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И НЕВРОННИ МРЕЖИ

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Systems of Differential Equations and Neural Networks with Delays and Impulses

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PREFACE

In the mathematical description of the evolution of real processes subject to short-time perturbations it is often convenient to neglect the duration of the perturbations and assume that these perturbations have an "instantaneous" character. Such idealization leads to the necessity to study dynamical systems with discontinuous trajectories or, as they are also called, *differential equations with impulses* or *impulsive differential equations*.

The impulsive differential equations are a rather new branch of the theory of ordinary and partial differential equations. They mark their beginning in 1960 with the paper by V. D. Mil'man and A. D. Myshkis [89]. The investigation of these equations was initially carried out extremely slowly. This was due to the great difficulties caused by the specific properties of the impulsive equations such as "beating" of the solutions, bifurcation, merging of the solutions, dying of the solutions and loss of the property of autonomy. Despite these difficulties, however, a boom in the development of this theory is observed in the last, say, quarter of a century. The interest in it is caused by the great possibilities of mathematical simulation by means of impulsive differential equations in important fields of science and technology such as the theory of optimal control, theoretical physics, population dynamics, biotechnologies, impulse techniques, industrial robotics, economics, etc.

In the period after 1988 important results were obtained, related with the impulsive differential equations with a small parameter. The development of the theory of impulsive differential equations with a small parameter is connected with the names of V. Lakshmikantham and his collaborators, A. M. Samoilenko, N. A. Perestyuk, A. A. Boichuk and many others in Ukraine, D. D. Bainov and his collaborators in Bulgaria, etc. I have kept this list short since any attempt to make a more complete list would leave out authors of significant contributions.

Neural network simulations appear to be a recent development. However, this field was established before the advent of computers, and has survived at least one major setback and several eras. Many important advances have been boosted by the use of inexpensive computer emulations. Following an initial period of enthusiasm, the field survived a period of frustration and disrepute. The first artificial neuron was produced in 1943 by the neurophysiologist Warren McCulloch and the logician Walter Pitts [87]. More information about neural networks can be found at the beginning of Chapter 3 of the present thesis.

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely discrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems. To the best of our knowledge impulsive neural networks first appeared in 1999 [64], yet I would mention that after the publication of our paper [4] in 2004 hundreds or maybe thousands of papers devoted to impulsive neural networks appear each year, mostly in China. Since it is impossible to list the most important contributions and their authors, I do not provide a list here.

The present thesis is based on the author's papers in the last two decades devoted to impulsive differential equations with a small parameter and the global asymptotic stability of equilibrium points and periodic solutions of continuous- and discrete-time neural networks with delays and impulses. It consists of three chapter.

Chapter 1 has an auxiliary character. It contains the necessary information about impulsive differential equations, periodic solutions of linear impulsive systems in the noncritical and critical cases, almost periodic solutions of linear impulsive systems, and differential equations with a deviating argument.

Chapter 2 deals mostly with periodic and almost periodic solutions of impulsive systems with delay. The role of a small parameter is played by the delay, or the amplitude of the oscillation of the delay about a constant value. In §2.1 we find sufficient conditions for the existence of a periodic solution of a periodic retarded or neutral system in a neighbourhood of an isolated periodic solution of the system without delay. In §2.2 we study a system with impulses and a small delay such that the corresponding system without delay is linear and has a family of periodic solutions. More generally, we study a boundary value problem for an impulsive differential system with many small delays such that the corresponding system without delay is linear and the boundary value problem for the homogeneous system has a family of nontrivial solutions. Finally, in §2.3 we consider retarded and neutral impulsive systems whose delay differs from a constant by a small-amplitude periodic perturbation, provided that the corresponding system without delay has an isolated ω–periodic solution. If the period of the small-amplitude perturbation of the delay is (a rational multiple of) ω, we find sufficient conditions for the existence of a periodic solution; if it is rationally independent with ω, we find sufficient conditions for the existence of an almost periodic solution.

Chapter 3 begins with some general information about neural networks and is concerned with the global asymptotic (in most cases, exponential) stability of equilibrium points and periodic solutions of continuous- and discrete-time neural networks with delays and impulses. In §3.1 we find sufficient conditions for the global exponential stability of unique equilibrium points of continuous-time neural networks. In §3.2 we obtain discrete-time analogues of continuous-time neural networks and find sufficient conditions for the global exponential stability of their unique equilibrium points or periodic solutions. Finally, in §3.3 we find sufficient conditions for global asymptotic stability of the unique equilibrium point of a continuous-time Cohen-Grossberg neural network of neutral type and its discrete-time counterpart provided with impulse conditions.

The present thesis was written during the author's work in Sultan Qaboos University, Muscat, Oman. However, some of the papers it is based on were written during my work in the Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria; Fatih University, Istanbul, Turkey; and my visits to Valencia Polytechnical University, Valencia, Spain and Université de Pau et du Pays de l'Adour, Pau, France in 1993.

I would like to express my gratitude to many colleagues from different countries, especially to Prof. Lucas Jódar (Spain), Prof. Alexander Boichuk (Ukraine), Prof. Haydar Akça (Turkey, now in UAE) and, *post mortem*, to Prof. Drumi Bainov (Bulgaria) and Prof. Ovide Arino (France).

Contents

Chapter 1

PRELIMINARIES

1.1 Differential Equations with Impulse Effect

In this section, following [103], we shall describe the general characteristics of systems of differential equations with impulse effect, in particular, when the impulses take place at fixed moments (instants) of time.

Let $\mathbf{M}$ be the phase space of some evolution process, *i.e.*, the set of all possible states of the process. Denote by $x(t)$ the point mapping the state of the given process at the moment t. The process is assumed to be finite dimensional, *i.e.*, the description of its state at a fixed moment of time requires a finite number of, say n, parameters. Thus the point $x(t)$ for any fixed value of t can be interpreted as an n–dimensional vector of the Euclidean space $\mathbb{R}^n$, and $\mathbf{M}$ can be regarded as a subset of $\mathbb{R}^n$. We shall call the topological product $\mathbb{R} \times \mathbf{M}$ of the real axis $\mathbb{R}$ and the phase space $\mathbf{M}$ an *extended phase space* of the evolution process considered. Let the law of evolution of the process considered be described by:

a) a system of differential equations

$$\frac{dx}{dt} \equiv \dot{x} = f(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbf{M}; \tag{1.1.1}$$

b) a set $\mathcal{J}_t$ in the extended phase space;

c) an operator $\mathcal{A}_t$ defined on the set $\mathcal{J}_t$ and mapping it onto the sat $\mathcal{J}'_t = \mathcal{A}_t \mathcal{J}_t$ in the extended phase space.

The process takes place as follows: the mapping point $P_t = (t, x(t))$ issuing from the point (t_0, x_0) moves along the curve $\{(t, x(t))\}_{t \geq t_0}$ defined by the solution $x(t) = x(t; t_0, x_0)$ of system (1.1.1). The motion along this curve continues till the moment of time $t = t_1 > t_0$ at which the point $(t, x(t))$ meets the set $\mathcal{J}_t$ (*i.e.*, coincides with a point of the set $\mathcal{J}_t$). At the moment of time $t = t_1$ the point P_t is "momentarily" transferred by the operator $\mathcal{A}_t$ from the position $P_{t_1} = (t_1, x(t_1))$ into the position $P_{t_1}^+ = \mathcal{A}_{t_1} P_{t_1} = (t_1, x^+(t_1)) \in \mathcal{J}'_{t_1}$ and moves along the curve $\{(t, x(t))\}_{t>t_1}$ described by the solution $x(t) = x(t; t_1, x^+(t_1))$ of system (1.1.1) issuing from the point $(t_1, x^+(t_1))$. The motion along this curve takes place till the moment of time $t_2 > t_1$ at which the point P_t again meets the set $\mathcal{J}_t$. At this moment under the action of the operator $\mathcal{A}_t$ the point P_t "momentarily" jumps from $P_{t_2} = (t_2, x(t_2))$ to $P_{t_2}^+ = \mathcal{A}_{t_2} P_{t_2} = (t_2, x^+(t_2)) \in \mathcal{J}'_{t_2}$ and moves further along the curve $\{(t, x(t))\}_{t>t_2}$ described by the solution $x(t) = x(t; t_2, x^+(t_2))$ of system (1.1.1) until it meets with the set $\mathcal{J}_t$, etc.

The set of relations a)–c) characterizing the evolution of the process is called a *system of differential equations with impulse effect* (*system of impulsive differential equations*, *impulsive differential system*). The curve $\{t, x(t))\}$ described by the point P_t in the extended phase space is called an *integral curve*, and the function $x = x(t)$ defining this curve — a *solution* of this system.

In dependence on the character of the impulse effect three essentially different classes (types) of impulsive differential systems can be distinguished:

1) systems subject to impulse effect at fixed moments of time;

2) systems subject to impulse effect at the moment of meeting the mapping point P_t with given hypersurfaces $t = \tau_i(x)$ in the extended phase space;

3) discontinuous dynamical systems.

In the present work we consider only systems of the first type.

If the real process described by the system of equations (1.1.1) is subject to impulse effect at fixed moments of time, then the mathematical model of the given process is the following system of differential equations with impulse effect

$$\begin{aligned} &\dot{x} = f(t, x), \quad t \neq t_i, \\ &\Delta x|_{t=t_i} = I_i(x). \end{aligned} \tag{1.1.2}$$

In such a system the set $\mathcal{J}_t$ consists of a sequence of hypersurfaces $t = t_i$ in the extended phase space, where $\{t_i\}$ is a given sequence (finite or infinite) of moments of time. It suffices to define the operator $\mathcal{A}_t$ only for $t = t_i$, *i.e.*, it suffices to consider only its restriction to the hypersurfaces $t = t_i$,

$\mathcal{A}_{t_i} : \mathbf{M} \to \mathbf{M}$. It is convenient to define a sequence of operators $\mathcal{A}_i : \mathbf{M} \to \mathbf{M}$ by the expressions

$$\mathcal{A}_i : \; x \mapsto \mathcal{A}_i x = x + I_i(x).$$

A solution of system (1.1.2) is a piecewise-continuous function $x = \varphi(t)$ with discontinuities of the first kind at $t = t_i$, such that $\dot{\varphi}(t) = f(t, \varphi(t))$ for all $t \neq t_i$ and

$$\Delta\varphi(t_i) \equiv \varphi(t_i + 0) - \varphi(t_i - 0) = I_i(\varphi(t_i - 0)).$$

Henceforth by the value of the function $\varphi(t)$ at the point t_i we mean $\lim\limits_{t \to t_i - 0} \varphi(t)$, *i.e.*, if t_i is a point of discontinuity of the first kind of the function $\varphi(t)$, then we assume that $\varphi(t)$ is continuous from the left and set

$$\varphi(t_i) = \varphi(t_i - 0) \equiv \lim_{t \to t_i - 0} \varphi(t).$$

This convention is used throughout the present work unless stated otherwise.

We shall recall a general theorem on the properties of the solutions of system (1.1.2). Suppose that the function $f(t, x)$ is defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$; the case when it is defined in some domain of this space is considered analogously. Moreover, suppose that the set of solutions of system (1.1.1) satisfies the following conditions:

A1.1.1. (noncontinuability) Each solution $x(t)$ is a continuous function defined on an interval (a, b) $(-\infty \leq a < b \leq \infty)$, which may be different for distinct solutions; moreover, if $a > -\infty$ $(b < \infty)$, then $|x(a + 0)| = \infty$ (respectively, $|x(b - 0)| = \infty$).

A1.1.2. (local character) If some function $x(t)$, $a < t < b$, satisfies condition **A1.1.1** and for any $t_0 \in (a, b)$ there exists $\varepsilon > 0$ such that on any interval $(t_0 - \varepsilon, t_0]$ and $[t_0, t_0 + \varepsilon)$ the function $x(t)$ coincides with some solution, then $x(t)$ is also a solution.

A1.1.3. (solvability of the Cauchy problem) For any t_0, x_0 there exists at least one solution $x(t)$, $a < t < b$, such that $t_0 \in (a, b)$ and $x(t_0) = x_0$.

The operators $\mathcal{A}_i$ are not, in general, assumed to be bijective, *i.e.*, for any $x \in \mathbb{R}^n$ and $i \in K$ (an index set), $\mathcal{A}_i x$ is some subset, possibly empty, of $\mathbb{R}^n$.

Theorem 1.1.1. ([103, Theorem 2.1]) *If the solutions of system* (1.1.1) *satisfy conditions* **A1.1.1**–**A1.1.3**, *then for any* $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ *there exists at least one solution* $x(t)$, $a < t < b$, *of the impulsive system* (1.1.2) *for which* $-\infty \le a < t_0 \le b \le \infty$, $x(t_0) = x_0$ *(for* $t_0 < b$*) or* $x(t_0 - 0) = x_0$ *(for* $t_0 = b$*); moreover,*

if $a > -\infty$, *then either* $|x(a+0)| = \infty$, *or* $a = t_i$ *for some* i, $x(a+0)$ *exists as a finite limit and* $x(a+0) \notin \mathcal{A}_i\mathbb{R}^n$*;*

if $b < \infty$, *then either* $|x(b-0)| = \infty$, *or* $b = t_i$ *for some* i, $x(b-0)$ *exists as a finite limit and* $\mathcal{A}_i x(b-0) = \emptyset$.

Such a solution $x(t)$ *is noncontinuable.*

Theorem 1.1.2. ([103, Theorem 2.2]) *For the uniqueness of the solution of the Cauchy problem for the impulsive system* (1.1.2) *for increasing* t *for arbitrary initial data it is necessary and sufficient that system* (1.1.1) *enjoys the same property for any* $t_0 \ne t_i$, *and that none of the sets* $\mathcal{A}_i x$ *contains more than one point. For the uniqueness of the solution of the Cauchy problem for system* (1.1.2) *for decreasing* t *it is necessary and sufficient that system* (1.1.1) *enjoys this property for any* $t_0 \ne t_i$, *and that none of the sets* $\mathcal{A}_i^{-1} x$ *contains more than one point.*

The above content can be found in [21, Chapter 1, §2]. More information about impulsive differential systems can be found in the monographs [82, 103].

1.2 Periodic Solutions of Linear Impulsive Systems

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}$ the set of all positive integers (also called *natural numbers*).

Provided that the sequence $\{t_i\}_{i\in\mathbb{Z}}$ is such that $t_{i+m} = t_i + \omega$ ($i \in \mathbb{Z}$) for some $\omega > 0$ and $m \in \mathbb{N}$, denote by $\widetilde{C}_{\omega,n}$ the space of all ω–periodic, piecewise continuous functions $w : \mathbb{R} \to \mathbb{R}^n$ with discontinuities of the first kind at t_i and $w(t_i - 0) = w(t_i)$, $i \in \mathbb{Z}$, equipped with the norm

$$\|w\| = \sup_{t\in\mathbb{R}} |w(t)|.$$

Consider the linear impulsive system

$$\begin{aligned} \dot{x} &= A(t)x + g(t), \quad & t \neq t_i, \\ \Delta x(t_i) &= B_i x(t_i) + a_i, \quad & i \in \mathbb{Z}, \end{aligned} \tag{1.2.1}$$

where $x \in \mathbb{R}^n$, $g : \mathbb{R} \to \mathbb{R}^n$, $a_i \in \mathbb{R}^n$, $A(t)$ and B_i are $(n \times n)$–matrices, $\{t_i\}_{i \in \mathbb{Z}}$ is a strictly increasing sequence such that $\lim_{i \to \pm\infty} t_i = \pm\infty$.

Suppose that the following conditions hold.

A1.2.1. There exists $\omega > 0$ and $m \in \mathbb{N}$ such that $t_{i+m} = t_i + \omega$, $B_{i+m} = B_i$, $a_{i+m} = a_i$ $(i \in \mathbb{Z})$.

A1.2.2. The matrix $A(\cdot) \in \tilde{C}_{\omega, n \times n}$.

A1.2.3. The function $g(\cdot) \in \tilde{C}_{\omega, n}$.

A1.2.4. The matrices $E + B_i$, $i \in \mathbb{Z}$, are nonsingular (E — the unit $(n \times n)$–matrix).

Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

Together with (1.2.1) we consider the respective homogeneous system

$$\begin{aligned} \dot{x} &= A(t)x, \quad & t \neq t_i, \\ \Delta x(t_i) &= B_i x(t_i), \quad & i \in \mathbb{Z}. \end{aligned} \tag{1.2.2}$$

First let us consider the so called *noncritical* case where

A1.2.5. System (1.2.2) has a unique ω–periodic solution $x(t) \equiv 0$.

Let the $(n \times n)$–matrix $X(t)$ be the fundamental solution of (1.2.2) (*i.e.*, $X(0) = E$). Then condition **A1.2.5** implies that the matrix $Q \equiv E - X(\omega)$ is nonsingular. In such a case (see [103]) the nonhomogeneous system (1.2.2) has a unique ω–periodic solution $\varphi(t)$ given by the formula

$$\varphi(t) = \int_0^\omega G(t, \tau) g(\tau)\, d\tau + \sum_{i=1}^{m} G(t, t_i + 0) a_i, \tag{1.2.3}$$

where *Green's function of the periodic problem for the nonhomogeneous system corresponding to* (1.2.2) is defined by

$$G(t,\tau) = \begin{cases} X(t)(E - X(\omega))^{-1}X^{-1}(\tau), & 0 \le \tau < t \le \omega, \\ X(t+\omega)(E - X(\omega))^{-1}X^{-1}(\tau), & 0 \le t \le \tau \le \omega, \end{cases} \tag{1.2.4}$$

and extended as ω–periodic with respect to t, τ. We may note the relation

$$G(t, t_i + 0) = G(t, t_i)(E + B_i)^{-1}, \quad i \in \mathbb{Z}.$$

The above content can be found in [21, Chapter 2, §7.1].

Now let us consider the *critical* case (see [32]) where

A1.2.6. $\operatorname{rank} Q = n_1 < n$.

If $r = n - n_1$, then the homogeneous impulsive system (1.2.2) has an r–parametric family of ω–periodic solutions. Let Q^* be the transpose of Q, and let Q^+ be its Moore-Penrose pseudoinverse [99]. Denote by $\mathcal{P} = \mathcal{P}_Q$ the orthoprojector $\mathbb{R}^n \to \operatorname{Ker}(Q)$ and by $\mathcal{P}^* = \mathcal{P}_{Q^*}$ the orthoprojector $\mathbb{R}^n \to \operatorname{Ker}(Q^*)$.

Then the nonhomogeneous system (1.2.1) has ω–periodic solutions if and only if

$$\mathcal{P}^* X(\omega) \left(\int_0^\omega X^{-1}(\tau) g(\tau)\, d\tau + \sum_{i=1}^m X_i^{-1} a_i \right) = 0. \tag{1.2.5}$$

Here, for the sake of brevity, we have denoted $X_i \equiv X(t_i+0) = (E+B_i)X(t_i)$.

Since $\operatorname{rank} \mathcal{P}^* = n - \operatorname{rank} Q^* = n - n_1 = r$, condition (1.2.5) consists of r linearly independent scalar equalities. Denote by $\mathcal{P}_r^* = \mathcal{P}_{Q_r^*}$ an $(r \times n)$–matrix whose rows are r linearly independent rows of $\mathcal{P}^*$. Then (1.2.5) takes the form

$$\mathcal{P}_r^* X(\omega) \left(\int_0^\omega X^{-1}(\tau) g(\tau)\, d\tau + \sum_{i=1}^m X_i^{-1} a_i \right) = 0. \tag{1.2.6}$$

If condition (1.2.6) is satisfied, then system (1.2.1) has an r–parametric family of ω–periodic solutions

$$x_0(t, c_r) = X_r(t) c_r + \int_0^\omega G(t,\tau) g(\tau)\, d\tau + \sum_{i=1}^m G(t, t_i) a_i, \tag{1.2.7}$$

where $X_r(t)$ is an $(n \times r)$–matrix whose columns are a complete system of r linearly independent ω–periodic solutions of (1.2.2), $c_r \in \mathbb{R}^r$ is an arbitrary vector, and $G(t,\tau)$ is the *generalized Green's function*

$$G(t,\tau) = \begin{cases} X(t)(E + Q^+X(\omega))X^{-1}(\tau), & 0 \le \tau \le t \le \omega, \\ X(t)Q^+X(\omega)X^{-1}(\tau), & 0 \le t < \tau \le \omega, \end{cases}$$

and extended as ω–periodic with respect to t, τ.

1.3 Almost Periodic Solutions of Linear Impulsive Systems

First we shall give some definitions and assertions concerning almost periodic sequences and functions, which can be found in [103, §§22–24].

Let $\{x_i\}_{i\in\mathbb{Z}}$ be a sequence of vectors in $\mathbb{R}^n$.

Definition 1.3.1. The sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *almost periodic* if for any $\varepsilon > 0$ it has a relatively dense set of ε–almost periods, *i.e.*, there exists $\mathcal{N} = \mathcal{N}(\varepsilon) > 0$ such that for any $k \in \mathbb{Z}$ there exists an integer $m \in [k, k+\mathcal{N}]$ such that $|x_{i+m} - x_i| < \varepsilon$ for all $i \in \mathbb{Z}$.

