

Fekete-Szegő problem for certain subclasses of analytic functions with respect to symmetric points

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Let $S_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$ be the class of normalized analytic functions defined in the open unit disk U satisfy the following subordination

$$(1 - \alpha) \frac{(1 - t)z (D_{\lambda,\mu}^n f(z))'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} + \alpha \frac{\left[(1 - t)z (D_{\lambda,\mu}^n f(z))' \right]'}{\left[D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz) \right]'} \prec \left(\frac{1 + Az}{1 + Bz} \right)^\delta,$$

where $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha \leq 1$, $0 < \delta \leq 1$, $-1 \leq B < A \leq 1$ and $D_{\lambda,\mu}^n$ is the Raducanu-Orhan multiplier differential operator. The object of the present investigation is to obtain sharp upper bounds for the Fekete-Szegő functional $|a_3 - \xi a_2^2|$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let Ω denote the class of bounded analytic functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. It is known [13] that

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \tag{1.2}$$

Let us recall the principle of subordination between analytic functions. Let the functions f and g be analytic in U . Then we say that f is subordinate to g , if there exists a Schwarz function $w(z)$, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in U),$$

such that

$$f(z) = g(w(z)) \quad (z \in U).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U).$$

In particular, if the function g is univalent in U , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For $f(z)$ belonging to $\mathcal{A}$, the multiplier differential operator $D_{\lambda,\mu}^n f$ was defined by Raducanu and Orhan (see [19]) as follows:

$$D_{\lambda,\mu}^0 f(z) = f(z),$$

$$D_{\lambda,\mu}^1 f(z) = D_{\lambda,\mu} f(z) = \lambda \mu z^2 (f(z))'' + (\lambda - \mu) z (f(z))' + (1 - \lambda + \mu) f(z),$$

$$D_{\lambda,\mu}^2 f(z) = D_{\lambda,\mu} \left(D_{\lambda,\mu}^1 f(z) \right),$$

$$\vdots$$

$$D_{\lambda,\mu}^n f(z) = D_{\lambda,\mu} \left(D_{\lambda,\mu}^{n-1} f(z) \right),$$

where $0 \leq \mu \leq \lambda \leq 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Very recently, the operator $D_{\lambda,\mu}^n f$ was extended to $\lambda \geq \mu \geq 0$ by Deniz and Orhan in [4]. We use this operator for $\lambda \geq \mu \geq 0$.

If f is given by (1.1), then from the definition of the operator $D_{\lambda,\mu}^n f(z)$ it is easy to see that

$$D_{\lambda,\mu}^n f(z) = z + \sum_{k=2}^{\infty} \Phi_k^n a_k z^k,$$

where $\Phi_k^n = [1 + (\lambda \mu k + \lambda - \mu)(k - 1)]^n$, $(\Phi_k^n = [\Phi_k]^n)$; $\lambda \geq \mu \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It should be remarked that $D_{\lambda,\mu}^n$ generalizes many other linear operators considered earlier by different authors. In particular, for $f \in \mathcal{A}$ we have the following:

- $D_{1,0}^n f(z) \equiv D^n f(z)$ the operator investigated by Sălăgean (see [21]).

- $D_{\lambda,0}^n f(z) \equiv D_{\lambda}^n f(z)$ the operator worked by Al-Oboudi (see [1]).

By the motivation of the paper in [12], we define the following new subclass of analytic functions.

A function $f(z) \in \mathcal{A}$ is said to be in the class $S_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$, if it satisfies for all $z \in U$

$$(1-\alpha) \frac{(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} + \alpha \frac{\left[(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)' \right]'}{\left[D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz) \right]'} \prec \left(\frac{1+Az}{1+Bz} \right)^{\delta},$$

where $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha \leq 1$, $0 < \delta \leq 1$ and $-1 \leq B < A \leq 1$.

For special cases of the parameters $n, \lambda, \mu; \alpha, t; \delta, A, B$, we get the following classes introduced and studied by other authors:

- $S_{\lambda,\mu}^0(0, -1; 1, 1, -1)$ by Sakaguchi [20],
- $S_{\lambda,\mu}^0(0, t; 1, 1, -1)$ by Owa et al. [17],
- $S_{\lambda,\mu}^0(1, -1; \delta, A, B)$ by Das and Singh [3],
- $S_{\lambda,\mu}^0(0, -1; 1, A, B)$ by Goel and Mehrok [7],
- $S_{\lambda,\mu}^0(1, -1; 1, A, B)$ by Janteng and Halim [8],
- $S_{\lambda,\mu}^0(\alpha, t; \delta, A, B)$ by Mehrok et al. [12].

