simple calculation leads to $n_2 = \left[\left(\frac{106}{\lambda_0(1-\mu_0)sin(1/2)}\right)^2\right] + 1$ (here $[a]$ means the integer part of a), with $\lambda_0 = \frac{32}{27\pi^3}$. Choosing above $\mu_0 = 1/2$, we obtain

$$\|B_n(\sin,\cdot) - \sin\| \geq \frac{1}{64}\omega_\varphi^2(\sin, \frac{1}{\sqrt{n}}), \quad \text{for all } n \geq \max\{2, n_2\} = n_2.$$

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