In particular, if the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is m–periodic, *i.e.*, $x_{i+m} = x_i$ for all $i \in \mathbb{Z}$, then for any $\varepsilon > 0$ the numbers nm, where $n \in \mathbb{Z}$, are ε–almost periods.

Since any almost periodic sequence is bounded [103], we can introduce the norm

$$\|\{x_i\}_{i\in\mathbb{Z}}\| = \sup_{i\in\mathbb{Z}} |x_i|$$

into the space ap_n of all almost periodic sequences with values in $\mathbb{R}^n$.

Lemma 1.3.1. [103, Lemma 22.2] *Let the numbers t_i, $i \in \mathbb{Z}$, be such that the sequence $\{\bar{t}_i\}_{i\in\mathbb{Z}}$, $\bar{t}_i = t_{i+1} - t_i$, $i \in \mathbb{Z}$, is almost periodic. Then uniformly with respect to $t \in \mathbb{R}$ there exists the limit*

$$\lim_{T\to\infty} \frac{i(t, t+T)}{T} = m,$$

where $i(t, t+T)$ is the number of points t_i in the interval $(t, t+T]$.

For a sequence $\{t_i\}_{i\in\mathbb{Z}}$ as in the above lemma set $t_i^j = t_{i+j} - t_i$ and consider the set of sequences $\{t_i^j\}_{i\in\mathbb{Z}}$, $j \in \mathbb{Z}$.

Definition 1.3.2. The set of sequences $\{t_i^j\}_{i\in\mathbb{Z}}$, $j \in \mathbb{Z}$, is said to be *uniformly almost periodic* if for any $\varepsilon > 0$ there exists a relatively dense set of ε–almost periods common for all sequences $\{t_i^j\}_{i\in\mathbb{Z}}$, $j \in \mathbb{Z}$.

Now we have to give a generalization of the usual notion of an *almost periodic in the sense of Bohr* function to cover the case of piecewise continuous functions with discontinuities of the first kind.

Let $\varphi(t) : \mathbb{R} \to \mathbb{R}^n$ be piecewise continuous with discontinuities of the first kind at the points of the sequence $\{t_i\}_{i\in\mathbb{Z}}$ which is such that the set of sequences $\{t_i^j\}_{i\in\mathbb{Z}}$, $j \in \mathbb{Z}$, is uniformly almost periodic.

Definition 1.3.3. The function $\varphi(t)$ is said to be *almost periodic* if

a) for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of $\varphi(t)$ and satisfy $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$;

b) for any $\varepsilon > 0$ there exists a relatively dense set Γ of ε–almost periods τ, *i.e.*, if $\tau \in \Gamma$, then $|\varphi(t+\tau) - \varphi(t)| < \varepsilon$ for all $t \in \mathbb{R}$ such that $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$.

If the function $\varphi(t)$ is as in Definition 1.3.3, we write $\varphi(t) \in AP(\mathbb{R}, \mathbb{R}^n; \{t_i\}_{i\in\mathbb{Z}})$ or, more briefly, $\varphi(t) \in AP_n(\{t_i\}_{i\in\mathbb{Z}})$.

Theorem 23.1 [103] claims that any almost periodic function is bounded. This allows us to introduce the norm

$$\|\varphi\| = \sup_{t\in\mathbb{R}} |\varphi(t)|$$

into the space $AP_n(\{t_i\}_{i\in\mathbb{Z}})$.

Now consider the linear impulsive system (1.2.1) and the respective homogeneous system (1.2.2) under the assumptions **A1.2.2**, **A1.2.4** as well as

A1.3.1. There exists $\omega > 0$ and $m \in \mathbb{N}$ such that $t_{i+m} = t_i + \omega$, $B_{i+m} = B_i$ $(i \in \mathbb{Z})$.

A1.3.2. The function $g(\cdot) \in AP_n(\{t_i\}_{i\in\mathbb{Z}})$.

A1.3.3. The sequence of vectors $\{a_i\}_{i\in\mathbb{Z}} \in ap_n$.

Let the matrix $X(t)$ be as in §1.2 (the fundamental solution of (1.2.2) with $X(0) = E$). Set

$$\Lambda = \frac{1}{\omega} \ln X(\omega), \quad \Phi(t) = X(t)e^{-\Lambda t}.$$

$\Phi(t)$ is a nonsingular matrix, $\Phi \in \tilde{C}_{\omega,n\times n}$.

Now we make the following additional assumption:

A1.3.4. The matrix Λ has no eigenvalues with real part zero.

Under these assumptions system (1.2.1) has a unique almost periodic solution (see [103, Theorem 25.3]).

We give only those fragments of the proof that will be used henceforth.

Without loss of generality we may assume that $\Lambda = \operatorname{diag}(P, N)$, where P and N are square matrices of order k and $n-k$ respectively, such that

$$\operatorname{Re}\lambda_j(P) > 0, \quad j = \overline{1,k}, \qquad \operatorname{Re}\lambda_j(N) < 0, \quad j = \overline{k+1,n}.$$

Denote

$$G(t) = \begin{cases} -\operatorname{diag}\left(e^{Pt}, 0\right) & \text{for } t < 0, \\ \ \ \operatorname{diag}\left(0, e^{Nt}\right) & \text{for } t > 0. \end{cases}$$

It can be shown that

$$\|G(t)\| \le Ce^{-\alpha|t|}, \tag{1.3.1}$$

where C and α are positive constants. Moreover,

$$x_0(t) = \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)g(\tau)\,d\tau + \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)a_i \tag{1.3.2}$$

is the unique almost periodic solution of (1.2.1).

The above content can be found in [21, Chapter 2, §7.2].

Further on, we will also need to estimate

$$\int_{-\infty}^{\infty} \|G(t-\tau)\|\,d\tau \qquad \text{and} \qquad \sum_{i\in\mathbb{Z}} \|G(t-t_i)\|$$

making use of (1.3.1). As in [44, 50] we have

$$\int_{-\infty}^{\infty} \|G(t-\tau)\|\,d\tau \le C\int_{-\infty}^{\infty} e^{-\alpha|t-\tau|}\,d\tau = C\int_{-\infty}^{\infty} e^{-\alpha|\sigma|}\,d\sigma = 2C\int_0^{\infty} e^{-\alpha\sigma}\,d\sigma = \frac{2C}{\alpha}. \tag{1.3.3}$$

Next we estimate

$$S(t) = \sum_{i \in \mathbb{Z}} e^{-\alpha|t-t_i|}$$

under the assumption that

$$\inf_{j \in \mathbb{Z}} (t_{j+1} - t_j) = \theta > 0.$$

In our case $\theta = \min_{0 \le j \le m-1} (t_{j+1} - t_j)$. Without loss of generality we may assume that $t_0 \le t < t_1$. Then

$$S(t) = \sum_{i=1}^{\infty} e^{-\alpha(t_i - t)} + \sum_{i=0}^{\infty} e^{-\alpha(t - t_{-i})}.$$

In the first sum

$$t_i - t \ge t_i - t_1 = \sum_{j=1}^{i-1} (t_{j+1} - t_j) \ge (i-1)\theta,$$

and in the second one

$$t - t_{-i} \ge t_0 - t_{-i} = \sum_{j=0}^{i-1} (t_{-j} - t_{-j-1}) \ge i\theta.$$

So we have

$$S(t) \le \sum_{i=1}^{\infty} e^{-\alpha\theta(i-1)} + \sum_{i=0}^{\infty} e^{-\alpha\theta i} = \frac{2}{1 - e^{-\alpha\theta}}$$

and

$$\sum_{i \in \mathbb{Z}} \|G(t - t_i)\| \le \frac{2C}{1 - e^{-\alpha\theta}}. \tag{1.3.4}$$

1.4 Differential Systems with Delay

Impulsive differential equations with delay describe models of real processes and phenomena where both dependence on the past and momentary disturbances are observed. For instance, the size of a given population may be normally described by a delay differential equation and, at certain moments, the number of individuals can be abruptly changed. The interaction of the impulsive perturbation and the delay makes difficult the qualitative investigation of such equations. In particular, the solutions are not smooth at the moments of impulse effect shifted by the delay [25].

Typical representatives of the class of differential equations with deviating argument are the equations with *concentrated* deviation of the argument

$$\begin{aligned} \dot{z}(t) &= F(t, z(h_1(t)), \ldots, z(x_k(t))), \quad t \in [a,b], \\ z(t) &= \psi(t) \quad \text{if } t \notin [a,b], \end{aligned} \tag{1.4.1}$$

where $h_j : [a,b] \to \mathbb{R}$, $j = \overline{1,k}$, are given functions, and with *distributed* deviation of the argument

$$\begin{aligned} \dot{z}(t) &= F\left(t, \int_a^b x(s)\, d_s R(t,s)\right), \quad t \in [a,b], \\ z(t) &= \psi(t) \quad \text{if } t \notin [a,b]. \end{aligned}$$

If in (1.4.1) $h_j(t) \le t$, $j = \overline{1,k}$ (*i.e.*, $j = 1, 2, \ldots, k$), then it is a system with *concentrated delay.*

In Chapter 2 we consider only systems with concentrated delays.

Suppose that there are finitely many argument deviations whose dependence on t is known. Then the general form of the functional differential equation is

$$x^{(m)}(t) = f(t, x^{(m_1)}(t - h_1(t)), \ldots, x^{(m_k)}(t - h_k(t))) \tag{1.4.2}$$

where $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ and m_i and $h_i(\cdot)$ $(i = \overline{1,k})$ are respectively nonnegative integers and nonnegative functions.

i) Equation (1.4.2) is called *a functional differential equation of retarded type*, or *retarded functional differential equation* if

$$\max\{m_1, m_2, \ldots, m_k\} < m.$$

ii) Equation (1.4.2) is called *a functional differential equation of neutral type* if

$$\max\{m_1, m_2, \ldots, m_k\} = m.$$

iii) Equation (1.4.2) is called *a functional differential equation of advanced type* if

$$\max\{m_1, m_2, \ldots, m_k\} > m.$$

In Chapter 2, §1, we consider retarded and neutral systems with one small constant delay. In §2 we consider retarded systems with small constant or variable delays. In §3 the problem of existence of periodic and almost periodic solutions for retarded and neutral impulsive systems is studied in the presence of a delay which differs from a constant by a small amplitude periodic perturbation. "... the use of a *periodic* delay is especially relevant since it can model daily, seasonal or annual fluctuations." [104]

In Chapter 3 distributed delays of different types are also considered.

Chapter 2

PERIODIC AND ALMOST PERIODIC SOLUTIONS OF IMPULSIVE SYSTEMS WITH DELAY

A classical problem of the qualitative theory of differential equations is the existence of periodic (or almost periodic) solutions. Numerous references on this matter concerning differential equations with delay and impulsive differential equations can be found in [23]. A traditional approach to this problem is the investigation of the linearized system (also called *system in variations*) with respect to a periodic solution of the unperturbed system satisfying certain nondegeneracy assumptions.

2.1 Periodic Solutions of Impulsive Systems with Small Constant Delays in the Noncritical Case

For an impulsive system with delay it is proved that if the corresponding system without delay has an isolated ω–periodic solution, then in any neighbourhood of this orbit the system considered also has an ω–periodic solution if the delay is small enough. This result is extended to the case of a neutral impulsive system with a small delay.

2.1.1 Retarded systems

In the present subsection we study a system with impulses at fixed moments and a small delay of the argument

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-h)), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x(t_i), x(t_i - h)), \quad i \in \mathbb{Z}, \end{aligned} \qquad (2.1.1.1)$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$, Ω is a domain in $\mathbb{R}^n$; $\Delta x(t_i)$ are the impulses at moments t_i and $\{t_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{i\to\pm\infty} t_i = \pm\infty$; $I_i : \Omega \times \Omega \to \mathbb{R}^n$ $(i \in \mathbb{Z})$, $h \geq 0$ is the delay.

It is clear that, in general, the derivatives $\dot{x}(t_i + kh)$, $k \in \mathbb{Z}$, do not exist. On the other hand, there do exist the limits $\dot{x}(t_i + kh \pm 0)$. According to the convention of §1.1, we assume $\dot{x}(t_i + kh) \equiv \dot{x}(t_i + kh - 0)$. We require the continuity of the solution $x(t)$ at the points $t_i + kh$ if they are distinct from the moments of impulse effect t_i.

For the sake of brevity we shall use the notation $x_i = x(t_i)$, $\bar{x}(t) = x(t-h)$ (thus, $\bar{x}_i = x(t_i - h)$).

In the sequel we require the fulfillment of the following assumptions:

A2.1.1.1. The function $f(t, x, \bar{x})$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω–periodic with respect to t, continuously differentiable with respect to $x, \bar{x}$, with locally Lipschitz continuous with respect to $x, \bar{x}$ first derivatives.

A2.1.1.2. The functions $I_i(x, \bar{x})$, $i \in \mathbb{Z}$, are continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ first derivatives.

A2.1.1.3. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, \bar{x}) = I_i(x, \bar{x})$ for $i \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

For $h = 0$, from (2.1.1.1) we obtain

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t)), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x_i, x_i), \quad i \in \mathbb{Z}, \end{aligned} \qquad (2.1.1.2)$$

so called *generating system*, and suppose that

A2.1.1.4. The generating system (2.1.1.2) has an ω–periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned} \dot{y} &= A(t)y, \quad t \neq t_i, \\ \Delta y(t_i) &= B_i y_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.1.3}$$

where

$$A(t) = \frac{\partial}{\partial x} f(t, x, x)\bigg|_{x=\psi(t)}, \quad B_i = \frac{\partial}{\partial x} I_i(x, x)\bigg|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t)$ be the fundamental solution of the system (2.1.1.3) (*i.e.*, $X(0) = E$ — the unit matrix). Now we make the following additional assumptions:

A2.1.1.5. The matrix $E - X(\omega)$ is nonsingular.

A2.1.1.6. The matrices $E + B_i,\ i \in \mathbb{Z}$, are nonsingular.

If the last two conditions hold, the nonhomogeneous system

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + g(t), \quad t \neq t_i, \\ \Delta y(t_i) &= B_i y_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.1.4}$$

where $g(\cdot) \in \tilde{C}_{\omega,n}$ and $a_{i+m} = a_i,\ i \in \mathbb{Z}$, has a unique ω–periodic solution given by the formula

$$y(t) = \int_0^\omega G(t,\tau)g(\tau)\,d\tau + \sum_{0<t_i<\omega} G(t, t_i + 0)a_i, \tag{2.1.1.5}$$

where $G(t,\tau)$ is Green's function (see §1.2). Let us denote

$$\mathcal{M} = \sup\left\{|G(t,\tau)| :\ t, \tau \in [0,\omega]\right\}.$$

Our result in the present subsection is the following

Theorem 2.1.1.1. *Let conditions* **A2.1.1.1**–**A2.1.1.6** *hold. Then there exists a number $h_* > 0$ such that for $h \in (0, h_*)$ system* (2.1.1.1) *has a unique ω–periodic solution $x(t,h)$ depending continuously on h and such that $x(t,h) \to \psi(t)$ as $h \to 0$.*

Proof. In system (2.1.1.1) we change the variables according to the formula

$$x = \psi(t) + y \tag{2.1.1.6}$$

and obtain the system

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + Q(t, y(t)) \\ &+ \delta f(t, \psi(t) + y(t), \bar{\psi}(t) + \bar{y}(t)), \quad t \neq t_i, \\ \Delta y(t_i) &= B_i y_i + J_i(y_i) + \delta I_i(\psi_i + y_i, \bar{\psi}_i + \bar{y}_i), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.1.7}$$

where

$$\begin{aligned} Q(t, y) &\equiv f(t, \psi(t) + y, \psi(t) + y) - f(t, \psi(t), \psi(t)) - A(t)y, \\ J_i(y_i) &\equiv I_i(\psi_i + y_i, \psi_i + y_i) - I_i(\psi_i, \psi_i) - B_i y_i \end{aligned}$$

are nonlinearities inherent to the generating system (2.1.1.2) and therefore independent of the small delay h, while

$$\begin{aligned} \delta f(t, x(t), \bar{x}(t)) &\equiv f(t, x(t), \bar{x}(t)) - f(t, x(t), x(t)), \\ \delta I_i(x_i, \bar{x}_i) &\equiv I_i(x_i, \bar{x}_i) - I_i(x_i, x_i) \end{aligned}$$

are increments due to the presence of the small delay.

We can formally consider (2.1.1.7) as a nonhomogeneous system of the form (2.1.1.4). Then its unique ω–periodic solution $y(t)$ must satisfy an equality of the form (2.1.1.5) which in this case is the operator equation

$$y = \mathcal{U}_h y, \tag{2.1.1.8}$$

where

$$\begin{aligned} \mathcal{U}_h y(t) &\equiv \int_0^\omega G(t,\tau)Q(\tau, y(\tau))\, d\tau + \int_0^\omega G(t,\tau)\delta f(\tau, x(\tau), \bar{x}(\tau))\, d\tau \\ &+ \sum_{0<t_i<\omega} G(t, t_i + 0)J_i(y_i) + \sum_{0<t_i<\omega} G(t, t_i + 0)\delta I_i(x_i, \bar{x}_i) \\ &\equiv \mathcal{I}_1 y(t) + \mathcal{I}_2 y(t) + \mathcal{S}_1 y(t) + \mathcal{S}_2 y(t). \end{aligned}$$

For brevity we still write x instead of $\psi(t) + y$ in $\delta f(t, x(\tau), \bar{x}(\tau))$, $\delta I_i(x_i, \bar{x}_i)$ as well as in $\mathcal{I}_2 y$, $\mathcal{S}_2 y$. Moreover, we will further transform the expressions

$\mathcal{I}_2 y(t)$ and $\mathcal{S}_2 y(t)$ under the assumption that $x(t)$ is a solution of system (2.1.1.1). This will considerably simplify some estimates henceforth.

An ω–periodic solution $x(t) = x(t,h)$ of system (2.1.1.1) corresponds to a fixed point y of the operator $\mathcal{U}_h$ in a suitable set of ω–periodic functions. To this end we shall prove that $\mathcal{U}_h$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. For $x, \bar{x} \in \Omega_1$ the functions $f(t,x,\bar{x})$ $(t \in [0,\omega])$ and $I_i(x,\bar{x})$ $(i = \overline{1,m})$ are bounded, together with their first derivatives with respect to $x, \bar{x}$. Let us denote

$$\begin{aligned} M_0 &= \max\Big\{\sup\big\{|f(t,x,\bar{x})| :\ t \in [0,\omega],\ x,\bar{x} \in \Omega_1\big\}, \\ &\qquad \sup\big\{|I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}\Big\}, \\ M_1 &= \max\Big\{\sup\big\{|\partial_x f(t,x,\bar{x})| :\ t \in [0,\omega],\ x,\bar{x} \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x})| :\ t \in [0,\omega],\ x,\bar{x} \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_x I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_{\bar{x}} I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}\Big\} \end{aligned}$$

Let L_1 and L_2 be respectively the greatest Lipschitz constants for the first derivatives of $f(t,x,\bar{x})$ and $I_i(x,\bar{x})$, whose existence is provided by conditions **A2.1.1.1**, **A2.1.1.2** and the compactness of the set Ω_1. Sometimes, for the sake of brevity, we shall use the Landau symbol $O(\mu^k)$ for a quantity whose module (norm) can be estimated by a constant times μ^k for μ small enough. The meaning of $O(h)$ is similar.

Let $h_0 > 0$ be so small that for any $h \in [0,h_0]$ we should have

$$t_i + h < t_{i+1}, \quad i = \overline{1,m-1}, \qquad t_m + h < \omega.$$

Further we define the "bad" set $\Delta_1^h = \bigcup_{i=1}^{m} (t_i, t_i + h)$ and the "good" set $\Delta_2^h = [0,\omega] \setminus \Delta_1^h$. Thus the "bad" set Δ_1^h is a disjoint union of m intervals.

For $\mu \in (0, \mu_0]$ define a set of functions

$$\mathcal{T}_\mu \;=\; \{\; y \in \tilde{C}_{\omega,n} : \quad \|y\| \le \mu \;\}.$$

We shall find a dependence between h and μ so that the operator $\mathcal{U}_h$ in (2.1.1.8) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h$. Let $y \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_h y(t)|$ using the representation

$$\mathcal{U}_h y(t) \;=\; \mathcal{I}_1 y(t) + \mathcal{I}_2 y(t) + \mathcal{S}_1 y(t) + \mathcal{S}_2 y(t)$$

and system (2.1.1.1).