Various properties of Sakaguchi type functions and Fekete-Szegő problem for certain classes of univalent functions were discussed by many authors including ([2, 5, 6, 9, 10, 11, 14, 15, 16, 18]). The functions in the class $S_{\lambda,\mu}^0(0, -1; 1, 1, -1) \equiv S_s^*$ are called starlike with respect to symmetrical points and the functions in the class $S_{\lambda,\mu}^0(1, -1; 1, 1, -1) \equiv K_s$ are called convex with respect to symmetrical points in U .

We also introduce the following classes for $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha \leq 1$, $0 < \delta \leq 1$ and $-1 \leq B < A \leq 1$:

A function $f(z) \in \mathcal{A}$ is said to be in the class $T_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$ if it satisfies

$$(1-\alpha) \frac{(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} + \alpha \frac{\left[(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)' \right]'}{\left[D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz) \right]'} \prec \left(\frac{1+Az}{1+Bz} \right)^{\delta}, g \in S_s^*,$$

and a function $f(z) \in \mathcal{A}$ is said to be in the class $H_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$ if it satisfies

$$(1-\alpha) \frac{(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n h(z) - D_{\lambda,\mu}^n h(tz)} + \alpha \frac{\left[(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)' \right]'}{\left[D_{\lambda,\mu}^n h(z) - D_{\lambda,\mu}^n h(tz) \right]'} \prec \left(\frac{1+Az}{1+Bz} \right)^\delta, h \in K_s.$$

The following lemma will be used.

Lemma 1.1. ([20]) Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{A}$. Then, for $k \geq 2$

$$|b_k| \leq \begin{cases} 1, & \text{if } g \in S_s^* \\ \frac{1}{k}, & \text{if } g \in K_s. \end{cases}$$

2. Coefficient Inequalities

For $\lambda \geq \mu \geq 0$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $0 \leq \alpha \leq 1$, $|t| \leq 1$, $t \neq 1$, $0 < \delta \leq 1$ and $-1 \leq B < A \leq 1$, we have the following main results.

Theorem 2.1. If $f \in S_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$, then

$$|a_2| \leq \frac{\delta(A-B)}{(1-t)(1+\alpha)\Phi_2^n},$$

$$|a_3| \leq \begin{cases} v, & \text{if } |\rho| \leq 1, \\ v|\rho|, & \text{if } |\rho| \geq 1, \end{cases}$$

and

$$|a_3 - \xi a_2^2| \leq \begin{cases} v, & \text{if } |\xi + \sigma| \leq \tau, \\ \frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi_2^{2n}} |\xi + \sigma|, & \text{if } |\xi + \sigma| \geq \tau, \end{cases}$$

where

$$v = \frac{\delta(A-B)}{(1-t)(2+t)(1+2\alpha)\Phi_3^n}, \quad (2.1)$$

$$\rho = B + \frac{(1-\delta)(A-B)}{2} - \frac{\delta(1+t)(1+3\alpha)(A-B)}{(1-t)(1+\alpha)^2}, \quad (2.2)$$

$$\tau = \frac{(1-t)(1+\alpha)^2\Phi_2^{2n}}{\delta(2+t)(1+2\alpha)(A-B)\Phi_3^n}, \quad (2.3)$$

$$\sigma = \tau\rho. \quad (2.4)$$

Proof. Since $f \in S_{\lambda, \mu}^n(\alpha, t; \delta, A, B)$, it follows that

$$(1 - \alpha) \frac{(1 - t)z \left(D_{\lambda, \mu}^n f(z) \right)'}{D_{\lambda, \mu}^n f(z) - D_{\lambda, \mu}^n f(tz)} + \alpha \frac{\left[(1 - t)z \left(D_{\lambda, \mu}^n f(z) \right)' \right]'}{\left[D_{\lambda, \mu}^n f(z) - D_{\lambda, \mu}^n f(tz) \right]'} = \left(\frac{1 + Aw(z)}{1 + Bw(z)} \right)^\delta. \quad (2.5)$$