First we have

$$\begin{aligned} J_i(y_i) \;&=\; \left\{ \int_0^1 \left(\partial_x I_i(\psi_i + s y_i, \psi_i + s y_i) \;-\; \partial_x I_i(\psi_i, \psi_i)\right) ds \right. \\ &+\; \left. \int_0^1 \left(\partial_{\bar{x}} I_i(\psi_i + s y_i, \psi_i + s y_i) \;-\; \partial_{\bar{x}} I_i(\psi_i, \psi_i)\right) ds \right\} y_i, \end{aligned}$$

thus

$$|J_i(y_i)| \le 2\int_0^1 2L_2 s|y_i|\, ds \cdot |y_i| = 2L_2 |y_i|^2$$

and

$$|\mathcal{S}_1 y(t)| \;\le\; 2L_2 \mathcal{M} \sum_{i=1}^{m} |y_i|^2 \;=\; O(\mu^2). \tag{2.1.1.9}$$

Similarly, we have

$$\begin{aligned} &Q(\tau, y(\tau)) \\ =\;& \left\{ \int_0^1 \big[\partial_x f(\tau, \psi(\tau) + s y(\tau), \psi(\tau) + s y(\tau)) - \partial_x f(\tau, \psi(\tau), \psi(\tau))\big] ds \right. \\ +\;& \left. \int_0^1 \big[\partial_{\bar{x}} f(\tau, \psi(\tau) + s y(\tau), \psi(\tau) + s y(\tau)) - \partial_{\bar{x}} f(\tau, \psi(\tau), \psi(\tau))\big] ds \right\} y(\tau), \end{aligned}$$

thus

$$|Q(\tau, y(\tau))| \le 2\int_0^1 2L_1 s|y(\tau)|\, ds \cdot |y(\tau)| = 2L_1 |y(\tau)|^2$$

and

$$|\mathcal{I}_1 y(t)| \;\le\; 2L_1\mathcal{M}\int_0^\omega |y(\tau)|^2\,d\tau \;=\; O(\mu^2). \tag{2.1.1.10}$$

Now we can choose $\tilde{\mu}_0 \in (0,\mu_0]$ so that for any $\mu \in (0,\tilde{\mu}_0]$ we have

$$|\mathcal{I}_1 y(t) + \mathcal{S}_1 y(t)| \le \mu/2. \tag{2.1.1.11}$$

If $x(t)$ is a solution of (2.1.1.1) in the set Ω_1, then

$$|\dot{x}(t)| = |f(t, x(t), \bar{x}(t))| \le M_0.$$

Further on, since the intervals $(t_i - h, t_i)$ contain none of the points t_j, we have

$$\begin{aligned}
\delta I_i(x_i,\bar{x}_i) &= \int_0^1 \frac{\partial}{\partial s}\, I_i(x_i, x(t_i - sh))\,ds \qquad (2.1.1.12)\\
&= \int_0^1 \partial_{\bar{x}} I_i(x_i, x(t_i - sh))\,\frac{\partial}{\partial s}\,x(t_i - sh)\,ds\\
&= -h\int_0^1 \partial_{\bar{x}}\, I_i(x_i, x(t_i - sh))\dot{x}(t_i - sh)\,ds,\\
&= \;-h\int_0^1 \partial_{\bar{x}}\, I_i(x_i, x(t_i - sh)) f(t_i - sh, x(t_i - sh), x(t_i - (s+1)h))\,ds,
\end{aligned}$$

thus $|\delta I_i(x_i,\bar{x}_i)| \le hM_1M_0$ and

$$|\mathcal{S}_2 y(t)| \;\le\; hm\mathcal{M}M_1M_0 \;=\; O(h). \tag{2.1.1.13}$$

Next we estimate the difference $x(t) - x(t-h)$. If t is not in Δ_1^h, then $x(t)$ is continuous in $[t-h,t]$ and $\dot{x}$ exists in this segment, with the possible exception of finitely many points. Then we have

$$|x(t) - x(t-h)| \le h \sup_{\tau\in[t-h,t]} |\dot{x}(\tau)| \le hM_0.$$

Now we shall obtain an analogous estimate for Δ_1^h. Then the interval $(t-h,t)$ contains just one point of discontinuity t_i of $x(t)$, thus

$$\begin{aligned}
|x(t) - x(t-h)| &\le |x(t) - x(t_i+0)| + |x(t_i+0) - x(t_i)| + |x(t_i) - x(t-h)|\\
&\le M_0(t - t_i) + M_0 + M_0(t_i - t + h) \;=\; M_0(1+h).
\end{aligned}$$

Using these estimates, we evaluate $\delta f(\tau, x(\tau), \bar{x}(\tau))$. If $\tau \in \Delta_2^h$, we have

$$
\begin{aligned}
\delta f(\tau, x(\tau), \bar{x}(\tau)) &= \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh))\, ds \\
&= -h \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh)) \dot{x}(\tau - sh)\, ds \qquad (2.1.1.14) \\
&= -h \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh)) f(\tau - sh, x(\tau - sh), x(\tau - (s+1)h))\, ds
\end{aligned}
$$

and

$$
|\delta f(\tau, x(\tau), \bar{x}(\tau))| \;\le\; h M_0 M_1. \qquad (2.1.1.15)
$$

Next, if $\tau \in \Delta_1^h$, we have

$$
\begin{aligned}
\delta f(\tau, x(\tau), \bar{x}(\tau)) &= \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), s x(\tau - h) + (1-s) x(\tau))\, ds \qquad (2.1.1.16) \\
&= \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), s x(\tau - h) + (1-s) x(\tau))\, ds \cdot (x(\tau - h) - x(\tau))
\end{aligned}
$$

and

$$
|\delta f(\tau, x(\tau), \bar{x}(\tau))| \le M_0 M_1 (1 + h). \qquad (2.1.1.17)
$$

Making use of the estimates (2.1.1.15) and (2.1.1.17), we find

$$
\begin{aligned}
|\mathcal{I}_2 y(t)| &\le \int_{\Delta_2^h} h \mathcal{M} M_0 M_1\, d\tau + \int_{\Delta_1^h} \mathcal{M} M_0 M_1 (1 + h)\, d\tau \\
&= \int_0^\omega h \mathcal{M} M_0 M_1\, d\tau \;+\; \int_{\Delta_1^h} \mathcal{M} M_0 M_1\, d\tau \\
&= h \mathcal{M} M_0 M_1 (\omega + m) = O(h). \qquad (2.1.1.18)
\end{aligned}
$$

From (2.1.1.13) and (2.1.1.18) it follows that we can choose $\tilde{h}(\mu) \in (0, h_0]$ so that for any $h \in (0, \tilde{h}(\mu)]$ we have

$$
|\mathcal{I}_2 y(t) + \mathcal{S}_2 y(t)| \;\le\; \mu/2. \qquad (2.1.1.19)
$$

Finally, by virtue of the estimates (2.1.1.11) and (2.1.1.19) we obtain

$$
|\mathcal{U}_h y(t)| \;\le\; \mu,
$$

i.e., the operator $\mathcal{U}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h \in (0, \tilde{h}(\mu)]$.

Contraction property of the operator $\mathcal{U}_h$. Let $y', y'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_h y'(t) - \mathcal{U}_h y''(t) &= (\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t)) + (\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)) \\ &+ (\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t)) + (\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)).\end{aligned}$$

First we consider

$$\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t) = \sum_{i=1}^{m} G(t, t_i + 0)\big(J_i(y_i') - J_i(y_i'')\big).$$

We have

$$\begin{aligned}&J_i(y_i') - J_i(y_i'') \\ = &\big(I_i(\psi_i + y_i', \psi_i + y_i') - I_i(\psi_i + y_i'', \psi_i + y_i'')\big) - B_i(y_i' - y_i'') \\ = &\bigg\{\int_0^1 \big(\partial_x I_i(\psi_i + s y_i' + (1-s) y_i'', \psi_i + s y_i' + (1-s) y_i'') - \partial_x I_i(\psi_i, \psi_i)\big)\, ds \\ + &\int_0^1 \big(\partial_{\bar{x}} I_i(\psi_i + s y_i' + (1-s) y_i'', \psi_i + s y_i' + (1-s) y_i'') - \partial_{\bar{x}} I_i(\psi_i, \psi_i)\big)\, ds\bigg\} \\ &\times (y_i' - y_i''),\end{aligned}$$

thus

$$\begin{aligned}|J_i(y_i') - J_i(y_i'')| &\leq 4L_2 \int_0^1 \big[s|y_i'| + (1-s)|y_i''|\big]\, ds \cdot |y_i' - y_i''| \\ &= 2L_2\big(|y_i'| + |y_i''|\big)|y_i' - y_i''| \leq 4\mu L_2 |y_i' - y_i''|\end{aligned}$$

and

$$|\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t)| \leq 4\mu \mathcal{M} L_2 \sum_{i=1}^{m} |y_i' - y_i''| = O(\mu)\|y' - y''\|. \quad (2.1.1.20)$$

Next,

$$\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t) = \int_0^\omega G(t, \tau)\big(Q(\tau, y'(\tau)) - Q(\tau, y''(\tau))\big)\, d\tau.$$

We have

$$
\begin{aligned}
&Q(\tau, y'(\tau)) \;-\; Q(\tau, y''(\tau)) \\
= \;& \left\{ \int_0^1 [\partial_x f(\tau, x_s(\tau), x_s(\tau)) - \partial_x f(\tau, \psi(\tau), \psi(\tau))] \; ds \right. \\
+ \;& \left. \int_0^1 [\partial_{\bar{x}} f(\tau, x_s(\tau), x_s(\tau)) - \partial_{\bar{x}} f(\tau, \psi(\tau), \psi(\tau))] \; ds \right\} \cdot \big(y'(\tau) - y''(\tau)\big),
\end{aligned}
$$

where $x_s(\tau) = \psi(\tau) + sy'(\tau) + (1-s)y''(\tau)$. Thus

$$
\begin{aligned}
&|Q(\tau, y'(\tau) - Q(\tau, y''(\tau))| \\
\leq \;& 4L_1 \int_0^1 [s|y'(\tau)| + (1-s)|y''(\tau)|] \; ds \cdot |y'(\tau) - y''(\tau)| \\
\leq \;& 2L_1 \big(\|y'\| + \|y''\|\big) \cdot \|y' - y''\| \leq 4\mu L_1 \; \|y' - y''\|
\end{aligned}
$$

and

$$
|\mathcal{I}_1 z'(t) - \mathcal{I}_1 y''(t)| \leq 4\mu \mathcal{M} \omega L_1 \; \|y' - y''\| = O(\mu)\|y' - y''\|. \qquad (2.1.1.21)
$$

In order to estimate $\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)$ we use the representation (2.1.1.12). If we denote $x' = \psi(t) + y'$, $x'' = \psi(t) + y''$, we have

$$
\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t) \;=\; \sum_{i=1}^{m} G(t, t_i + 0)(\delta I_i(x_i', \bar{x}_i') \;-\; \delta I_i(x_i'', \bar{x}_i''))
$$

and

$$
\begin{aligned}
&\delta I_i(x_i', \bar{x}_i') \;-\; \delta I_i(x_i'', \bar{x}_i'') \\
= \;& -h \int_0^1 \Big[\partial_{\bar{x}}\, I_i(x_i', x'(t_i - sh)) f(t_i - sh, x'(t_i - sh), x'(t_i - (s+1)h)) \\
& -\partial_{\bar{x}}\, I_i(x_i'', x''(t_i - sh)) f(t_i - sh, x''(t_i - sh), x''(t_i - (s+1)h))\Big] \, ds.
\end{aligned}
$$

Further on,

$$
\begin{aligned}
&\Big|\partial_{\bar{x}}\, I_i(x_i', x'(t_i - sh)) f(t_i - sh, x'(t_i - sh), x'(t_i - (s+1)h)) \\
&\quad -\partial_{\bar{x}}\, I_i(x_i'', x''(t_i - sh)) f(t_i - sh, x''(t_i - sh), x''(t_i - (s+1)h))\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left|\partial_{\bar{x}}\, I_i(x_i', x'(t_i - sh)) - \partial_{\bar{x}}\, I_i(x_i'', x''(t_i - sh))\right| \\
&\qquad\qquad\qquad \times |f(t_i - sh, x'(t_i - sh), x'(t_i - (s+1)h))| \\
&+ \ |\partial_{\bar{x}}\, I_i(x_i'', x''(t_i - sh))| \cdot \Big| f(t_i - sh, x'(t_i - sh), x'(t_i - (s+1)h)) \\
&\qquad\qquad\qquad - f(t_i - sh, x''(t_i - sh), x''(t_i - (s+1)h)) \Big| \\
&\qquad\qquad \leq 2(L_2 M_0 + M_1^2)\|y' - y''\|
\end{aligned}
$$

and thus

$$
|\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)| = O(h)\|y' - y''\|. \tag{2.1.1.22}
$$

Similarly, in order to estimate $\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)$ we use the representations (2.1.1.14) and (2.1.1.16). If $\tau \in \Delta_2^h$, then

$$
|\delta f(\tau, x'(\tau), \bar{x}'(\tau)) - \delta f(\tau, x''(\tau), \bar{x}''(\tau))| \leq O(h)\|y' - y''\|.
$$

If, however, $\tau \in \Delta_1^h$, we have

$$
|\delta f(\tau, x'(\tau), \bar{x}'(\tau)) - \delta f(\tau, x''(\tau), \bar{x}''(\tau))| \leq O(1)\|y' - y''\|
$$

and as above we obtain

$$
|\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)| \leq O(h)\|y' - y''\|. \tag{2.1.1.23}
$$

Choose an arbitrary number $q \in (0,1)$. Then by virtue of (2.1.1.21) and (2.1.1.20) we can find $\mu_1 \in (0, \tilde{\mu}_0]$ so that for any $\mu \in (0, \mu_1]$ we have

$$
|\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t)| \ + \ |\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t)| \leq \frac{q}{2}\|y' - y''\|.
$$

Next, by virtue of (2.1.1.23) and (2.1.1.22) we find $h_* \in (0, \tilde{h}(\mu_1)]$ so that for any $h \in (0, h_*]$ we have

$$
|\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)| \ + \ |\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)| \leq \frac{q}{2}\|y' - y''\|.
$$

Then for any $\mu \in (0, \mu_1]$ and $h \in [0, h_*]$ the estimate

$$
\|\mathcal{U}_h y' \ - \ \mathcal{U}_h y''\| \ \leq \ q\|y' \ - \ y''\|, \qquad q \in (0,1),
$$

is valid for any $y', y'' \in \mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an ω–periodic solution $y(t,h)$ of system (2.1.1.7). Since $y(t) \equiv 0$ is the unique ω–periodic solution of system (2.1.1.7) for $h = 0$, then $y(t,0) \equiv 0$. Now $x(t,h) = \psi(t) + y(t,h)$ is the unique ω–periodic solution of system (2.1.1.1) and $x(t,0) = \psi(t)$. This completes the proof of Theorem 2.1.1.1. □

A problem similar to (2.1.1.1) in the one-dimensional case was touched in [74, 75]. The results of the present subsection were published in [21, Chapter 2, §8] and [22]. The proofs therein were based on the same idea but contained a lot of unnecessary notations and calculations. A proof using an implicit function theorem was given in [23].

2.1.2 Neutral systems

In the present subsection we study a neutral system with impulses at fixed moments and a small delay of the argument of the derivative

$$\begin{aligned} \dot{x}(t) &= D(t)\dot{x}(t-h) + f(t, x(t), x(t-h)), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x(t_i), x(t_i - h)), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.2.1}$$

where $D : \mathbb{R} \to \mathbb{R}^{n\times n}$ and all the other notation is as in §2.1.1.

In the sequel we require the fulfillment of the following assumptions:

A2.1.2.1. The function $f(t, x, \bar{x})$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω–periodic with respect to t, twice continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ second derivatives.

A2.1.2.2. The matrix $D(t)$ is ω– periodic, $\sup\limits_{t\in[0,\omega]} |D(t)| = \eta < 1$, its first derivative is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and its second derivative is bounded on each interval of continuity.

A2.1.2.3. The functions $I_i(x, \bar{x})$, $i \in \mathbb{Z}$, are twice continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ second derivatives.

A2.1.2.4. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, \bar{x}) = I_i(x, \bar{x})$ for $i \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

We may note that the invertibility of the matrix $E - D(t)$ (E is the unit matrix) follows from the inequality $\eta < 1$ (condition **A2.1.2.2**). Moreover, $\sup\limits_{t\in[0,\omega]} |(E - D(t))^{-1}| \le (1-\eta)^{-1}$.

For $h = 0$, from (2.1.2.1) we obtain

$$\begin{aligned} \dot{x}(t) &= (E - D(t))^{-1} f(t, x(t), x(t)), \quad t \ne t_i, \\ \Delta x(t_i) &= I_i(x_i, x_i), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.2.2}$$

so called *generating system*, and suppose that

A2.1.2.5. The generating system (2.1.2.2) has an ω–periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned} \dot{y}(t) &= (E - D(t))^{-1} A(t) y(t), \quad t \ne t_i, \\ \Delta y(t_i) &= B_i y_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.2.3}$$

where

$$A(t) = \frac{\partial}{\partial x} f(t, x, x)\bigg|_{x=\psi(t)}, \qquad B_i = \frac{\partial}{\partial x} I_i(x, x)\bigg|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t)$ be the fundamental solution of the system (2.1.2.3) (*i.e.*, $X(0) = E$). Now we make two additional assumptions:

A2.1.2.6. The matrix $E - X(\omega)$ is nonsingular.

A2.1.2.7. The matrices $E + B_i$, $i \in \mathbb{Z}$, are nonsingular.

If the last two conditions hold, the nonhomogeneous system

$$\begin{aligned} \dot{y}(t) &= (E - D(t))^{-1} A(t) y(t) + g(t), \quad t \ne t_i, \\ \Delta y(t_i) &= B_i y_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.2.4}$$

where $g(\cdot) \in \tilde{C}_{\omega,n}$ and $a_{i+m} = a_i$, $i \in \mathbb{Z}$, has a unique ω–periodic solution given by the formula

$$y(t) = \int_0^\omega G(t,\tau) g(\tau)\, d\tau + \sum_{0<t_i<\omega} G(t, t_i + 0) a_i, \tag{2.1.2.5}$$

where $G(t,\tau)$ is Green's function (see §1.2). Let us denote

$$\mathcal{M} = \sup\left\{|G(t,\tau)| : \ t,\tau \in [0,\omega]\right\}, \qquad \beta = \sum_{i=1}^{m} |B_i|.$$

Our result in the present subsection is the following

Theorem 2.1.2.1. *Let conditions* **A2.1.2.1**–**A2.1.2.7** *hold. If*

$$\eta(3 + 2\beta\mathcal{M}) < 1, \tag{2.1.2.6}$$

then there exists a number $h_* > 0$ such that for $h \in (0, h_*)$ system (2.1.2.1) *has a unique ω–periodic solution $x(t,h)$ depending continuously on h and such that $x(t,h) \to \psi(t)$ as $h \to 0$.*

Proof. In system (2.1.2.1) we change the variables according to the formula

$$x = \psi(t) + y \tag{2.1.2.7}$$

and obtain the system

$$\begin{aligned} \dot{y}(t) &= (E - D(t))^{-1}\big\{A(t)y(t) + Q(t, y(t)) \\ &+ \ \delta f(t, x(t), x(t-h)) - D(t)(\dot{x}(t) - \dot{x}(t-h))\big\}, \quad t \neq t_i, \\ \Delta y(t_i) &= B_i y_i + J_i(y_i) + \delta I_i(x_i, \bar{x}_i), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1.2.8}$$

where

$$\begin{aligned} Q(t,y) &\equiv f(t, \psi(t) + y, \psi(t) + y) - f(t, \psi(t), \psi(t)) - A(t)y, \\ J_i(y_i) &\equiv I_i(\psi_i + y_i, \psi_i + y_i) - I_i(\psi_i, \psi_i) - B_i y_i \end{aligned}$$

are nonlinearities inherent to the generating system (2.1.2.2) and therefore independent of the small delay h, while

$$\begin{aligned} \delta f(t, x(t), \bar{x}(t)) &\equiv f(t, x(t), \bar{x}(t)) - f(t, x(t), x(t)), \\ \delta I_i(x_i, \bar{x}_i) &\equiv I_i(x_i, \bar{x}_i) - I_i(x_i, x_i) \end{aligned}$$

are increments due to the presence of the small delay.

We can formally consider (2.1.2.8) as a nonhomogeneous system of the form (2.1.2.4). Then its unique ω–periodic solution $y(t)$ must satisfy an equality of the form (2.1.2.5) which in this case is the operator equation

$$y \;=\; \mathcal{U}_h y - \mathcal{V}_h y, \tag{2.1.2.9}$$

where

$$\begin{aligned}
\mathcal{U}_h y(t) \;&\equiv\; \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} Q(\tau, y(\tau))\, d\tau \\
&+\; \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} \delta f(\tau, x(\tau), \bar{x}(\tau))\, d\tau \\
&+\; \sum_{0<t_i<\omega} G(t, t_i + 0) J_i(y_i) + \sum_{0<t_i<\omega} G(t, t_i + 0)\delta I_i(x_i, \bar{x}_i) \\
&\equiv\; \mathcal{I}_1 z(t) \;+\; \mathcal{I}_2 z(t) \;+\; \mathcal{S}_1 z(t) \;+\; \mathcal{S}_2 z(t),
\end{aligned}$$

$$\mathcal{V}_h y(t) \equiv \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} D(\tau)\big(\dot{x}(\tau) - \dot{x}(\tau - h)\big)\, d\tau. \tag{2.1.2.10}$$

For brevity we still write x instead of $\psi(t) + y$ in $\delta f(t, x(t), \bar{x}(t))$, $\delta I_i(x_i, \bar{x}_i)$ as well as in $\mathcal{I}_2 y$, $\mathcal{S}_2 y$ and in $\mathcal{V}_h y$. Moreover, we will further estimate the expressions $\mathcal{I}_2 y(t)$ and $\mathcal{S}_2 y(t)$ under the assumption that $x(t)$ is a solution of system (2.1.2.1).

An ω–periodic solution $x(t) = x(t, h)$ of system (2.1.2.1) corresponds to a fixed point y of the operator $\mathcal{U}_h - \mathcal{V}_h$ in a suitable set of ω–periodic functions. To this end we shall prove that $\mathcal{U}_h - \mathcal{V}_h$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. Let us denote

$$\begin{aligned}
M_0 \;&=\; \max\Big\{ \sup\big\{ |f(t, x, \bar{x})| :\ t \in [0, \omega],\ x, \bar{x} \in \Omega_1 \big\}, \\
&\qquad \sup\big\{ |I_i(x, \bar{x})| :\ i = \overline{1, m},\ x, \bar{x} \in \Omega_1 \big\} \Big\},
\end{aligned}$$

$$
\begin{aligned}
M_1 \;\; = \;\; \max\Big\{ & \sup\big\{|\partial_x f(t,x,\bar{x})| : \; t \in [0,\omega], \; x,\bar{x} \in \Omega_1\big\}, \\
& \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x})| : \; t \in [0,\omega], \; x,\bar{x} \in \Omega_1\big\}, \\
& \sup\big\{|\partial_x I_i(x,\bar{x})| : \; i = \overline{1,m}, \; x,\bar{x} \in \Omega_1\big\}, \\
& \sup\big\{|\partial_{\bar{x}} I_i(x,\bar{x})| : \; i = \overline{1,m}, \; x,\bar{x} \in \Omega_1\big\}\Big\}
\end{aligned}
$$

and, similarly, let M_2 be the maximum of the suprema of the matrices of the second derivatives of $f(t,x,\bar{x})$ with respect to $x, \bar{x}$ for $t \in [0,\omega], \; x,\bar{x} \in \Omega_1$ and of the second derivatives of $I_i(x,\bar{x})$ for $i = \overline{1,m}, \; x,\bar{x} \in \Omega_1$. We shall not explicitly denote the Lipschitz constants for the second derivatives of $f(t,x,\bar{x})$ and $I_i(x,\bar{x})$.

Let $h_0 > 0$ be so small that for any $h \in [0,h_0]$ we should have

$$
t_i + h < t_{i+1}, \quad i = \overline{1,m-1}, \qquad t_m + h < \omega.
$$

Further on, as in §2.1.1 we define the "bad" set $\Delta_1^h = \bigcup_{i=1}^{m} (t_i, t_i + h)$ and the "good" set $\Delta_2^h = [0,\omega] \setminus \Delta_1^h$. Thus the "bad" set Δ_1^h is a disjoint union of m intervals.

For $\mu \in (0,\mu_0]$ define a set of functions

$$
\mathcal{T}_\mu \;=\; \{\; y \in \tilde{C}_{\omega,n} : \quad \|y\| \le \mu \;\}.
$$

We shall find a dependence between h and μ so that the operator $\mathcal{U}_h - \mathcal{V}_h$ in (2.1.2.9) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h - \mathcal{V}_h$. Let $y \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_h y(t)|$ using the representation

$$
\mathcal{U}_h y(t) \;=\; \mathcal{I}_1 y(t) + \mathcal{I}_2 y(t) + \mathcal{S}_1 y(t) + \mathcal{S}_2 y(t)
$$

and system (2.1.2.1).

First we have

$$
\begin{aligned}
J_i(y_i) \;\; = \;\; & \Big\{ \int_0^1 \left(\partial_x I_i(\psi_i + s y_i, \psi_i + s y_i) \;-\; \partial_x I_i(\psi_i,\psi_i)\right) ds \\
+ \;\; & \int_0^1 \left(\partial_{\bar{x}} I_i(\psi_i + s y_i, \psi_i + s y_i) \;-\; \partial_{\bar{x}} I_i(\psi_i,\psi_i)\right) ds \Big\}\, y_i,
\end{aligned}
$$

thus

$$|J_i(y_i)| \le 2\int_0^1 2M_2 s|y_i|\,ds\cdot|y_i| = 2M_2|y_i|^2$$

and

$$|\mathcal{S}_1 y(t)| \;\le\; 2M_2\mathcal{M}\sum_{i=1}^{m}|y_i|^2 \;=\; O(\mu^2). \qquad (2.1.2.11)$$

Similarly, we have

$$\begin{aligned} & Q(\tau, y(\tau)) \\ = & \;\Big\{\int_0^1 \big[\partial_x f(\tau,\psi(\tau)+sy(\tau),\psi(\tau)+sy(\tau)) - \partial_x f(\tau,\psi(\tau),\psi(\tau))\big]ds \\ + & \int_0^1 \big[\partial_{\bar{x}} f(\tau,\psi(\tau)+sy(\tau),\psi(\tau)+sy(\tau)) - \partial_{\bar{x}} f(\tau,\psi(\tau),\psi(\tau))\big]ds\Big\}\, y(\tau), \end{aligned}$$

thus

$$|Q(\tau, y(\tau))| \le 2\int_0^1 M_2 s 2|y(\tau)|\,ds\cdot|y(\tau)| = 2M_2|y(\tau)|^2$$

and

$$|\mathcal{I}_1 y(t)| \;\le\; 2M_2\mathcal{M}(1-\eta)^{-1}\int_0^\omega |y(\tau)|^2\,d\tau \;=\; O(\mu^2). \qquad (2.1.2.12)$$

Now let us estimate $|\dot{x}(t)|$, where $x(t)$ is a solution of (2.1.2.1). We have

$$|\dot{x}(t)| \le |D(t)||\dot{x}(t-h)| + |f(t,x(t),x(t-h))| \le \eta\sup|\dot{x}(t)| + M_0.$$

Thus

$$\sup|\dot{x}(t)| \le \eta\sup|\dot{x}(t)| + M_0$$

and, finally,

$$\sup|\dot{x}(t)| \le M_0(1-\eta)^{-1}.$$

Further on, since the intervals $(t_i - h, t_i)$ contain none of the points t_j, we have

$$\begin{aligned} \delta I_i(x_i,\bar{x}_i) &= \int_0^1 \frac{\partial}{\partial s} I_i(x_i, x(t_i - sh))\,ds \qquad (2.1.2.13) \\ &= \int_0^1 \partial_{\bar{x}} I_i(x_i, x(t_i-sh))\,\frac{\partial}{\partial s}\,x(t_i - sh)\,ds \\ &= -h\int_0^1 \partial_{\bar{x}}\, I_i(x_i, x(t_i-sh))\dot{x}(t_i - sh)\,ds, \end{aligned}$$

thus $|\delta I_i(x_i, \bar{x}_i)| \le hM_1M_0(1-\eta)^{-1}$ and

$$|\mathcal{S}_2 z(t)| \;\le\; hm\mathcal{M}M_1M_0(1-\eta)^{-1} \;=\; O(h). \qquad (2.1.2.14)$$

Then we estimate the difference $x(t)-x(t-h)$. If t is not in Δ_1^h, then $x(t)$ is continuous in $[t-h,t]$ and $\dot{x}$ exists in this segment, with the possible exception of finitely many points. Then we have

$$|x(t)-x(t-h)| \le hM_0(1-\eta)^{-1}.$$

Now we shall obtain an analogous estimate for Δ_1^h. Then the interval $(t-h,t)$ contains just one point of discontinuity t_i of $x(t)$, thus

$$|x(t)-x(t-h)| \le |x(t)-x(t_i+0)| + |x(t_i+0)-x(t_i)| + |x(t_i)-x(t-h)|$$

$$\le M_0(1-\eta)^{-1}(t-t_i) + M_0 + M_0(1-\eta)^{-1}(t_i-t+h) \;=\; M_0(1+h(1-\eta)^{-1}).$$

Using these estimates, we evaluate $\delta f(\tau, x(\tau), \bar{x}(\tau))$. If $\tau \in \Delta_2^h$, we have

$$\delta f(\tau, x(\tau), \bar{x}(\tau)) \;=\; \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh))\, ds$$

$$=\; -h\int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau-sh))\dot{x}(\tau-sh)\, ds$$

and

$$|\delta f(\tau, x(\tau), \bar{x}(\tau))| \;\le\; hM_0M_1(1-\eta)^{-1}. \qquad (2.1.2.15)$$

Next, if $\tau \in \Delta_1^h$, we have

$$\delta f(\tau, x(\tau), \bar{x}(\tau)) = \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), sx(\tau-h) + (1-s)x(\tau))\, ds$$

$$=\; \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), sx(\tau-h)+(1-s)x(\tau))\, ds \cdot (x(\tau-h) - x(\tau))$$

and

$$|\delta f(\tau, x(\tau), \bar{x}(\tau))| \le M_0M_1(1+h(1-\eta)^{-1}). \qquad (2.1.2.16)$$

Making use of the estimates (2.1.2.15) and (2.1.2.16), we find

$$
\begin{aligned}
|\mathcal{I}_2 y(t)| \ \leq \ & \int_{\Delta_2^h} h\mathcal{M}M_0M_1(1-\eta)^{-2}\,d\tau \\
+ \ & \int_{\Delta_1^h} \mathcal{M}(1-\eta)^{-1}M_0M_1(1+h(1-\eta)^{-1})\,d\tau \\
= \ & \int_0^\omega h\mathcal{M}M_0M_1(1-\eta)^{-2}\,d\tau + \int_{\Delta_1^h} \mathcal{M}M_0M_1(1-\eta)^{-1}\,d\tau \\
\leq \ & h\omega\mathcal{M}M_0M_1(1-\eta)^{-2} + hm\mathcal{M}M_0M_1(1-\eta)^{-1} = O(h).
\end{aligned} \tag{2.1.2.17}
$$

Adding together the estimates (2.1.2.11), (2.1.2.12), (2.1.2.14) and (2.1.2.17), we obtain

$$|\mathcal{U}_h z(t)| \ = \ O(\mu^2) + O(h). \tag{2.1.2.18}$$

Henceforth, we shall repeatedly use the following lemma or arguments of its proof.

Lemma 2.1.2.1. *Let $y(t) \in \tilde{C}_{\omega,n}$ be such that in each of the intervals $(0, t_1)$, $(t_1, t_2), \ldots, (t_m, \omega)$ $\dot{y}(t)$ exists, except for a finite number of points, and is bounded. Then there exists a constant $C > 0$ such that for any function $\chi(t)$ integrable by Riemann in $[0, \omega]$ and for any $h > 0$ we have*

$$\left| \int_0^\omega \chi(t)\big(y(t) - y(t-h)\big)\,dt \right| \leq Ch \sup_{t\in[0,\omega]} |\chi(t)|.$$

Proof. It suffices to prove the assertion for h small enough. Then we can define the "bad" set Δ_1^h as above. If $t \notin \Delta_1^h$, then

$$|y(t) - y(t-h)| \leq h \sup_{\tau\in[0,\omega]} |\dot{y}(\tau)|.$$

If $t \in \Delta_1^h$, then we have

$$|y(t) - y(t-h)| \leq h \sup_{\tau\in[0,\omega]} |\dot{y}(\tau)| + \sup_{i=\overline{1,m}} |\Delta y(t_i)|.$$

Since the measures of the sets $[0, \omega] \setminus \Delta_1^h$ and Δ_1^h are respectively $\omega - mh$ and mh, we obtain

$$\left| \int_0^\omega \chi(t)\big(y(t) - y(t-h)\big)\,dt \right|$$

$$\le \sup_{t\in[0,\omega]} |\chi(t)| \left\{ (\omega - mh)h \sup_{t\in[0,\omega]} |\dot{y}(t)| + mh \left[h \sup_{t\in[0,\omega]} |\dot{y}(t)| + \sup_{i=\overline{1,m}} |\Delta y(t_i)| \right] \right\}$$

$$= h \sup_{t\in[0,\omega]} |\chi(t)| \left(\omega \sup_{t\in[0,\omega]} |\dot{y}(t)| + m \sup_{i=\overline{1,m}} |\Delta y(t_i)| \right),$$

thus it suffices to choose $C = \omega \sup_{t\in[0,\omega]} |\dot{y}(t)| + m \sup_{i=\overline{1,m}} |\Delta y(t_i)|$. □

We can note that in the proof of this lemma we have used arguments which are just simplified versions of those used in the evaluation of $\mathcal{I}_2 y(t)$.

In order to estimate $-\mathcal{V}_h y(t)$, we represent the integral in (2.1.2.10) as a difference of two integrals and change the integration variable in the first one to obtain

$$\begin{aligned} -\mathcal{V}_h y(t) \;\; = \;\; & \int_0^\omega \big\{ G(t,\tau)(E - D(\tau))^{-1} D(\tau) \\ & \quad - G(t,\tau-h)(E - D(\tau-h))^{-1} D(\tau-h) \big\} \bar{\dot{x}}(\tau)\, d\tau. \end{aligned}$$

We apply to the last integral Lemma 2.1.2.1 with $\bar{\dot{x}}(\tau)$ instead of χ and $G(t,\tau)(E-D(\tau))^{-1}D(\tau)$ considered as a function of τ for any fixed t instead of y (with points of discontinuity $t_1, t_2, \ldots, t_m$ and t if distinct from the moments of impulse effect). Then

$$| - \mathcal{V}_h y(t)| \le ChM_0(1-\eta)^{-1},$$

where the constant C can be chosen independent of t, and

$$|\mathcal{U}_h y(t) - \mathcal{V}_h y(t)| = O(\mu^2) + O(h),$$

i.e.,

$$|\mathcal{U}_h y(t) - \mathcal{V}_h y(t)| \le K_1 \mu^2 + K_2 h \tag{2.1.2.19}$$

for some positive constants K_1 and K_2.

To provide the validity of the inequality $|\mathcal{U}_h y(t) - \mathcal{V}_h y(t)| \le \mu$, we first choose

$$\tilde{\mu}_0 \;=\; \min\{\, \mu_0,\; \frac{1}{2K_1}\,\}.$$

Then for any $\mu \in (0, \tilde{\mu}_0]$ we have $K_1\mu^2 \;\le\; \mu/2$ and inequality (2.1.2.19) takes on the form

$$|\mathcal{U}_h y(t) - \mathcal{V}_h y(t)| \le \mu/2 \;+\; K_2 h.$$

If we choose

$$\tilde{h}(\mu) \;=\; \min\,\{\; h_0,\; \frac{\mu}{2K_2}\; \},$$

then for any $h \in (0, \tilde{h}(\mu)]$ we have $K_2 h \;\le\; \mu/2$ and thus

$$|\mathcal{U}_h y(t) - \mathcal{V}_h y(t)| \;\le\; \mu,$$

i.e., the operator $\mathcal{U}_h - \mathcal{V}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h \in (0, \tilde{h}(\mu)]$.

Contraction property of the operator $\mathcal{U}_h - \mathcal{V}_h$. Let $y', y'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_h y'(t) - \mathcal{U}_h y''(t) \;&=\; (\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t)) \;+\; (\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t))\\ &+\; (\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t)) \;+\; (\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)).\end{aligned}$$

First we consider

$$\mathcal{S}_1 y'(t) - \mathcal{S}_1 y''(t) \;=\; \sum_{i=1}^{m} G(t, t_i + 0)\big(J_i(y_i') \;-\; J_i(y_i'')\big).$$

We have

$$\begin{aligned}&J_i(y_i') \;-\; J_i(y_i'')\\ =\;& \big(I_i(\psi_i + y_i', \psi_i + y_i') \;-\; I_i(\psi_i + y_i'', \psi_i + y_i'')\big) \;-\; B_i(y_i' - y_i'')\\ =\;& \Big\{\int_0^1 \big(\partial_x I_i(\psi_i + s y_i' + (1-s)y_i'', \psi_i + y_i' + (1-s)y_i'') \;-\; \partial_x I_i(\psi_i, \psi_i)\big)\, ds\\ +\;& \int_0^1 \big(\partial_{\bar{x}} I_i(\psi_i + s y_i' + (1-s)y_i'', \psi_i + s y_i' + (1-s)y_i'') - \partial_{\bar{x}} I_i(\psi_i, \psi_i)\big)\, ds\Big\}\\ &\times (y_i' - y_i''),\end{aligned}$$

thus

$$\begin{aligned}|J_i(y_i') \;-\; J_i(y_i'')| \;&\le\; 4M_2 \int_0^1 \big[s|y_i'| + (1-s)|y_i''|\big]\, ds \cdot |y_i' - y_i''|\\ &\le\; 2M_2\big(|y_i'| + |y_i''|\big)|y_i' - y_i''| \;\le\; 4\mu M_2 |y_i' - y_i''|\end{aligned}$$

and

$$|\mathcal{S}_1 y'(t) \;-\; \mathcal{S}_1 y''(t)| \;\le\; 4\mu \mathcal{M} m M_2 \|y' \;-\; y''\|. \tag{2.1.2.20}$$

Next,

$$\begin{aligned}&\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t)\\ =\ & \int_0^{\omega} G(t,\tau)(E-D(\tau))^{-1}\big(Q(\tau,y'(\tau)) - Q(\tau,y''(\tau))\big)\,d\tau.\end{aligned}$$

We have

$$\begin{aligned}&Q(\tau,y'(\tau)) - Q(\tau,y''(\tau))\\ =\ & \left\{\int_0^1 [\partial_x f(\tau,x_s(\tau),x_s(\tau)) - \partial_x f(\tau,\psi(\tau),\psi(\tau))]\,ds\right.\\ +\ & \left.\int_0^1 [\partial_{\bar{x}} f(\tau,x_s(\tau),x_s(\tau)) - \partial_{\bar{x}} f(\tau,\psi(\tau),\psi(\tau))]\,ds\right\}\cdot\big(y'(\tau)-y''(\tau)\big),\end{aligned}$$

where $x_s(\tau) = \psi(\tau) + sy'(\tau) + (1-s)y''(\tau)$. Thus

$$\begin{aligned}&|Q(\tau,y'(\tau)) - Q(\tau,y''(\tau))|\\ \le\ & 2M_2\int_0^1 [2s|y'(\tau)| + 2(1-s)|y''(\tau)|]\,ds\cdot|y'(\tau)-y''(\tau)|\\ \le\ & 2M_2\big(\|y'\|+\|y''\|\big)\cdot\|y'-y''\| \ \le\ 4\mu M_2\,\|y' - y''\|\end{aligned}$$

and

$$|\mathcal{I}_1 y'(t) - \mathcal{I}_1 y''(t)| \ \le\ 4\mu\mathcal{M}\omega M_2(1-\eta)^{-1}\,\|y' - y''\|. \qquad (2.1.2.21)$$

For the estimation of $\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)$ and $\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)$ we denote $x' = \psi(t) + y',\ x'' = \psi(t) + y''$. Now

$$\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t) \ =\ \sum_{i=1}^{m} G(t,t_i+0)(\delta I_i(x_i',\bar{x}_i') \ -\ \delta I_i(x_i'',\bar{x}_i''))$$

and

$$\begin{aligned}&\delta I_i(x_i',\bar{x}_i') \ -\ \delta I_i(x_i'',\bar{x}_i'')\\ =\ & \big(I_i(x_i',\bar{x}_i') - I_i(x_i',x_i')\big) \ -\ \big(I_i(x_i'',\bar{x}_i'') - I_i(x_i'',x_i'')\big)\\ =\ & \big(I_i(x_i',\bar{x}_i') - I_i(\psi_i,\psi_i)\big) \ -\ \big(I_i(x_i',x_i') - I_i(\psi_i,\psi_i)\big)\\ -\ & \big(I_i(x_i'',\bar{x}_i'') - I_i(\psi_i,\psi_i)\big) \ +\ \big(I_i(x_i'',x_i'') - I_i(\psi_i,\psi_i)\big)\end{aligned}$$

$$
\begin{aligned}
= \; & -\int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s y_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i'))\, d(1-s) \\
& + \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s y_i', \psi_i + s y_i')\, d(1-s) \\
& + \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s y_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i''))\, d(1-s) \\
& - \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s y_i'', \psi_i + s y_i'')\, d(1-s).
\end{aligned}
$$