By expanding (2.5), we get

$$\begin{aligned} & 1 + (1 - t)(1 + \alpha)\Phi_2^n a_2 z \\ &= (1 - t) \left[(2 + t)(1 + 2\alpha)\Phi_3^n a_3 - (1 + t)(1 + 3\alpha)\Phi_2^{2n} a_2^2 \right] z^2 + \dots \\ &= 1 + \delta(A - B)c_1 z + \delta(A - B) \left(c_2 - \left[B + \frac{1}{2}(1 - \delta)(A - B) \right] c_1^2 \right) z^2 + \dots \end{aligned}$$

From this equality we have

$$a_2 = \frac{\delta(A - B)}{(1 - t)(1 + \alpha)\Phi_2^n} c_1 \quad (2.6)$$

and

$$a_3 = v \left(c_2 - \rho c_1^2 \right), \quad (2.7)$$

where v and ρ are defined in (2.1) and (2.2) respectively. Now, by using (2.6) and (2.7) we obtain

$$a_3 - \xi a_2^2 = v c_2 - \frac{\delta^2(A - B)^2}{(1 - t)^2(1 + \alpha)^2 \Phi_2^{2n}} [\xi + \sigma] c_1^2,$$

where σ is defined in (2.4). By using $|c_1| \leq 1$ from (1.2) we get

$$|a_2| \leq \frac{\delta(A - B)}{(1 - t)(1 + \alpha)\Phi_2^n}.$$

Applying the well known triangle inequality, we arrive at

$$|a_3| \leq v \left(|c_2| + |\rho| |c_1|^2 \right) \quad (2.8)$$

and

$$\left| a_3 - \xi a_2^2 \right| \leq v |c_2| + \frac{\delta^2(A - B)^2}{(1 - t)^2(1 + \alpha)^2 \Phi_2^{2n}} |\xi + \sigma| |c_1|^2. \quad (2.9)$$

If we use the inequality $|c_2| \leq 1 - |c_1|^2$ in (1.2), the inequalities (2.8) and (2.9) lead us to

$$|a_3| \leq v \left[1 + (|\rho| - 1) |c_1|^2 \right], \quad (2.10)$$

$$\left| a_3 - \xi a_2^2 \right| \leq v + \frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi_2^{2n}} (|\xi + \sigma| - \tau) |c_1|^2, \quad (2.11)$$

where τ is defined in (2.3).

If $|\rho| \leq 1$, then

$$|a_3| \leq v$$

and for $|\rho| \geq 1$, by using $|c_1| \leq 1$ from (1.2), the inequality (2.10) implies that

$$|a_3| \leq v |\rho|.$$

Similarly, for $|\xi + \sigma| \leq \tau$, we have

$$\left| a_3 - \xi a_2^2 \right| \leq v.$$

The bound is sharp for $w(z) = z^2$.

If $|\xi + \sigma| \geq \tau$, again by using $|c_1| \leq 1$ we get

$$\left| a_3 - \xi a_2^2 \right| \leq \frac{\delta^2(A-B)^2}{(1-t)^2(1+\alpha)^2\Phi_2^{2n}} |\xi + \sigma|.$$

This bound is sharp for $w(z) = z$. ■

When $n = 0$ and $t = -1$ from Theorem 2.1 we can state the following result given by Mehrotra et al. [12].

Corollary 2.2. *If $f \in S_{\lambda, \mu}^0(\alpha, -1; \delta, A, B)$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \begin{cases} \frac{\delta(A-B)}{2(1+2\alpha)}, & \text{if } |\xi + \sigma_1| \leq \tau_1, \\ \frac{\delta^2(A-B)^2}{4(1+\alpha)^2} |\xi + \sigma|, & \text{if } |\xi + \sigma_1| \geq \tau_1, \end{cases}$$

where

$$\begin{aligned} \sigma_1 &= \frac{(1+\alpha)^2[2B + (1-\delta)(A-B)]}{(1+2\alpha)\delta(A-B)}, \\ \tau_1 &= \frac{2(1+\alpha)^2}{\delta(1+2\alpha)(A-B)}. \end{aligned}$$

For special cases of the parameters $n, \lambda, \mu; \alpha, t; \delta, A, B$, in Theorem 2.1 we obtain the following corollaries.