Making use of the continuity of the second derivatives of $I_i(x, \bar{x})$ (condition **A2.1.2.2**), we integrate by parts and rearrange the terms to obtain

$$
\begin{aligned}
& \delta I_i(x_i', \bar{x}_i') \,-\, \delta I_i(x_i'', \bar{x}_i'') \\
= \; & \left\{ \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} I_i(x,x)|_{x=\psi_i + s y_i''}\, ds \cdot y_i'', y_i'' \right\rangle \right. \\
- \; & \left. \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} I_i(x,x)|_{x=\psi_i + s y_i'}\, ds \cdot y_i', y_i' \right\rangle \right\} \\
+ \; & \left\{ \left\langle \int_0^1 (1-s) \partial^2_{xx} I_i(\psi_i + s y_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i'))\, ds \cdot y_i', y_i' \right\rangle \right. \\
- \; & \left. \left\langle \int_0^1 (1-s) \partial^2_{xx} I_i(\psi_i + s y_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i''))\, ds \cdot y_i'', y_i'' \right\rangle \right\} \\
+ \; & 2 \left\{ \left\langle \int_0^1 (1-s) \partial^2_{x\bar{x}} I_i(\psi_i + s y_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i'))\, ds \cdot y_i', \bar{\psi}_i - \psi_i + \bar{y}_i' \right\rangle \right. \\
- \; & \left. \left\langle \int_0^1 (1-s) \partial^2_{x\bar{x}} I_i(\psi_i + s y_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i''))\, ds \cdot y_i'', \bar{\psi}_i - \psi_i + \bar{y}_i'' \right\rangle \right\} \\
+ \; & \left\{ \left\langle \int_0^1 (1-s) \partial^2_{\bar{x}\bar{x}} I_i(\psi_i + s y_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i'))\, ds \cdot (\bar{\psi}_i - \psi_i + \bar{y}_i'), \right. \right. \\
& \qquad \left. \bar{\psi}_i - \psi_i + \bar{y}_i' \right\rangle
\end{aligned}
$$

$$- \quad \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} I_i(\psi_i + sy_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{y}_i''))\, ds \cdot (\bar{\psi}_i - \psi_i + \bar{y}_i''), \right.$$

$$\left. \bar{\psi}_i - \psi_i + \bar{y}_i'' \right\rangle \Bigg\} .$$

Now we estimate separately the four addends in the braces making use also of the Lipschitz continuity of the second derivatives of $I_i(x_i, \bar{x}_i)$ according to condition **A2.1.2.2**. It is easy to see that the first two addends are estimated by $O(\mu)\|y' - y''\|$, while the other two terms are estimated by $(O(\mu) + O(h))\|y' - y''\|$. Thus we obtain

$$|\mathcal{S}_2 y'(t) - \mathcal{S}_2 y''(t)| \;\leq\; (O(\mu) + O(h))\|y' \;-\; y''\|. \tag{2.1.2.22}$$

We estimate $\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)$ in a similar way, using condition **A2.1.2.1**. Now we have (the argument τ is dropped for brevity)

$$\begin{aligned}
&\delta f(\cdot, x', \bar{x}') \;-\; \delta f(\cdot, x'', \bar{x}'') \\
&= \; \Bigg\{ \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} f(\cdot, x, x)|_{x=\psi+sy''}\, ds \cdot y'', y'' \right\rangle \\
&- \; \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} f(\cdot, x, x)|_{x=\psi+sy'}\, ds \cdot y', y' \right\rangle \Bigg\} \\
&+ \; \Bigg\{ \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sy'', \psi + sy'')\, ds \cdot y'', y'' \right\rangle \\
&- \; \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sy', \psi + sy')\, ds \cdot y', y' \right\rangle \Bigg\} \\
&+ \; \Bigg\{ \left\langle \int_0^1 (1-s)\partial^2_{xx} f(\cdot, \psi + sy', \psi + s(\bar{\psi} - \psi + \bar{y}'))\, ds \cdot y', y' \right\rangle \\
&- \; \left\langle \int_0^1 (1-s)\partial^2_{xx} f(\cdot, \psi + sy'', \psi + s(\bar{\psi} - \psi + \bar{y}''))\, ds \cdot y'', y'' \right\rangle \Bigg\} \\
&+ \; 2\Bigg\{ \left\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} f(\cdot, \psi + sy', \psi + s(\bar{\psi} - \psi + \bar{y}'))\, ds \cdot y', \bar{\psi} - \psi + \bar{y}' \right\rangle \\
&- \; \left\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} f(\cdot, \psi + sy'', \psi + s(\bar{\psi} - \psi + \bar{y}''))\, ds \cdot y'', \bar{\psi} - \psi + \bar{y}'' \right\rangle \Bigg\}
\end{aligned}$$

$$+\quad \left\{ \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sy', \psi + s(\bar{\psi} - \psi + \bar{y}'))\, ds \cdot (\bar{\psi} - \psi + \bar{y}'), \right.\right.$$

$$\left. \bar{\psi} - \psi + \bar{y}' \right\rangle$$

$$-\quad \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sy'', \psi + s(\bar{\psi} - \psi + \bar{y}''))\, ds \cdot (\bar{\psi} - \psi + \bar{y}''), \right.$$

$$\left.\left. \bar{\psi} - \psi + \bar{y}'' \right\rangle \right\}.$$

The first three expressions in the braces are estimated by $O(\mu)\|y' - y''\|$ for all $\tau \in [0, \omega]$. The fourth and fifth expressions are estimated by

$$\begin{cases} (O(\mu) + O(h))\|y' - y''\| & \text{for} \quad \tau \notin \Delta_1^h, \\ (O(\mu) + O(1))\|y' - y''\| & \text{for} \quad \tau \in \Delta_1^h. \end{cases}$$

Using these estimates, by arguments similar to those in the proof of Lemma 2.1.2.1 we find

$$|\mathcal{I}_2 y'(t) - \mathcal{I}_2 y''(t)| \le (O(\mu) + O(h))\|y' - y''\|. \tag{2.1.2.23}$$

Now by virtue of the estimates (2.1.2.20), (2.1.2.21), (2.1.2.22) and (2.1.2.23) we obtain

$$\|\mathcal{U}_h y' - \mathcal{U}_h y''\| \le (O(\mu) + O(h))\|y' - y''\|.$$

In order to estimate $\mathcal{V}_h y' - \mathcal{V}_h y''$, we integrate by parts the expression for $\mathcal{V}_h y(t)$ taking into account that the function $G(t,\tau)$ is discontinuous at $\tau = t_1, \ldots, t_m$ and $\tau = t$ while $x(\tau)$ is discontinuous at $t_1, \ldots, t_m$ making use of the equalities

$$\frac{\partial G(t,\tau)}{\partial \tau} = -G(t,\tau)(E - D(\tau))^{-1} A(\tau)$$

and $G(t, t_i) = G(t, t_i + 0)(E + B_i)$. We obtain

$$\begin{aligned} \mathcal{V}_h y(t) \;=&\; (E - D(t))^{-1} D(t)\big(x(t) - x(t-h)\big) \\ +& \sum_{i=1}^m \Big\{ G(t, t_i + h)(E - D(t_i + h))^{-1} D(t_i + h) \\ &\quad - G(t, t_i + 0)(E - D(t_i))^{-1} D(t_i) \Big\} I_i(x_i, \bar{x}_i) \\ +& \sum_{i=1}^m G(t, t_i + 0) B_i (E - D(t_i))^{-1} D(t_i)(x_i - \bar{x}_i) \end{aligned}$$

$$
\begin{aligned}
- & \int_0^\omega \Big\{ G(t,\tau)(E - D(\tau))^{-1} A(\tau)(E - D(\tau))^{-1} D(\tau) - G(t, \tau - h) \\
\times & \ (E - D(\tau - h))^{-1} A(\tau - h)(E - D(\tau - h))^{-1} D(\tau - h) \Big\} x(\tau - h)\, d\tau \\
+ & \int_0^\omega \Big\{ G(t,\tau)(E - D(\tau))^{-1} \dot{D}(\tau)(E - D(\tau))^{-1} \\
& \quad -G(t,\tau - h)(E - D(\tau - h))^{-1} \dot{D}(\tau - h)(E - D(\tau - h))^{-1} \Big\} x(\tau - h)\, d\tau.
\end{aligned}
$$

In view of Lemma 2.1.2.1 the two integral terms are estimated by $O(h)\|x\|$. The sum of the first and third terms can be estimated by $2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}\|x\|$. The difference

$$
G(t, t_i + h)(E - D(t_i + h))^{-1} D(t_i + h) - G(t, t_i + 0)(E - D(t_i))^{-1} D(t_i)
$$

is estimated by $O(h)$ for $t \notin \Delta_1^h$, and by $\eta(1-\eta)^{-1} + O(h)$ otherwise. At last, similarly to estimate (2.1.2.22), we note that

$$
|I_i(x_i', \bar{x}_i') - I_i(x_i'', \bar{x}_i'')| \le \big(O(\mu) + O(h)\big)\|y' - y''\|,
$$

thus we have

$$
|\mathcal{V}_h y'(t) - \mathcal{V}_h y''(t)| \le \big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} + O(\mu) + O(h)\big)\|y' - y''\|
$$

and

$$
\begin{aligned}
& |\big(\mathcal{U}_h y'(t) - \mathcal{V}_h y'(t)\big) - \big(\mathcal{U}_h y''(t) - \mathcal{V}_h y''(t)\big)| \\
\le & \ \big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} + \gamma_1 \mu + \gamma_2 h\big)\|y' - y''\|,
\end{aligned}
$$

where γ_1 and γ_2 are some positive constants.

By condition (2.1.2.6) we have

$$
\tilde{\eta} \equiv 2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} < 1.
$$

Choose a number $q \in (\tilde{\eta}, 1)$ and denote $r = q - \tilde{\eta}$, and $\mu_1 = \min\{\tilde{\mu}_0, \frac{r}{2\gamma_1}\}$ and $h_1 = \min\{\tilde{h}(\mu_1), \frac{r}{2\gamma_2}\}$. Then for any $\mu \in (0, \mu_1]$ and $h \in [0, h_1]$ we have

$$
\|\big(\mathcal{U}_h y' - \mathcal{V}_h y'\big) - \big(\mathcal{U}_h y'' - \mathcal{V}_h y''\big)\| \le q\|y' - y''\|, \qquad q \in (0,1),
$$

for any $y', y'' \in \mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_h - \mathcal{V}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an ω–periodic solution $y(t,h)$ of system (2.1.2.8). Since $y(t) \equiv 0$ is the unique ω–periodic solution of system (2.1.2.8) for $h = 0$, then $y(t,0) \equiv 0$. Now $x(t,h) = \psi(t) + y(t,h)$ is the unique ω–periodic solution of system (2.1.2.1) and $x(t,0) = \psi(t)$. This completes the proof of Theorem 2.1.2.1. $\square$

The results of the present subsection were reported at the Fourth International Colloquium on Differential Equations, Plovdiv, Bulgaria, 18–23 August 1993, and published in its proceedings [76]. The existence of a periodic solution under somewhat weaker assumptions using the Schauder Fixed Point Theorem was reported at the 20-th Summer School "Applications of Mathematics in Engineering", Varna, Bulgaria, 26 August – 2 September 1994 and published in its proceedings [24].

2.2 Periodic Solutions of Impulsive Systems with Small Delays in the Critical Case

We consider an impulsive differential-difference system such that the corresponding system without delay is linear and has an r–parametric family of ω–periodic solutions. For this case, an equation for the generating amplitudes is derived, and sufficient conditions are obtained for the existence of ω–periodic solutions of the initial system in the critical cases of first and second order if the delay is small enough.

As an application of these results, for an age-dependent model with a dominant age class an ω–periodic regime of the population size is sought by means of impulsive perturbations. For both noncritical case and critical case of first order, the problem is reduced to operator systems solvable by a convergent simple iteration method.

Finally, we consider a nonlinear boundary value problem for an impulsive system of ordinary differential equations with concentrated delays in the general case when the number of the boundary conditions does not coincide with the order of the system. It is assumed that the corresponding boundary value problem without delay is linear and has an r–parametric family of solutions. Then the equation for the generating amplitudes is derived, and sufficient conditions for the existence and an iteration algorithm for the construction of a solution of the initial problem are obtained in the critical case of first order if the delays are sufficiently small.

2.2.1 Critical case of first order

In the present subsection we study a system with impulses at fixed moments and a small delay of the argument

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t) + H(t, x(t), x(t-h)), \quad t \neq t_i, \\ \Delta x(t_i) &= B_i x(t_i) + a_i + I_i(x(t_i), x(t_i - h)), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.2.1.1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}^n$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $H(t, x, \bar{x}) = g(t, x, \bar{x})(x - \bar{x})$, $g : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^{n\times n}$, Ω is a domain in $\mathbb{R}^n$; $\Delta x(t_i)$ are the impulses at moments t_i and $\{t_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{i\to\pm\infty} t_i = \pm\infty$; $a_i \in \mathbb{R}^n$, $B_i \in \mathbb{R}^{n\times n}$, $I_i(x, \bar{x}) = J_i(x, \bar{x})(x - \bar{x})$, $J_i : \Omega \times \Omega \to \mathbb{R}^{n\times n}$ ($i \in \mathbb{Z}$), $h \geq 0$ is the delay.

As in §2.1.1 we shall use the notation $x_i = x(t_i)$, $\bar{x}(t) = x(t-h)$.

In the sequel we require the fulfillment of the following assumptions:

A2.2.1.1. The function $g(t, x, \bar{x})$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω–periodic with respect to t, continuously differentiable with respect to $x, \bar{x}$.

A2.2.1.2. The functions $J_i \in C^1(\Omega \times \Omega, \mathbb{R}^{n\times n})$, $i \in \mathbb{Z}$.

A2.2.1.3. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $B_{i+m} = B_i$, $a_{i+m} = a_i$, $J_{i+m}(x, \bar{x}) = J_i(x, \bar{x})$ for $i \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

A2.2.1.4. $A(\cdot) \in \tilde{C}_{\omega,n\times n}$, $f(\cdot) \in \tilde{C}_{\omega,n}$.

A2.2.1.5. The matrices $E + B_i$, $i \in \mathbb{Z}$, are nonsingular.

Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

Let $h_1 > 0$ be so small that, for any $h \in [0, h_1]$, we have

$$h < t_1, \quad t_i + h < t_{i+m}, \quad i = \overline{1, m-1}, \quad t_m + h < \omega.$$

Together with (2.2.1.1), we consider the so called *generating* system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t), \quad t \neq t_i, \\ \Delta x(t_i) &= B_i x_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.2.1.2}$$

obtained from (2.2.1.1) for $h = 0$.

Let $X(t)$ be the fundamental solution (*i.e.*, $X(0) = E$) of the homogeneous system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad t \neq t_i, \\ \Delta x(t_i) &= B_i x_i, \quad i \in \mathbb{Z}. \end{aligned} \tag{2.2.1.3}$$

Denote $Q = E - X(\omega)$ and consider the critical case where

A2.2.1.6. $\operatorname{Rank} Q = n_1 < n$.

Let $r = n - n_1$ and Q^*, Q^+, $\mathcal{P}^*$, $\mathcal{P}_r^*$ be as in §1.2. Then the nonhomogeneous system (2.2.1.2) has ω–periodic solutions if and only if

$$\mathcal{P}_r^* X(\omega) \left(\int_0^\omega X^{-1}(\tau) f(\tau)\, d\tau + \sum_{i=1}^m X_i^{-1} a_i \right) = 0. \qquad (2.2.1.4)$$

Recall that $X_i \equiv X(t_i + 0)$. If condition (2.2.1.4) is satisfied, then system (2.2.1.2) has an r–parametric family of ω–periodic solutions

$$x_0(t, c_r) = X_r(t) c_r + \int_0^\omega G(t, \tau) f(\tau)\, d\tau + \sum_{i=1}^m G(t, t_i) a_i, \qquad (2.2.1.5)$$

where $X_r(t)$ is an $(n \times r)$–matrix whose columns are a complete system of r linearly independent ω–periodic solutions of (2.2.1.3), $c_r \in \mathbb{R}^r$ is an arbitrary vector, and $G(t, \tau)$ is the generalized Green's function (see §1.2).

Let us find conditions for the existence of ω–periodic solutions $x(t, h)$ of system (2.2.1.1) depending continuously on h and such that, for some $c_r \in \mathbb{R}^r$, we have $x(t, 0) = x_0(t, c_r)$. A necessary condition for the existence of such solutions is given by the following statement:

Theorem 2.2.1.1. *Let system* (2.2.1.1) *satisfying conditions* **A2.2.1.1** – **A2.2.1.6** *and* (2.2.1.4) *have an ω–periodic solution $x(t, h)$ which, for $h = 0$, turns into a generating solution $x_0(t, c_r^*)$. Then the vector $c_r^* \in \mathbb{R}^r$ satisfies the equation*

$$\begin{aligned} F(c_r^*) \;\equiv\; & \mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau) g(\tau, x_0(\tau, c_r^*), x_0(\tau, c_r^*))(A(\tau) x_0(\tau, c_r^*) + f(\tau))\, d\tau \right. \\ + & \sum_{i=1}^m X_i^{-1} \left[J_i(x_0(t_i, c_r^*), x_0(t_i, c_r^*))(A_i x_0(t_i, c_r^*) + f_i) \right. \qquad (2.2.1.6) \\ + & \left. \left. g(t_i + 0, (E + B_i) x_0(t_i, c_r^*) + a_i, x_0(t_i, c_r^*))(B_i x_0(t_i, c_r^*) + a_i) \right] \right\} = 0 \end{aligned}$$

known as *equation for the generating amplitudes* (see, for instance, [63] or [29, 35]) of the problem of finding ω–periodic solutions of the impulsive system with delay (2.2.1.1).

Proof. In (2.2.1.1), we change the variables according to the formula

$$x(t, h) = x_0(t, c_r^*) + y(t, h) \qquad (2.2.1.7)$$

and are led to the problem of finding ω–periodic solutions $y(t) = y(t,h)$ of the impulsive system of functional differential equations

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + H(t, x(t,h), x(t-h,h)), \quad t \neq t_i, \\ \Delta y(t_i) &= B_i y_i + I_i(x(t_i,h), x(t_i - h, h)), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.2.1.8}$$

such that $y(t,h) \to 0$ as $h \to 0$.

We can formally consider (2.2.1.8) as a nonhomogeneous system of the form (2.2.1.2). Then the solvability condition (2.2.1.4) becomes

$$\mathcal{P}_r^* X(\omega) \left(\int_0^\omega X^{-1}(\tau) H(\tau, x(\tau,h), x(\tau - h, h))\, d\tau + \sum_{i=1}^m X_i^{-1} I_i(x_i, \bar{x}_i) \right) = 0. \tag{2.2.1.9}$$

For the sake of later convenience, we denote by $\varepsilon(h,x)$ expressions tending to 0 as $h \to 0$, and satisfying the Lipschitz condition with respect to x with a constant tending to 0 as $h \to 0$. We shall sometimes write $\varepsilon(h)$ instead of $\varepsilon(h,x)$ and $x(t)$ instead of $x(t,h)$ if this does not lead to misunderstanding. Thus, for instance, we write x_i and $\bar{x}_i$ instead of $x(t_i,h)$ and $x(t_i - h, h)$, respectively.

Since the left-hand side of equality (2.2.1.9) tends to 0 as $h \to 0$, we first divide it by h and then study its behaviour as $h \to 0$. First we note that

$$(x_i - \bar{x}_i)/h = \dot{x}_i + \varepsilon(h) = A_i x_i + f_i + g(t_i, x_i, \bar{x}_i)(x_i - \bar{x}_i) + \varepsilon(h)$$

since the interval $(t_i - h, t_i)$ contains no points of discontinuity of the function $x(t,h)$ or its derivative.

This equality implies that

$$(E - hg(t_i, x_i, \bar{x}_i))(x_i - \bar{x}_i)/h = A_i x_i + f_i + \varepsilon(h).$$

If h is small enough, we have

$$(x_i - \bar{x}_i)/h = (E - hg(t_i, x_i, \bar{x}_i))^{-1}(A_i x_i + f_i + \varepsilon(h)) = A_i x_i + f_i + \varepsilon(h).$$

Thus,

$$I_i(x_i, \bar{x}_i)/h = J_i(x_i, \bar{x}_i)(x_i - \bar{x}_i)/h = J_i(x_i, x_i)(A_i x_i + f_i) + \varepsilon(h). \tag{2.2.1.10}$$

We can represent the integral $\int_0^\omega$ in (2.2.1.9) as a sum of integrals containing no points of discontinuity of the integrand. It is obvious that, for

$\tau \in (t_i, t_i + h)$, the interval $(\tau - h, \tau)$ contains the point of discontinuity t_i, while for τ inside the remaining intervals, the interval $(\tau - h, \tau)$ contains no such points. We denote $\Delta_1^h = \bigcup_{i=1}^{m} (t_i, t_i + h)$, $\Delta_2^h = [0, \omega] \setminus \Delta_1^h$ and use the representation $\int_0^\omega = \int_{\Delta_1^h} + \int_{\Delta_2^h}$.

We first begin with the "bad" set Δ_1^h. For any $i \in \{1, 2, \dots, m\}$, we can find a constant $\theta_i \in (0, 1)$ such that

$$
\begin{aligned}
& h^{-1} \int_{t_i}^{t_i+h} X^{-1}(\tau) H(\tau, x(\tau, h), x(\tau - h, h))\, d\tau \\
= \;& X^{-1}(t_i + \theta_i h) H(t_i + \theta_i h, x(t_i + \theta_i h, h), x(t_i - (1 - \theta_i) h, h)) \\
= \;& X_i^{-1} g(t_i + 0, (E + B_i) x_i + a_i, x_i)(B_i x_i + a_i) + \varepsilon(h). \qquad (2.2.1.11)
\end{aligned}
$$

On the other hand, for τ in the "good" set Δ_2^h, we have as above (with $x \equiv x(\tau, h)$, $\bar{x} \equiv x(\tau - h, h)$, $A \equiv A(\tau)$, etc.) $(x - \bar{x})/h = Ax + f + \varepsilon(h)$, thus

$$
\begin{aligned}
& h^{-1} \int_{\Delta_2^h} X^{-1}(\tau) H(\tau, x(\tau, h), x(\tau - h, h))\, d\tau \\
= \;& \int_{\Delta_2^h} X^{-1}(\tau) g(\tau, x(\tau, h), x(\tau, h))(A(\tau) x(\tau, h) + f(\tau))\, d\tau + \varepsilon(h).
\end{aligned}
$$

Since $\int_{\Delta_2^h} = \int_0^\omega - \int_{\Delta_1^h}$ and

$$
\int_{\Delta_1^h} X^{-1}(\tau) g(\tau, x(\tau, h), x(\tau, h))(A(\tau) x(\tau, h) + f(\tau))\, d\tau = \varepsilon(h),
$$

we obtain

$$
\begin{aligned}
& h^{-1} \int_{\Delta_2^h} X^{-1}(\tau) H(\tau, x(\tau, h), x(\tau - h, h))\, d\tau \qquad (2.2.1.12) \\
= \;& \int_0^\omega X^{-1}(\tau) g(\tau, x(\tau, h), x(\tau, h))(A(\tau) x(\tau, h) + f(\tau))\, d\tau + \varepsilon(h).
\end{aligned}
$$

In view of (2.2.1.10), (2.2.1.11) and (2.2.1.12), we can write the solvability condition (2.2.1.9) in the form

$$\begin{aligned} & \mathcal{P}_r^* X(\omega) \Big\{ \int_0^\omega X^{-1}(\tau) g(\tau, x(\tau,h), x(\tau,h)) (A(\tau)x(\tau,h) + f(\tau))\, d\tau \\ + & \sum_{i=1}^m X_i^{-1} J_i(x_i, x_i)(A_i x_i + f_i) \\ + & \sum_{i=1}^m X_i^{-1} g(t_i+0, (E+B_i)x_i + a_i, x_i)(B_i x_i + a_i) + \varepsilon(h,x) \Big\} = 0. \end{aligned} \tag{2.2.1.13}$$

Passing to the limit in (2.2.1.13), in view of $\lim_{h\to 0} x(t,h) = x_0(t, c_r^*)$ we obtain (2.2.1.6). □

Now suppose that c_r^* is a solution of equation (2.2.1.6). Then the ω–periodic solution $y(t,h)$ of system (2.2.1.8) such that $y(t,0) \equiv 0$ can be represented in the form

$$y(t,h) = X_r(t)c + hy^{(1)}(t,h), \tag{2.2.1.14}$$

where the unknown constant vector $c = c(h) \in \mathbb{R}^r$ must satisfy an equation derived below from (2.2.1.13), while the unknown ω–periodic vector-valued function $y^{(1)}(t,h)$ can be represented as

$$\begin{aligned} y^{(1)}(t,h) &= \Big\{ \int_0^\omega G(t,\tau) H(\tau, x(\tau,h), x(\tau-h,h))\, d\tau \\ &+ \sum_{i=1}^m G(t,t_i) I_i(x(t_i,h), x(t_i-h,h)) \Big\} / h. \end{aligned}$$

By arguments similar to those above, we find

$$\begin{aligned} y^{(1)}(t,h) &= \int_0^\omega G(t,\tau) g(\tau, x(\tau,h), x(\tau,h)) (A(\tau)x(\tau,h) + f(\tau))\, d\tau \\ &+ \sum_{i=1}^m G(t,t_i) J_i(x_i, x_i)(A_i x_i + f_i) \\ &+ \sum_{i=1}^m G(t,t_i) g(t_i+0, (E+B_i)x_i + a_i, x_i)(B_i x_i + a_i) + \varepsilon(h,x). \end{aligned} \tag{2.2.1.15}$$

In equalities (2.2.1.13) and (2.1.1.15) we replace $x(t,h)$ by $x_0(t,c_r^*)+y(t,h)$. Thus we obtain

$$\begin{aligned} & \mathcal{P}_r^* X(\omega)\Bigg\{\int_0^\omega X^{-1}(\tau) g(\tau, x_0(t,c_r^*)+y(t,h), x_0(t,c_r^*)+y(t,h)) \\ & \qquad \times (A(\tau)(x_0(t,c_r^*)+y(t,h))+f(\tau))\,d\tau \qquad (2.2.1.16) \\ + & \sum_{i=1}^m X_i^{-1} J_i(x_0(t_i,c_r^*)+y_i, x_0(t_i,c_r^*)+y_i)(A_i(x_0(t_i,c_r^*)+y_i)+f_i) \\ + & \sum_{i=1}^m X_i^{-1} g(t_i+0,(E+B_i)(x_0(t_i,c_r^*)+y_i)+a_i, x_0(t_i,c_r^*)+y_i) \\ & \qquad \times (B_i(x_0(t_i,c_r^*)+y_i)+a_i)+\varepsilon(h,x)\Bigg\}=0 \end{aligned}$$

and

$$\begin{aligned} y^{(1)}(t,h) &= \int_0^\infty G(t,\tau) g(\tau, x_0(t,c_r^*)+y(t,h), x_0(t,c_r^*)+y(t,h)) \\ & \qquad \times (A(\tau)(x_0(t,c_r^*)+y(t,h))+f(\tau))\,d\tau \qquad (2.2.1.17) \\ &+ \sum_{i=1}^m G(t,t_i) J_i(x_0(t_i,c_r^*)+y_i, x_0(t_i,c_r^*)+y_i)(A_i(x_0(t_i,c_r^*)+y_i)+f_i) \\ &+ \sum_{i=1}^m G(t,t_i) g(t_i+0,(E+B_i)(x_0(t_i,c_r^*)+y_i)+a_i, x_0(t_i,c_r^*)+y_i) \\ & \qquad \times (B_i(x_0(t_i,c_r^*)+y_i)+a_i)+\varepsilon(h,x). \end{aligned}$$

We expand the left-hand side of (2.2.1.16) about the point $y=0$. We have

$$\begin{aligned} & g(\tau, x_0(t,c_r^*)+y, x_0(t,c_r^*)+y)(A(\tau)(x_0(t,c_r^*)+y)+f(\tau)) \\ = \; & g(\tau, x_0(t,c_r^*), x_0(t,c_r^*))(A(\tau)x_0(t,c_r^*)+f(\tau))+g_1(\tau)y+g_2(\tau,y), \end{aligned} \qquad (2.2.1.18)$$

where $g_2(\tau,y)$ is such that

$$g_2(\tau,0)=0, \qquad \frac{\partial}{\partial y} g_2(\tau,0)=0.$$

Analogously, we have

$$\begin{aligned} & J_i(x_0(t_i,c_r^*)+y, x_0(t_i,c_r^*)+y)(A_i(x_0(t_i,c_r^*)+y)+f_i) \\ + \; & g(t_i+0,(E+B_i)(x_0(t_i,c_r^*)+y)+a_i, x_0(t_i,c_r^*)+y)(B_i(x_0(t_i,c_r^*)+y)+a_i) \\ = \; & J_i(x_0(t_i,c_r^*), x_0(t_i,c_r^*))(A_i x_0(t_i,c_r^*)+f_i) \qquad (2.2.1.19) \\ + \; & g(t_i+0,(E+B_i)x_0(t_i,c_r^*)+a_i, x_0(t_i,c_r^*))(B_i x_0(t_i,c_r^*)+a_i)+J_{1i}y+J_{2i}(y), \end{aligned}$$

where $J_{2i}(y)$ is such that

$$J_{2i}(0) = 0, \qquad \frac{\partial}{\partial y} J_{2i}(0) = 0.$$

In view of the assumption $F(c_r^*) = 0$, equality (2.2.1.16) now takes the form

$$\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)(g_1(\tau)y(\tau,h) + g_2(\tau, y(\tau,h)))\, d\tau \right. \quad (2.2.1.20)$$
$$\left. + \sum_{i=1}^{m} X_i^{-1}(J_{1i}y_i + J_{2i}(y_i)) + \varepsilon_1(h,x) \right\} = 0.$$

In view of representation (2.2.1.14), denote

$$\mathcal{B}_0 = \mathcal{P}_r^* X(\omega) \left(\int_0^\omega X^{-1}(\tau) g_1(\tau) X_r(\tau)\, d\tau + \sum_{i=1}^{m} X_i^{-1} J_{1i} X_r(t_i) \right), \quad (2.2.1.21)$$

which is an $(r \times r)$–matrix. Then (2.2.1.20) can be written in the form

$$\begin{aligned} \mathcal{B}_0 c \;=\; & -\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)(h g_1(\tau) y^{(1)}(\tau,h) + g_2(\tau, y(\tau,h)))\, d\tau \right. \\ & \left. + \sum_{i=1}^{m} X_i^{-1}(h J_{1i} y^{(1)}(t_i,h) + J_{2i}(y_i)) + \varepsilon_1(h,x) \right\}. \qquad (2.2.1.22) \end{aligned}$$

In the representation (2.2.1.17) we use the same expansions (2.2.1.18) and (2.2.1.19) and obtain

$$\begin{aligned} y^{(1)}(t,h) \;=\; & \int_0^\infty G(t,\tau) \left\{ g(\tau, x_0(t,c_r^*), x_0(t,c_r^*))(A(\tau)x_0(t,c_r^*) + f(\tau)) \right. \\ & \left. + g_1(\tau)[X_r(\tau)c + h y^{(1)}(\tau,h)] + g_2(\tau, y(\tau,h)) \right\} d\tau \qquad (2.2.1.23) \\ + & \sum_{i=1}^{m} G(t,t_i) \left\{ J_i(x_0(t_i,c_r^*), x_0(t_i,c_r^*))(A_i x_0(t_i,c_r^*) + f_i) \right. \\ & + g(t_i + 0, (E + B_i)x_0(t_i,c_r^*) + a_i, x_0(t_i,c_r^*))(B_i x_0(t_i,c_r^*) + a_i) \\ & \left. + J_{1i}[X_r(t_i)c + h y^{(1)}(t_i,h)] + J_{2i}(y_i) \right\} + \varepsilon_2(h,x). \end{aligned}$$

Thus we have reduced problem (2.2.1.1) to the equivalent operator system (2.2.1.7), (2.2.1.14), (2.2.1.22), (2.2.1.23).

Suppose that $\det \mathcal{B}_0 \neq 0$. It is easy to see [29, 63] that this condition is equivalent to the simplicity of the root $c_r = c_r^*$ of the equation for the generating amplitudes:

$$F(c_r^*) = 0, \qquad \det\left(\frac{\partial F(c_r)}{\partial c_r}\right)\Bigg|_{c_r=c_r^*} \neq 0.$$

This is the so called *critical case of the first order*. Then equation (2.2.1.21) can be solved with respect to c and we obtain [63] a Fredholm operator system of the second type to which a convergent simple iteration method can be applied.

Theorem 2.2.1.2. *For system* (2.2.1.1)*, let conditions* **A2.2.1.1**–**A2.2.1.6** *and* (2.2.1.4) *hold. Then for any simple* ($\det \mathcal{B}_0 \neq 0$) *root* $c_r = c_r^* \in \mathbb{R}^r$ *of equation* (2.2.1.6)*, there exists a constant* $h_0 > 0$ *such that, for* $h \in [0, h_0]$*, system* (2.2.1.1) *has a unique* ω*–periodic solution* $x(t,h)$ *depending continuously on* h *and such that* $x(t,0) = x_0(t, c_r^*)$*. This solution is determined by a simple iteration method convergent for* $h \in [0, h_0]$*.*

The results of the present subsection were reported at the Conference on Nonlinear Differential Equations, Kyiv, Ukraine, 1995, and the Conference on Functional Differential-Difference Equations and Applications, Antalya, Turkey, 1997. They first appeared in the latter's proceedings [31]. Since the respective site soon became unavailable, they were later included in [32].

2.2.2 Critical case of second order

In the present subsection we consider system (2.2.1.1) satisfying conditions **A2.2.1.1**–**A2.2.1.6** and (2.2.1.4) in the critical case of second order when

$$\det \mathcal{B}_0 = 0. \tag{2.2.2.1}$$

Now in equality (2.2.1.22) we shall also need the terms linear in h. To this end we shall need the following additional assumption.

A2.2.2.1. Conditions **A2.2.1.1** and **A2.2.1.4** still hold if the functions $g(\cdot, x, \bar{x})$, A, and f are replaced by $\frac{\partial g}{\partial t}(\cdot, x, \bar{x})$, $\dot{A}$, and $\dot{f}$, respectively.

Now we obtain more precise versions of some representations of the previous subsection. First we note that

$$\begin{aligned} &(x_i - \bar{x}_i)/h = \dot{x}_i - h\ddot{x}_i/2 + h\varepsilon(h) \\ = \quad &A_i x_i + f_i + g(t_i, x_i, \bar{x}_i)(x_i - \bar{x}_i) - (\dot{A}_i x_i + A_i \dot{x}_i + \dot{f}_i)h/2 + h\varepsilon(h). \end{aligned}$$

This equality implies that

$$(E-hg(t_i,x_i,\bar{x}_i))(x_i-\bar{x}_i)/h = A_ix_i+f_i-[\dot{A}_ix_i+\dot{f}_i+A_i(A_ix_i+f_i)]h/2+h\varepsilon(h).$$

If h is small enough, we have

$$\begin{aligned} & (x_i-\bar{x}_i)/h && (2.2.2.2)\\ = \; & (E-hg(t_i,x_i,\bar{x}_i))^{-1}\{A_ix_i+f_i-[\dot{A}_ix_i+\dot{f}_i+A_i(A_ix_i+f_i)]h/2+h\varepsilon(h)\} \\ = \; & A_ix_i+f_i+h\{g(t_i,x_i,x_i)(A_ix_i+f_i)-[\dot{A}_ix_i+\dot{f}_i+A_i(A_ix_i+f_i)]/2\}+h\varepsilon(h). \end{aligned}$$

Further on,

$$\begin{aligned} & I_i(x_i,\bar{x}_i)/h = J_i(x_i,\bar{x}_i)(x_i-\bar{x}_i)/h \\ = \; & \left[J_i(x_i,x_i)-\frac{\partial}{\partial\bar{x}}J_i(x_i,x_i)(x_i-\bar{x}_i)+h\varepsilon(h)\right](x_i-\bar{x}_i)/h \end{aligned}$$

and in view of (2.2.2.2) we obtain

$$I_i(x_i,\bar{x}_i)/h = J_i(x_i,x_i)(A_ix_i+f_i)+h\mathbf{J}_i(x_i)+h\varepsilon(h), \qquad (2.2.2.3)$$

where

$$\begin{aligned} \mathbf{J}_i(x_i) \; = \; & J_i(x_i,x_i)\Big\{g(t_i,x_i,x_i)(A_ix_i+f_i)-[\dot{A}_ix_i+\dot{f}_i+A_i(A_ix_i+f_i)]/2\Big\} \\ - \; & \frac{\partial}{\partial\bar{x}}J_i(x_i,x_i)(A_ix_i+f_i)\cdot(A_ix_i+f_i). \qquad (2.2.2.4) \end{aligned}$$

The equality (2.2.1.11) is replaced by

$$\begin{aligned} & h^{-1}\int_{t_i}^{t_i+h} X^{-1}(\tau)H(\tau,x(\tau,h),x(\tau-h,h))\,d\tau \qquad (2.2.2.5) \\ = \; & X_i^{-1}(g(t_i+0,(E+B_i)x_i+a_i,x_i)(B_ix_i+a_i)+h\mathbf{G}_i(x_i))+h\varepsilon(h), \end{aligned}$$

where

$$\begin{aligned} \mathbf{G}_i(x_i) \; = \; & \big\{\theta_i(\partial_t-A_i^+)g_i^+ + \partial_x g_i^+[J_i(x_i,x_i)(A_ix_i+f_i)+\theta_i(A_i^+(B_ix_i+a_i)+f_i^+)] \\ & -(1-\theta_i)\partial_{\bar{x}}g_i^+(A_ix_i+f_i)\big\} + g_i^+[J_i(x_i,x_i)(A_ix_i+f_i) \\ & +\theta_i(A_i^+(B_ix_i+a_i)+f_i^+)+(1-\theta_i)(A_ix_i+f_i)]. \qquad (2.2.2.6) \end{aligned}$$

Here $g_i^+ \equiv g(t_i+0,(E+B_i)x_i+a_i,x_i)$, $A_i^+ \equiv A(t_i+0)$, $f_i^+ \equiv f(t_i+0)$ (but $X_i \equiv X(t_i+0)$).

Further on,

$$h^{-1} \int_{\Delta_2^h} X^{-1}(\tau) H(\tau, x(\tau,h), x(\tau-h,h))\, d\tau \qquad (2.2.2.7)$$

$$= \int_0^\omega X^{-1}(\tau)[g(\tau, x(\tau,h), x(\tau,h))(A(\tau)x(\tau,h) + f(\tau)) + h\mathcal{H}(\tau, x(\tau,h))]\, d\tau$$

$$- \int_{\Delta_1^h} X^{-1}(\tau) g(\tau, x(\tau,h), x(\tau,h))(A(\tau)x(\tau,h) + f(\tau))\, d\tau + h\varepsilon(h),$$

where

$$\begin{aligned}\mathcal{H}(\tau, x) \;=\; & g(\tau,x,x)\{g(\tau,x,x)(A(\tau)x + f(\tau)) \qquad (2.2.2.8)\\ & -[\dot{A}(\tau)x + \dot{f}(\tau) + A(\tau)(A(\tau)x + f(\tau))]/2\}\\ & -\partial_{\bar{x}} g(\tau,x,x)(A(\tau)x + f(\tau)) \cdot (A(\tau)x + f(\tau)),\end{aligned}$$

$$\int_{\Delta_1^h} X^{-1}(\tau) g(\tau, x(\tau,h), x(\tau,h))(A(\tau)x(\tau,h) + f(\tau))\, d\tau = h \sum_{i=1}^m X_i^{-1} \mathbf{g}_i(x_i) + h\varepsilon(h)$$

and

$$\mathbf{g}_i(x_i) = g_i^+[A_i^+((E + B_i)x_i + a_i) + f_i^+].$$

In view of (2.2.2.3), (2.2.2.5) and (2.2.2.7), we can write the solvability condition (2.2.1.9) in the form

$$\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)[g(\tau, x(\tau,h), x(\tau,h))(A(\tau)x(\tau,h) + f(\tau)) \right.$$

$$+ h\mathcal{H}(\tau, x(\tau,h))]\, d\tau + \sum_{i=1}^m X_i^{-1}[J_i(x_i,x_i)(A_i x_i + f_i) + h\mathbf{J}_i(x_i)$$

$$\left. + g_i^+(B_i x_i + a_i) + h(\mathbf{G}_i(x_i) - \mathbf{g}_i(x_i))] + h\varepsilon(h,x) \right\} = 0. \quad (2.2.2.9)$$

In view of the assumption $F(c_r^*) = 0$, equality (2.2.1.16) now takes the form

$$\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)[g_1(\tau)y(\tau,h) + g_2(\tau, y(\tau,h)) + h\mathcal{H}(\tau, x(\tau,h))]\, d\tau \right.$$

$$\left. + \sum_{i=1}^m X_i^{-1}[J_{1i}y_i + J_{2i}(y_i) + h\mathcal{G}_i(x_i)] + h\varepsilon_1(h,x) \right\} = 0, \quad (2.2.2.10)$$

where $\mathcal{G}_i(x_i) \equiv \mathbf{J}_i(x_i) + \mathbf{G}_i(x_i) - \mathbf{g}_i(x_i)$, $i = \overline{1, m}$.

With $\mathcal{B}_0$ defined by (2.2.1.21), equality (2.2.2.10) can be written in the form

$$\begin{aligned}\mathcal{B}_0 c &= -\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(1)}(\tau, h) + g_2(\tau, y(\tau, h)) + h\mathcal{H}(\tau, x(\tau, h))]\, d\tau \right. \\ &\quad \left. + \sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i, h) + J_{2i}(y_i) + h\mathcal{G}_i(x_i)] + h\varepsilon_1(h, x) \right\}. \qquad (2.2.2.11)\end{aligned}$$

Thus we have reduced problem (2.2.1.1) to the equivalent operator system (2.2.1.7), (2.2.1.14), (2.2.2.11), (2.2.1.23), where $\det \mathcal{B}_0 = 0$. Denote by $\mathcal{B}_0^*$ the matrix transpose to $\mathcal{B}_0$, by $\mathcal{B}_0^+$ its Moore-Penrose pseudoinverse, and by $\mathcal{P}_{\mathcal{B}_0} \neq 0$ and $\mathcal{P}_{\mathcal{B}_0^*}$ the orthoprojectors of $\mathbb{R}^r$ onto $\mathrm{Ker}(\mathcal{B}_0)$ and $\mathrm{Ker}(\mathcal{B}_0^*)$, respectively.