Corollary 2.3. *If $f \in S_s^*$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \max \{1, |\xi - 1|\}.$$

Corollary 2.4. *If $f \in K_s$, then*

$$|a_3 - \xi a_2^2| \leq \max \left\{ \frac{1}{3}, \frac{1}{4} \left| \xi - \frac{4}{3} \right| \right\}.$$

Corollary 2.5. *If $f \in S_{\lambda, \mu}^0(0, t; 1, 1, -1)$, then*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{2}{(1-t)(2+t)}, & \text{if } |\xi + \sigma_2| \leq \tau_2, \\ \frac{4}{(1-t)^2} |\xi + \sigma_2|, & \text{if } |\xi + \sigma_2| \geq \tau_2, \end{cases}$$

where

$$\begin{aligned} \sigma_2 &= -\frac{3+t}{2(2+t)}, \\ \tau_2 &= \frac{1-t}{2(2+t)}. \end{aligned}$$

Corollary 2.6. *If $f \in S_{\lambda, \mu}^0(1, -1; \delta, A, B)$, then*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{\delta(A-B)}{6}, & \text{if } |\xi + \sigma_3| \leq \tau_3, \\ \frac{\delta^2(A-B)^2}{16} |\xi + \sigma_3|, & \text{if } |\xi + \sigma_3| \geq \tau_3, \end{cases}$$

where

$$\begin{aligned} \sigma_3 &= \frac{4[2B + (1-\delta)(A-B)]}{3\delta(A-B)}, \\ \tau_3 &= \frac{8}{3\delta(A-B)}. \end{aligned}$$

Corollary 2.7. *If $f \in S_{\lambda, \mu}^0(0, -1; 1, A, B)$, then $f \in S_{\lambda, \mu}^n(\alpha, t; \delta, A, B)$*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{\delta(A-B)}{2}, & \text{if } |\xi + \sigma_4| \leq \tau_4, \\ \frac{\delta^2(A-B)^2}{4} |\xi + \sigma_4|, & \text{if } |\xi + \sigma_4| \geq \tau_4, \end{cases}$$

where

$$\begin{aligned} \sigma_4 &= \frac{2B + (1-\delta)(A-B)}{\delta(A-B)}, \\ \tau_4 &= \frac{2}{\delta(A-B)}. \end{aligned}$$

Corollary 2.8. *If $f \in S_{\lambda,\mu}^0(1, -1; 1, A, B)$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \begin{cases} \frac{A-B}{6}, & \text{if } |\xi + \sigma_5| \leq \tau_5, \\ \frac{(A-B)^2}{16} |\xi + \sigma_5|, & \text{if } |\xi + \sigma_5| \geq \tau_5, \end{cases}$$

where

$$\begin{aligned} \sigma_5 &= \frac{8B}{3(A-B)}, \\ \tau_5 &= \frac{8}{3(A-B)}. \end{aligned}$$

Corollary 2.9. *If $f \in S_{\lambda,\mu}^n(0, -1; 1, 1, -1)$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \max \left\{ \frac{1}{\Phi_3^n}, \frac{1}{\Phi_2^{2n}} |\xi - \sigma_6| \right\}$$

where

$$\sigma_6 = \frac{\Phi_2^{2n}}{\Phi_3^n}.$$

Corollary 2.10. *If $f \in S_{\lambda,\lambda}^1(0, -1; 1, 1, -1)$ and $\lambda \geq 0$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \max \left\{ \frac{1}{1+6\lambda^2}, \frac{1}{(1+2\lambda^2)^2} |\xi - \tau_6| \right\},$$

where

$$\tau_6 = \frac{(1+2\lambda^2)^2}{1+6\lambda^2}.$$

Corollary 2.11. *If $f \in S_{1,1}^1(0, -1; 1, 1, -1)$, then*

$$\left| a_3 - \xi a_2^2 \right| \leq \max \left\{ \frac{1}{7}, \frac{1}{9} \left| \xi - \frac{9}{7} \right| \right\}.$$

Corollary 2.12. *If $f \in S_{1,0}^1(0, -1; 1, 1, -1)$ then,*

$$\left| a_3 - \xi a_2^2 \right| \leq \max \left\{ \frac{1}{3}, \frac{1}{4} \left| \xi - \frac{4}{3} \right| \right\}.$$

Our next result is as follows.