Equation (2.2.2.11) is solvable with respect to $c \in \mathbb{R}^r$ if and only if its right-hand side belongs to the orthocomplement of $\mathrm{Ker}(\mathcal{B}_0^*)$, *i.e.*,

$$\begin{aligned}&\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(1)}(\tau, h) + g_2(\tau, y(\tau, h)) + h\mathcal{H}(\tau, x(\tau, h))]\, d\tau \right. \\ &\left. + \sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i, h) + J_{2i}(y_i) + h\mathcal{G}_i(x_i)] + h\varepsilon_1(h, x) \right\} = 0. \qquad (2.2.2.12)\end{aligned}$$

If this equality is satisfied, then from (2.2.2.11) we determine

$$\begin{aligned}c &= -\mathcal{B}_0^+\mathcal{P}_r^* X(\omega) \left\{ \int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(1)}(\tau, h) + g_2(\tau, y(\tau, h)) + h\mathcal{H}(\tau, x(\tau, h))]\, d\tau \right. \\ &\qquad\qquad (2.2.2.13) \\ &\quad \left. + \sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i, h) + J_{2i}(y_i) + h\mathcal{G}_i(x_i)] + h\varepsilon_1(h, x) \right\} + c^{(1)} \equiv c^{(0)} + c^{(1)},\end{aligned}$$

where $c^{(1)}$ is an arbitrary constant vector in $\mathrm{Ker}(\mathcal{B}_0)$, $c^{(1)} = \mathcal{P}_{\mathcal{B}_0}c$, $c^{(0)} = (\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c$. Then equality (2.2.1.23) can be rewritten in the form

$$y^{(1)}(t, h) = G_1(t)c^{(1)} + y^{(2)}(t, h), \qquad (2.2.2.14)$$

where

$$G_1(t) = \int_0^\omega G(t, \tau)g_1(\tau)X_r(\tau)\, d\tau + \sum_{i=1}^m G(t, t_i)J_{1i}X_r(t_i)$$

and

$$\begin{aligned} y^{(2)}(t,h) &= \int_0^\infty G(t,\tau)\left\{g(\tau,x_0(t,c_r^*),x_0(t,c_r^*))(A(\tau)x_0(t,c_r^*)+f(\tau))\right. \\ &\quad \left. + g_1(\tau)[X_r(\tau)c^{(0)}+hy^{(1)}(\tau,h)] + g_2(\tau,y(\tau,h))\right\} d\tau \quad (2.2.2.15) \\ &+ \sum_{i=1}^m G(t,t_i)\left\{J_i(x_0(t_i,c_r^*),x_0(t_i,c_r^*))(A_ix_0(t_i,c_r^*)+f_i)\right. \\ &\quad + g(t_i+0,(E+B_i)x_0(t_i,c_r^*)+a_i,x_0(t_i,c_r^*))(B_ix_0(t_i,c_r^*)+a_i) \\ &\quad \left. + J_{1i}[X_r(t_i)c^{(0)}+hy^{(1)}(t_i,h)] + J_{2i}(y_i)\right\} + \varepsilon_2(h,x). \end{aligned}$$

Now let us linearize the solvability condition (2.2.2.12) with respect to y. We use the expansion

$$\mathcal{H}(\tau,x_0(\tau,c_r^*)+y) = \mathcal{H}(\tau,x_0(\tau,c_r^*)) + \mathcal{H}_1(\tau)y + \mathcal{H}_2(\tau,y),$$

where

$$\mathcal{H}_1(\tau) = \partial_x\mathcal{H}(\tau,x)\big|_{x=x_0(\tau,c_r^*)}, \qquad \mathcal{H}_2(\tau,0)=0, \quad \partial_y\mathcal{H}_2(\tau,0)=0,$$

and, analogously,

$$\begin{aligned} &\mathcal{G}_i(x_0(t_i,c_r^*)+y_i) = \mathcal{G}_i(x_0(t_i,c_r^*)) + \mathcal{G}_{1i}y_i + \mathcal{G}_{2i}(y_i), \\ &\mathcal{G}_{1i} = \partial_x\mathcal{G}_i(x)\big|_{x=x_0(t_i,c_r^*)}, \qquad \mathcal{G}_{2i}(0)=0, \quad \partial_y\mathcal{G}_{2i}(0)=0. \end{aligned}$$

In view of equalities (2.2.2.4), (2.2.2.6), (2.2.2.8), these expansions require the existence of the continuous derivatives $\partial_{x\bar{x}}g(t,x,\bar{x})$, $\partial_{\bar{x}\bar{x}}g(t,x,\bar{x})$, $\partial_{x\bar{x}}J_i(x,\bar{x})$, $\partial_{\bar{x}\bar{x}}J_i(x,\bar{x})$ ensured by the following condition:

A2.2.2.2. Conditions **A2.2.1.1** and **A2.2.1.2** still hold if the functions g and J_i are replaced by the partial derivatives $\partial_{\bar{x}}g$ and $\partial_{\bar{x}}J_i$, respectively.

Let us denote

$$\begin{aligned} &\gamma(h) \equiv \int_0^\omega X^{-1}(\tau)\mathcal{H}(\tau,x_0(\tau,c_r^*))\,d\tau + \sum_{i=1}^m X_i^{-1}\mathcal{G}_i(x_0(t_i,c_r^*)) + \varepsilon_1(h,x_0), \\ &\tilde{\varepsilon}_1(h,y) \equiv \varepsilon_1(h,x_0+y) - \varepsilon_1(h,x_0). \end{aligned}$$

Thus, $\tilde{\varepsilon}_1(h,0) = \tilde{\varepsilon}_1(0,y) = 0$, while the quantity $\varepsilon_1(h,x_0)$ depends just on the generating solution $x_0(t,c_r^*)$, and so thus $\gamma(h)$.

We can now rewrite equality (2.2.2.12) in the form

$$\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(1)}(\tau,h)+g_2(\tau,y(\tau,h)) + h\mathcal{H}_1(\tau)y(\tau,h)+h\mathcal{H}_2(\tau,y(\tau,h))]\,d\tau + \sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(1)}(t_i,h)+J_{2i}(y_i)+h(\mathcal{G}_{1i}y_i+\mathcal{G}_{2i}(y_i))]+h\gamma(h)+h\tilde{\varepsilon}_1(h,y)\right\}=0. \tag{2.2.2.16}$$

We substitute (2.2.2.14) into (2.2.2.16) to obtain a system with respect to $c^{(1)} = \mathcal{P}_{\mathcal{B}_0}c \in \mathrm{Ker}(\mathcal{B}_0)$:

$$\begin{aligned} h\mathcal{B}_1 c^{(1)} \;=\; & -\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(2)}(\tau,h)+g_2(\tau,y(\tau,h)) \right.\\ & + h\mathcal{H}_1(\tau)(X_r(\tau)c^{(0)}+hy^{(1)}(\tau,h))+h\mathcal{H}_2(\tau,y(\tau,h))]\,d\tau \\ & + \sum_{i=1}^m X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h)+J_{2i}(y_i)+h(\mathcal{G}_{1i}(X_r(t_i)c^{(0)}+hy^{(1)}(t_i,h))+\mathcal{G}_{2i}(y_i))] \\ & \left. + h\gamma(h)+h\tilde{\varepsilon}_1(h,y)\right\}, \end{aligned} \tag{2.2.2.17}$$

where

$$\mathcal{B}_1 = \mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[g_1(\tau)G_1(\tau)+\mathcal{H}_1(\tau)X_r(\tau)]\,d\tau + \sum_{i=1}^m X_i^{-1}[J_{1i}G_1(t_i)+\mathcal{G}_{1i}X_r(t_i)]\right\}\mathcal{P}_{\mathcal{B}_0}$$

is an $(r\times r)$–matrix.

As above, denote by $\mathcal{B}_1^*$ the matrix transpose to $\mathcal{B}_1$, by $\mathcal{B}_1^+$ its Moore-Penrose pseudoinverse, and by $\mathcal{P}_{\mathcal{B}_1}$ and $\mathcal{P}_{\mathcal{B}_1^*}$ the orthoprojectors of $\mathbb{R}^r$ onto $\mathrm{Ker}(\mathcal{B}_1)$ and $\mathrm{Ker}(\mathcal{B}_1^*)$, respectively. System (2.2.2.17) is solvable with respect to $hc^{(1)} \in \mathrm{Ker}(\mathcal{B}_0)$ if and only if its right-hand side belongs to the orthocomplement of $\mathrm{Ker}(\mathcal{B}_1^*)$, *i.e.*,

$$\mathcal{P}_{\mathcal{B}_1^*}\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(2)}(\tau,h)+g_2(\tau,y(\tau,h)) + h\mathcal{H}_1(\tau)(X_r(\tau)c^{(0)}+hy^{(1)}(\tau,h))+h\mathcal{H}_2(\tau,y(\tau,h))]\,d\tau\right.$$

$$+\sum_{i=1}^{m} X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h) + J_{2i}(y_i) + h(\mathcal{G}_{1i}(X_r(t_i)c^{(0)} + hy^{(1)}(t_i,h)) + \mathcal{G}_{2i}(y_i))]$$

$$\left. + h\gamma(h) + h\tilde{\varepsilon}_1(h,y)\right\} = 0.$$

Since no additional constraints are imposed on the solution sought $y(t,h)$, the above condition is fulfilled if $\mathcal{P}_{\mathcal{B}_1^*}\mathcal{P}_{\mathcal{B}_0^*} = 0$, *i.e.*, $\mathrm{Ker}(\mathcal{B}_1^*) \cap \mathrm{Ker}(\mathcal{B}_0^*) = \{0\}$. It is easy to see that this condition is equivalent to $\mathcal{P}_{\mathcal{B}_0}\mathcal{P}_{\mathcal{B}_1} = 0$.

Thus, if

$$\mathcal{P}_{\mathcal{B}_0} \neq 0, \qquad \mathcal{P}_{\mathcal{B}_0}\mathcal{P}_{\mathcal{B}_1} = 0, \tag{2.2.2.18}$$

then system (2.2.2.17) is uniquely solvable with respect to $hc^{(1)} \in \mathrm{Ker}(\mathcal{B}_0)$, and the operator system (2.2.1.7), (2.2.1.14), (2.2.2.11), (2.2.1.23) is reduced to the operator system (2.2.1.7),

$$\begin{aligned}
y(t,h) &= X_r(t)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + hG_1(t)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + hy^{(2)}(t,h),\\
c^{(0)} &= -\mathcal{B}_0^+\mathcal{P}_r^*X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[hg_1(\tau)(G_1(\tau)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(\tau,h))\right. \qquad (2.2.2.19)\\
&\qquad + g_2(\tau,y(\tau,h)) + h\mathcal{H}(\tau,x(\tau,h))]\,d\tau\\
&+ \left.\sum_{i=1}^{m} X_i^{-1}[hJ_{1i}(G_1(t_i)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(t_i,h) + J_{2i}(y_i) + h\mathcal{G}_i(x_i)] + h\varepsilon_1(h,x)\right\},\\
hc^{(1)} &= -\mathcal{B}_1^+\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^*X(\omega)\left\{\int_0^\omega X^{-1}(\tau)[hg_1(\tau)y^{(2)}(\tau,h) + g_2(\tau,y(\tau,h))\right.\\
&+ h\mathcal{H}_1(\tau)(X_r(\tau)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + h(G_1(\tau)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(\tau,h))) + h\mathcal{H}_2(\tau,y(\tau,h))]\,d\tau\\
&\quad + \sum_{i=1}^{m} X_i^{-1}[hJ_{1i}y^{(2)}(t_i,h) + J_{2i}(y_i) + h(\mathcal{G}_{1i}(X_r(t_i)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)}\\
&\quad \left. + h(G_1(t_i)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(t_i,h))) + \mathcal{G}_{2i}(y_i))] + h\gamma(h) + h\tilde{\varepsilon}_1(h,y)\right\}, \qquad (2.2.2.20)\\
y^{(2)}(t,h) &= \int_0^\infty G(t,\tau)\left\{g(\tau,x_0(t,c_r^*),x_0(t,c_r^*))(A(\tau)x_0(t,c_r^*) + f(\tau))\right.\\
&\quad \left. + g_1(\tau)[X_r(\tau)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + h(G_1(\tau)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(\tau,h))] + g_2(\tau,y(\tau,h))\right\}\,d\tau\\
&\quad + \sum_{i=1}^{m} G(t,t_i)\left\{J_i(x_0(t_i,c_r^*),x_0(t_i,c_r^*))(A_ix_0(t_i,c_r^*) + f_i)\right.\\
&\quad + g(t_i+0,(E+B_i)x_0(t_i,c_r^*) + a_i, x_0(t_i,c_r^*))(B_ix_0(t_i,c_r^*) + a_i)\\
&+ \left. J_{1i}[X_r(t_i)(\mathrm{Id} - \mathcal{P}_{\mathcal{B}_0})c^{(0)} + h(G_1(t_i)\mathcal{P}_{\mathcal{B}_0}c^{(1)} + y^{(2)}(t_i,h))] + J_{2i}(y_i)\right\} + \varepsilon_2(h,x).
\end{aligned}$$

To this system, a convergent simple iteration method can be applied [63] starting with, say,

$$y_k(t,h) = y_k^{(2)}(t,h) = 0, \qquad k = 0,1, \tag{2.2.2.21}$$

and the following assertion is valid:

Theorem 2.2.2.1. *For system* (2.2.1.1)*, let conditions* **A2.2.1.1**–**A2.2.1.6**, **A2.2.2.1**, **A2.2.2.2** *and* (2.2.1.4) *hold. Let* $c_r = c_r^* \in \mathbb{R}^r$ *be a root of equation* (2.2.1.6) *such that* (2.2.2.18) *is satisfied as well as the condition*

$$\mathcal{P}_{\mathcal{B}_0^*}\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)\mathcal{H}(\tau, x_0(\tau, c_r^*))\,d\tau \right. \tag{2.2.2.22}$$
$$\left. + \sum_{i=1}^m X_i^{-1}\mathcal{G}_i(x_0(t_i, c_r^*)) + \varepsilon_1(h, x_0)\right\} = 0.$$

Then there exists a constant $h_0 > 0$ *such that, for* $h \in [0, h_0]$*, system* (2.2.1.1) *has a unique* ω*–periodic solution* $x(t,h)$ *such that* $x(t,0) = x_0(t, c_r^*)$*. This solution can be determined by a simple iteration method convergent for* $h \in [0, h_0]$.

Condition (2.2.2.22) is obtained from the solvability condition (2.2.2.16) by virtue of (2.2.2.21). It is a necessary and sufficient condition for finding

$$c_0^{(0)} = -h\mathcal{B}_0^+\mathcal{P}_r^* X(\omega)\left\{\int_0^\omega X^{-1}(\tau)\mathcal{H}(\tau, x_0(\tau, c_r^*))\,d\tau \right.$$
$$\left. + \sum_{i=1}^m X_i^{-1}\mathcal{G}_i(x_0(t_i, c_r^*)) + \varepsilon_1(h, x_0)\right\}$$

from (2.2.2.11). The finding of the subsequent iterations of $c^{(0)}$ from (2.2.2.19) is enabled by the choice of the corresponding iterations of $c^{(1)}$ from (2.2.2.20).

The results of the present subsection were published in [32].

2.2.3 Periodicity of the population size of an age-dependent model with a dominant age class by means of impulsive perturbations

The following model is described in the papers of T. Kostova [80], T. Kostova & F. Milner [81], where the existence of oscillatory solutions is proved.

For two fixed ages σ_1, σ_2 such that $0 \leq \sigma_1 < \sigma_2 < \infty$ the age distribution $u(a,t)$ of a population is considered, where a is the age and t the time, with dynamics described by the following integro-differential equation with age-boundary condition in integral form,

$$\begin{cases} \frac{\partial u}{\partial a} + \frac{\partial u}{\partial t} = -\delta(a,Q)u(a,t), & a,t > 0, \\ u(0,t) = \int_0^\infty \beta(a,Q)u(a,t)\,da, & t \geq 0, \\ u(a,0) = u_0(a), & a \geq 0, \end{cases} \tag{2.2.3.1}$$

where

$$Q = Q(t) = \int_{\sigma_1}^{\sigma_2} u(a,t)\,da$$

is the *dominant age cohort* size and $\delta(a,Q)$ and $\beta(a,Q)$ are, respectively, the age-specific death rate and birth modulus when the dominant age group is of size Q. It is assumed that δ, β and u_0 are nonnegative, and that u_0 is integrable (so that the initial population is finite). This model is a generalization of the classical one of Gurtin & MacCamy [67], which is obtained by setting $\sigma_1 = 0$ and $\sigma_2 = \infty$.

Further on in [80, 81] the special case

$$\beta(a,Q) = \begin{cases} \beta(Q), & a \in [\sigma_1, \sigma_2], \\ 0, & \text{otherwise}, \end{cases}$$

is considered. This means that the dominant age class is the only one capable of having offspring, *i.e.*, births are possible only in the age interval $[\sigma_1, \sigma_2]$ and the fertility rate depends just on the size of the dominant age group itself (and not on the age within the group). Moreover, $\beta(Q) \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ and the mortality rate $\delta > 0$ is assumed constant. Then for the total population size,

$$P(t) = \int_0^\infty u(a,t)\,da,$$

the equation

$$\dot{P} + \delta P = \beta(Q)Q \tag{2.2.3.2}$$

is derived, where

$$Q(t) = P(t-\sigma_1)e^{-\sigma_1\delta} - P(t-\sigma_2)e^{-\sigma_2\delta} \quad \text{for} \quad t > \sigma_2, \tag{2.2.3.3}$$

$$Q(t) = e^{-\delta t}\int_{\sigma_1 - t}^{\sigma_2 - t} u_0(a)\,da \qquad \text{for} \qquad t \le \sigma_1,$$

while for $\sigma_1 < t \le \sigma_2$ $Q(t)$ satisfies the integral equation (assumed uniquely solvable)

$$e^{\delta t}Q(t) = \int_0^{t-\sigma_1} e^{\delta a}\beta(Q(a))Q(a)\,da + \int_0^{\sigma_2 - t} u_0(a)\,da. \tag{2.2.3.4}$$

Thus, for $t > \sigma_2$, $P(t)$ satisfies a nonlinear scalar delay equation (2.2.3.2) with Q given by (2.2.3.3), while for $t \in [0, \sigma_2]$ $Q(t)$ and eventually $P(t)$ can be expressed in terms of the initial function $u_0(a)$ of the age-dependent model (2.2.3.1):

$$P(t) = e^{-\delta t}\left[\int_0^{\infty} u_0(a)\,da + \int_0^t e^{\delta a}\beta(Q(a))Q(a)\,da\right].$$

Thus we find the initial function $P_0(t),\ t \in [0, \sigma_2]$ of the above mentioned delay equation.

We fix a number $\omega > 0$ much larger than the age σ_2, and try to obtain an ω–periodic regime of the population size by means of impulsive perturbations for a suitably chosen initial function u_0. More precisely, suppose that at certain moments t_i such that $t_{i+2} = t_i + \omega$ for all $i \in \mathbb{Z}$, the population size $P(t)$ is abruptly changed while for the moment equation (2.2.3.2) with (2.2.3.3) is assumed to hold for all $t \in \mathbb{R}$, $t \ne t_i$. We normalize the quantities in equation (2.2.3.2) as follows:

$$s = t/\omega, \quad \Pi(s) = P(\omega s), \quad D = \omega\delta, \quad B(Q) = \omega\beta(Q).$$

Henceforth we write again t, δ and β instead of s, D and B, respectively, x instead of Π, and $h = \sigma_2/\omega$ will be the small parameter, while the still smaller quantity σ_1/ω will be assumed 0, for the sake of simplicity. We suppose that the time interval between two successive abrupt changes (impulse effects)

$t_{i+1} - t_i$ is large in comparison with the "age" h for all $i \in \mathbb{Z}$, and look for 1–periodic solutions of the problem

$$\begin{cases} \dot{x} = -\delta x + H(h, x, \bar{x}), & t \neq t_i, \\ \Delta x(t_i) = B_i x_i + a_i, & i \in \mathbb{Z}, \end{cases} \tag{2.2.3.5}$$

where $\bar{x}(t) \equiv x(t-h)$, $\Delta x(t_i) \equiv x(t_i + 0) - x(t_i - 0)$ is the magnitude of the impulse effect at the moment t_i, $x_i \equiv x(t_i) \equiv x(t_i - 0)$, $H(h, x, \bar{x}) = \beta(Q)Q$, $Q = Q(h, x, \bar{x}) = x - \bar{x}e^{-h\delta}$, $0 < t_1 < t_2 < 1$.