Theorem 2.13. *If $f \in T_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$, then*

$$|a_2| \leq \frac{1}{2} \left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} + (1+t) \right),$$

and

$$|a_3| \leq \begin{cases} v_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}, & \text{if } |\rho_1| \leq 1, \\ v_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} |\rho_1|, & \text{if } |\rho_1| \geq 1, \end{cases}$$

where

$$v_1 = \frac{1+t+t^2}{3} + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n}, \quad (2.12)$$

$$\rho_1 = B + \frac{1}{2}(1-\delta)(A-B). \quad (2.13)$$

Proof. Since $f \in T_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$, by the definition of subordination

$$(1-\alpha) \frac{(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n g(z) - D_{\lambda,\mu}^n g(tz)} + \alpha \frac{\left[(1-t)z \left(D_{\lambda,\mu}^n f(z) \right)' \right]'}{\left[D_{\lambda,\mu}^n g(z) - D_{\lambda,\mu}^n g(tz) \right]'} = \left(\frac{1+Aw(z)}{1+Bw(z)} \right)^\delta \quad (2.14)$$

for some $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*$.

By expanding (2.14), we get

$$\begin{aligned} & 1 + (1+\alpha)\Phi_2^n [2a_2 - (1+t)b_2] z \\ & \left\{ (1+2\alpha)\Phi_3^n [3a_3 - (1+t+t^2)b_3] \right. \\ & \left. - (1+t)(1+3\alpha)\Phi_2^{2n} b_2 [2a_2 - (1+t)b_2] \right\} z^2 + \dots \\ & = 1 + \delta(A-B)c_1 z + \delta(A-B) \left(c_2 - [B + \frac{1}{2}(1-\delta)(A-B)]c_1^2 \right) z^2 + \dots \end{aligned}$$

After equating the terms, we obtain

$$a_2 = \frac{1}{2} \left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} c_1 + (1+t)b_2 \right), \quad (2.15)$$

$$a_3 = \frac{1+t+t^2}{3} b_3 + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n} b_2 c_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} \{c_2 - \rho_1 c_1\}, \quad (2.16)$$

where ρ_1 is defined in (2.13). Applying the triangle inequality, we have

$$|a_2| \leq \frac{1}{2} \left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} |c_1| + (1+t) |b_2| \right), \quad (2.17)$$

and

$$\begin{aligned} |a_3| &\leq \frac{1+t+t^2}{3} |b_3| + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{3(1+\alpha)(1+2\alpha)\Phi_3^n} |b_2| |c_1| \\ &\quad + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} \{|c_2| + |\rho_1| |c_1|\}. \end{aligned} \quad (2.18)$$

By using Lemma 1.1 and the inequalities in (1.2), then (2.17) and (2.18) lead us to

$$|a_2| \leq \frac{1}{2} \left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} + (1+t) \right)$$

and

$$|a_3| \leq v_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} \left[1 + (|\rho_1| - 1) |c_1|^2 \right],$$

where v_1 is defined in (2.12).

For $|\rho_1| \leq 1$,

$$|a_3| \leq v_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}$$

and for $|\rho_1| \geq 1$ we have

$$|a_3| \leq v_1 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} |\rho_1|.$$

■

Corollary 2.14. *If $f \in T_{\lambda,\mu}^0(\alpha, -1; 1, 1, -1)$, then*

$$|a_2| \leq \frac{1}{1+\alpha},$$

and

$$|a_3| \leq \frac{1}{3} + \frac{2}{3(1+2\alpha)}.$$

Corollary 2.15. *If $f \in T_{1,1}^1(0, -1; 1, 1, -1)$, then*

$$|a_2| \leq \frac{1}{3},$$

and

$$|a_3| \leq \frac{3}{7}.$$

Corollary 2.16. *If $f \in T_{1,0}^1(0, -1; 1, 1, -1)$, then*

$$|a_2| \leq \frac{1}{2},$$

and

$$|a_3| \leq \frac{5}{9}.$$

Next, we can give the following theorem.

Theorem 2.17. *If $f \in H_{\lambda,\mu}^n(\alpha, t; \delta, A, B)$, then*

$$|a_2| \leq \frac{1}{2} \left(\frac{\delta(A-B)}{(1+\alpha)\Phi_2^n} + \frac{1+t}{2} \right),$$

and

$$|a_3| \leq \begin{cases} v_2 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n}, & \text{if } |\rho_1| \leq 1, \\ v_2 + \frac{\delta(A-B)}{3(1+2\alpha)\Phi_3^n} |\rho_1|, & \text{if } |\rho_1| \geq 1, \end{cases}$$

where

$$v_2 = \frac{1+t+t^2}{9} + \frac{\delta(1+t)(1+3\alpha)(A-B)\Phi_2^n}{6(1+\alpha)(1+2\alpha)\Phi_3^n},$$

and ρ_1 is defined in (2.13).

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