Remark 2.2.3.1. The assumption that $\sigma_1/\omega = 0$ is of a technical character. It leads to system (2.2.3.5) with just one small delay of the argument. Similar systems were studied in [23, 22, 31]. The case of several small delays of the argument is considered in the next subsection.

Remark 2.2.3.2. We see that the nonlinearity $H(h, x, \bar{x})$ is not precisely of the form studied in [31]. However, its particular form much simplifies our calculations.

Suppose that $x(t)$ is a 1–periodic solution to the problem (2.2.3.5). Thus we find an ω–periodic solution $P(t)$ of equation (2.2.3.2) with (2.2.3.3) satisfying the respective impulse conditions. However, if $P(t)$ is the population size of the age-dependent model (2.2.3.1), for $0 \leq t \leq \sigma_2$ it must satisfy equation (2.2.3.2) with Q in the right-hand side determined from (2.2.3.4), *i.e.*,

$$P(t)e^{\delta t} = \int_0^\infty u_0(a)\, da + \int_0^t e^{\delta a}\beta(Q(a))Q(a)\, da$$

or, by virtue of (2.2.3.4),

$$\int_{\sigma_2 - t}^\infty u_0(a)\, da + Q(t)e^{\delta t} = P(t)e^{\delta t}. \tag{2.2.3.6}$$

Since $Q(t)$ can be expressed from (2.2.3.4) in terms of $u_0(a)$ by means of an integral operator, the initial function $u_0(a)$ of (2.2.3.1) must satisfy an integral equation (2.2.3.6). This is a counterpart of the well known fact that if a delay differential equation has a periodic solution, then its initial function must satisfy an integral equation.

We can easily find that the fundamental solution $X(t)$ of the homogeneous system

$$\begin{cases} \dot{x} = -\delta x, & t \neq t_i, \\ \Delta x(t_i) = B_i x_i, & i \in \mathbb{Z}, \end{cases} \tag{2.2.3.7}$$

is 1–periodic if and only if

$$(1+B_1)(1+B_2) = e^{\delta}. \tag{2.2.3.8}$$

If (2.2.3.8) is violated, we have to do with the so called *noncritical case* considered in [22, 23]. Green's function $G(t,\tau)$ of the periodic problem for a nonhomogeneous system corresponding to (2.2.3.7) is given by the formula

$$G(t,\tau) \;=\; \begin{cases} X(t)(1-X(1))^{-1}X^{-1}(\tau), & 0 \le \tau \le t \le 1, \\ X(1+t)(1-X(1))^{-1}X^{-1}(\tau), & 0 \le t < \tau \le 1. \end{cases}$$

A 1–periodic solution to system (2.2.3.5) must satisfy

$$x(t) = \int_0^1 G(t,\tau)H(h, x(\tau), x(\tau-h))\,d\tau + \sum_{i=1}^{2} G(t, t_i+0)a_i.$$

For h small enough this equation has a unique solution provided by the Implicit Function Theorem [23] (or the Contraction Mapping Principle [22]). This yields the existence of a unique 1–periodic solution to equation (2.2.3.2).

Next, we consider the *critical case* when (2.2.3.8) holds. Then

$$X(t) \;=\; \begin{cases} e^{-\delta t}, & 0 \le t \le t_1, \\ (1+B_1)e^{-\delta t}, & t_1 < t \le t_2, \\ e^{\delta(1-t)}, & t_2 < t \le 1. \end{cases}$$

Since $X_1 \equiv X(t_1+0) = (1+B_1)e^{-\delta t_1}$, $X_2 \equiv X(t_2+0) = e^{\delta(1-t_2)}$, then the nonhomogeneous system

$$\begin{cases} \dot{x} = -\delta x, & t \ne t_i, \\ \Delta x(t_i) = B_i x_i + a_i, & i \in \mathbb{Z}, \end{cases}$$

has a 1–parametric family of 1–periodic solutions $x_0(t,c)$ if and only if the nonhomogeneities a_1 and a_2 satisfy

$$e^{\delta t_1}a_1 + e^{\delta(t_2-1)}(1+B_1)a_2 = 0, \tag{2.2.3.9}$$

and

$$x_0(t,c) \;=\; \begin{cases} ce^{-\delta t}, & 0 \le t \le t_1, \\ c(1+B_1)e^{-\delta t} + a_1 e^{\delta(t_1-t)}, & t_1 < t \le t_2, \\ ce^{\delta(1-t)}, & t_2 < t \le 1. \end{cases}$$

Let us find conditions for the existence of 1–periodic solutions $x(t,h)$ of (2.2.3.5) depending continuously on h and such that for some $c \in \mathbb{R}$ we have $x(t,0) = x_0(t,c)$. In (2.2.3.5) we change the variables according to the formula

$$x(t,h) = x_0(t,c) + y(t,h) \tag{2.2.3.10}$$

and are led to the problem of finding 1–periodic solutions $y = y(t,h)$ of the impulsive system of functional differential equations

$$\begin{cases} \dot{y} = -\delta y + H(h, x(t,h), x(t-h,h)), & t \neq t_i, \\ \Delta y(t_i) = B_i y_i, & i \in \mathbb{Z}, \end{cases} \tag{2.2.3.11}$$

such that $y(t,h) \to 0$ as $h \to 0$.

We can formally consider (2.2.3.11) as a linear nonhomogeneous system. Then it has a 1–periodic solution y if and only if

$$\int_0^1 X^{-1}(\tau) H(h, x(\tau,h), x(\tau-h,h))\, d\tau = 0. \tag{2.2.3.12}$$

We divide the left-hand side of equality (2.2.3.12) by h and then study its behaviour as $h \to 0$. We can represent the integral in (2.2.3.12) by a sum of integrals over intervals containing no points of discontinuity of the integrand. It is obvious that for $\tau \in (t_i, t_i+h)$, $i = 1, 2$, the interval $(\tau-h, \tau)$ contains the point of discontinuity t_i while for τ inside the remaining intervals the interval $(\tau-h, \tau)$ contains no such points. We denote $\Delta_1^h = (t_1, t_1+h) \cup (t_2, t_2+h)$, $\Delta_2^h = [0,1] \setminus \Delta_1^h$ and make use of the representation $\int_0^1 = \int_{\Delta_1^h} + \int_{\Delta_2^h}$.

As in [31], denote by $\varepsilon(h,x)$ expressions tending to 0 as $h \to 0$, and satisfying a Lipschitz condition with respect to x with a constant tending to 0 as $h \to 0$. Making use of the particular form of the nonlinearity $H(h,x,\bar{x})$ and of the linear part of the equation, for τ in the “good” set Δ_2^h we find

$$Q(h, x(\tau,h), x(\tau-h,h))/h \;=\; \varepsilon(h,x).$$

Thus,

$$h^{-1} \int_{\Delta_2^h} X^{-1}(\tau) H(h, x(\tau,h), x(\tau-h,h))\, d\tau \;=\; \varepsilon(h,x).$$

On the other hand, for $i = 1, 2$ we have

$$h^{-1} \int_{t_i}^{t_i+h} X^{-1}(\tau) H(h, x(\tau,h), x(\tau-h,h))\, d\tau \;=\; X_i^{-1} \varphi_i(c,h) \beta(\varphi_i(c,h)) + \varepsilon(h,x),$$

where $\varphi_i(c,h)$ are linear with respect to c functions,

$$\varphi_i(c,h) = B_i x(t_i,h) + a_i.$$

Thus equality (2.2.3.12) takes on the form

$$\sum_{i=1}^{2} X_i^{-1} \varphi_i(c,h)\beta(\varphi_i(c,h)) + \varepsilon(h,x) = 0. \tag{2.2.3.13}$$

We have

$$\varphi_i(c,h) = \varphi_i(c,0) + B_i y_i, \qquad i = 1,2.$$

For $\varphi_i(c) \equiv \varphi_i(c,0)$, $i = 1,2$, we find

$$\varphi_1(c) = B_1 e^{-\delta t_1} c + a_1, \quad \varphi_2(c) = B_2 e^{-\delta t_2}(1 + B_1)c + B_2 e^{\delta(t_1 - t_2)} a_1 + a_2.$$

Passing to the limit in equation (2.2.3.13) as $h \to 0$, we derive the equation for the generating amplitudes

$$e^{\delta t_1} \varphi_1(c)\beta(\varphi_1(c)) + e^{\delta(t_2 - 1)}(1 + B_1)\varphi_2(c)\beta(\varphi_2(c)) = 0. \tag{2.2.3.14}$$

If in the solvability condition (2.2.3.9) a_1 and a_2 are not 0, then (2.2.3.14) can be also written as

$$\varphi_1(c)\beta(\varphi_1(c))/(\varphi_2(c)\beta(\varphi_2(c))) = a_1/a_2. \tag{2.2.3.15}$$

On the other hand, if $a_1 = a_2 = 0$, neglecting the solution $c = 0$ of equation (2.2.3.14) which yields the trivial solution $x = 0$ of problem (2.2.3.5), by virtue of relation (2.2.3.8) we derive the equation

$$B_1(1 + B_2)\beta(B_1 e^{-\delta t_1} c) + (1 + B_1)B_2 \beta(B_2 e^{-\delta t_2}(1 + B_1)c) = 0. \tag{2.2.3.16}$$

The precise form of equation (2.2.3.15) or (2.2.3.16) depends on the form of the function β.

Now suppose that c^* is a real root of equation (2.2.3.14). Then the 1–periodic solution $y(t,h)$ of system (2.2.3.11) such that $y(t,0) \equiv 0$ can be represented in the form

$$y(t,h) = X(t)c + hy^{(1)}(t,h), \tag{2.2.3.17}$$

where the unknown real constant $c = c(h)$ must satisfy an equation derived below from (2.2.3.13), while the unknown 1–periodic function $y^{(1)}(t, h)$ can be represented as

$$y^{(1)}(t,h) = h^{-1}\int_0^1 G(t,\tau)H(h, x_0(\tau, c^*)+y(\tau,h), x_0(\tau-h, c^*)+y(\tau-h,h))\,d\tau \tag{2.2.3.18}$$

in terms of the *generalized Green's function*

$$G(t,\tau) = \begin{cases} e^{\delta(\tau-t)}g(t,\tau), & 0 \le \tau \le t \le 1, \\ 0, & 0 \le t < \tau \le 1, \end{cases}$$

and $g(t,\tau)$ is a piecewise constant function:

$$g(t,\tau) = \begin{cases} 1, & [\tau,t] \subset [0,t_1] \cup (t_1,t_2] \cup (t_2,1], \\ 1+B_1, & 0 \le \tau \le t_1 < t \le t_2, \\ e^{\delta}, & 0 \le \tau \le t_1 < t_2 < t \le 1, \\ 1+B_2, & t_1 \le \tau \le t_2 < t \le 1. \end{cases}$$

By arguments similar to those above we find

$$y^{(1)}(t,h) = \sum_{i=1}^{2} G(t, t_i+0)\varphi_i(c^*,h)\beta(\varphi_i(c^*,h)) + \varepsilon(h, x(t,h)).$$

We linearize with respect to y making use of the expansions

$$\begin{aligned} \varphi_i(c^*,h)\beta(\varphi_i(c^*,h)) &\equiv (\varphi_i(c^*) + B_i y_i)\beta(\varphi_i(c^*) + B_i y_i) \\ &= \varphi_i(c^*)\beta(\varphi_i(c^*)) + \beta_{1i}B_i y_i + \beta_{2i}(y_i), \end{aligned}$$

where

$$\beta_{1i} = \beta(\varphi_i(c^*)) + \beta'(\varphi_i(c^*))\varphi_i(c^*),$$

β' is the derivative of β, while $\beta_{2i}(y)$ is such that

$$\beta_{2i}(0) = 0, \quad \frac{\partial}{\partial y}\beta_{2i}(0) = 0.$$

Now, since c^* satisfies (2.2.3.14), equality (2.2.3.13) becomes

$$\sum_{i=1}^{2} X_i^{-1}\{\beta_{1i}B_i y(t_i,h) + \beta_{2i}(y(t_i,h))\} + \varepsilon(h,x) = 0.$$

In view of the representation (2.2.3.17) let us denote

$$\mathcal{B}_0 \;=\; \sum_{i=1}^{2} \beta_{1i} \frac{B_i}{1+B_i}.$$

Then we have

$$\mathcal{B}_0 c = -\sum_{i=1}^{2} X_i^{-1}\{h\beta_{1i}B_i y^{(1)}(t_i,h) + \beta_{2i}(y(t_i,h))\} \;+\; \varepsilon(h,x) \qquad (2.2.3.19)$$

and

$$\begin{aligned} y^{(1)}(t,h) \;&=\; \sum_{i=1}^{2} G(t,t_i+0)\{\varphi_i(c^*)\beta(\varphi_i(c^*)) & (2.2.3.20)\\ &+\; \beta_{1i}B_i[X(t_i)c + hy^{(1)}(t_i,h)] + \beta_{2i}(y(t_i,h))\} \;+\; \varepsilon(h,x(t,h)). \end{aligned}$$

Thus problem (2.2.3.5) is reduced to the equivalent operator system $(2.2.3.10)|_{c=c^*-\text{a root of (2.2.3.14)}}$, (2.2.3.17), (2.2.3.19), (2.2.3.20).

It is easy to see that $\mathcal{B}_0 \neq 0$ is equivalent to the simplicity of the root c^* of equation (2.2.3.14). This is the so called critical case of first order. Then equation (2.2.3.19) can be solved with respect to c and we obtain a Fredholm operator system of the second type to which a convergent simple iteration method can be applied [63]. Thus to any simple real root of equation (2.2.3.14) for h small enough (*i.e.*, for σ_2 small enough with respect to ω) there corresponds a 1–periodic solution to problem (2.2.3.5) tending to $x_0(t,c^*)$ as $h \to 0$, *i.e.*, an ω–periodic solution to equation (2.2.3.2).

We could also apply the same method to the 2− and 3−dimensional systems obtained in [109] and [117].

The results of the present subsection were reported at the Fifth International Conference on Mathematical Population Dynamics, Zakopane, Poland, 1998, and published in [45].

2.2.4 Boundary value problems for impulsive systems with small concentrated delays

In the present subsection we study a boundary value problem (BVP) for an impulsive differential system with retarded argument

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t) + H(t, x(t), y(t)), \qquad t \in [a,b], \quad t \neq t_i, \\ \Delta x(t_i) &= B_i x(t_i) + a_i + I_i(x(t_i), y(t_i)), \quad i = \overline{1,p}, \\ x(t) &= \varphi(t) \quad \text{for} \quad t \in [a-\varepsilon_0, a], \\ \ell x &= \alpha + \varepsilon J(x, \varepsilon), \end{aligned} \tag{2.2.4.1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}^n$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $y = (y^1, y^2, \dots, y^k) \in \Omega^k$, $H(t,x,y) = \sum_{j=1}^{k} H_j(t,x,y)(x - y^j)$, $H_j : \mathbb{R} \times \Omega \times \Omega^k \to \mathbb{R}^{n\times n}$, Ω is a domain in $\mathbb{R}^n$; $y^j(t) = x(t - \varepsilon\omega^j(t))$, $j = \overline{1,k}$, $\omega^j : [a,b] \to [0,1]$; $\Delta x(t_i)$ are the impulses at moments t_i, and $a \equiv t_0 < t_1 < t_2 < \cdots < t_p < b$; $a_i \in \mathbb{R}^n$, $B_i \in \mathbb{R}^{n\times n}$, $I_i(x,y) = \sum_{j=1}^{k} I_{ij}(x,y)(x - y^j)$, $I_{ij} : \Omega \times \Omega^k \to \mathbb{R}^{n\times n}$ $(i = \overline{1,p},\ j = \overline{1,k})$, $\varepsilon \in [0, \varepsilon_0)$ is a small parameter; $\alpha = [\alpha_1, \dots, \alpha_m]^T \in \mathbb{R}^m$; $\ell = [\ell_1, \dots, \ell_m]^T$ and $J(x,\varepsilon)$ are, respectively, a linear and nonlinear with respect to x $m-$dimensional functionals; the initial function $\varphi \in C[a-\varepsilon_0, a]$, ε_0 will be specified later.

Remark 2.2.4.1. We could replace H, I_i, εJ by nonlinearities of a more general form, vanishing for $\varepsilon = 0$.

Remark 2.2.4.2. In the Russian-language literature BVP's with $m \neq n$ ($m = n$) are usually called BVP's of *Noether* (respectively *Fredholm*) type. Here we use the second term but not the first one. Note that in the non-Russian literature *Fredholm* BVP's are such that m does *not necessarily* coincide with m; for $m = n$ we have *Fredholm BVP's of zero index.*

In contrast with the convention of §1.1, we assume $x(a) \equiv x(a+0)$. In general, $x(a) \neq x(a-0) = \varphi(a)$. The nonlinearity $H(t,x,y)$ is discontinuous at the points t that are solutions to the equations

$$t - \varepsilon\omega^j(t) = t_i, \qquad i = \overline{0,p}, \quad j = \overline{1,k}. \tag{2.2.4.2}$$

We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_i or a.

For the sake of brevity, if not stated otherwise, we shall use the notation $x_i = x(t_i)$, $y_i = y(t_i)$ (*i.e.*, $y_i^j = x(t_i - \varepsilon^j(t_i))$).

In the sequel we require the fulfillment of the following assumptions:

A2.2.4.1. The components of $A(t)$, $f(t)$ belong to the space $C([a,b] \setminus \{t_i\})$ of functions which are continuous or piecewise continuous, with discontinuities of the first kind at the points t_i.

A2.2.4.2. The functions $H_j(t,x,y)$ are continuously differentiable with respect to x, y, and their components belong to $C([a,b]\setminus\{t_i\})$ with respect to t.

A2.2.4.3. The functions $I_{ij}(x,y) \in C^1(\Omega \times \Omega^k, \mathbb{R}^{n\times n})$, $i = \overline{1,p}$, $j = \overline{1,k}$.

A2.2.4.4. The functions ω^j are Lipschitz continuous:

$$|\omega^j(t') - \omega^j(t'')| \le K|t' - t''|, \qquad j = \overline{1,k}, \quad t', t'' \in [a,b],$$

and satisfy

$$0 \le \omega^1(t) \le \omega^2(t) \le \cdots \le \omega^k(t) \le 1, \qquad t \in [a,b].$$

A2.2.4.5. The matrices $E + B_i$, $i = \overline{1,p}$, are nonsingular.

A2.2.4.6. The functional ℓ is bounded on the space $C([a,b] \setminus \{t_i\})$.

A2.2.4.7. The functional $J(x,\varepsilon)$ is Fréchet continuously differentiable with respect to x and is continuous with respect to ε.

We assume that

$$\varepsilon_0 = \min\{t_{i+1} - t_i,\ i = \overline{0,p-1},\ b - t_p,\ 1/K\}.$$

Then for $\varepsilon \in (0, \varepsilon_0)$ each equation (2.2.4.2) has a unique solution $t_i^j = t_i^j(\varepsilon)$. It obviously satisfies $t_i \le t_i^j \le t_i + \varepsilon$.

Together with (2.2.4.1) we consider the so called *generating system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t), \qquad t \in [a,b], \quad t \ne t_i, \\ \Delta x(t_i) &= B_i x_i + a_i, \qquad i = \overline{1,p}, \end{aligned} \qquad (2.2.4.3)$$

$$\ell x = \alpha \qquad (2.2.4.4)$$

obtained from (2.2.4.1) for $\varepsilon = 0$.

The general solution of (2.2.4.3) is given by

$$x(t,c) = X(t)\left(c + \int_a^t X^{-1}(\tau)f(\tau)\,d\tau + \sum_{a<t_i<t} X_i^{-1}a_i\right), \qquad (2.2.4.5)$$

$c = x(0,c) \in \mathbb{R}^n$, where $X(t)$ is the fundamental solution (*i.e.*, $X(a) = E$) of the homogeneous system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \qquad t \in [a,b], \quad t \neq t_i, \\ \Delta x(t_i) &= B_i x_i, \qquad i = \overline{1,p}, \end{aligned} \qquad (2.2.4.6)$$

and $X_i \equiv X(t_i + 0) = (E + B_i)X(t_i)$ again for the sake of brevity.

If we introduce Green's function $K(t,\tau)$ for the Cauchy problem related to (2.2.4.3):

$$K(t,\tau) = \begin{cases} X(t)X^{-1}(\tau), & a \le \tau \le t \le b, \\ 0, & a \le t < \tau \le b, \end{cases}$$

then (2.2.4.5) takes the form

$$x(t,c) = X(t)c + \int_a^b K(t,\tau)f(\tau)\,d\tau + \sum_{i=1}^p K(t,t_i+0)a_i. \qquad (2.2.4.7)$$

A solution of the form (2.2.4.7) satisfies the boundary condition (2.2.4.4) if and only if the initial condition c satisfies

$$\alpha = \ell X(\cdot)c + \ell\int_a^b K(\cdot,\tau)f(\tau)\,d\tau + \sum_{i=1}^p \ell K(\cdot,t_i+0)a_i. \qquad (2.2.4.8)$$

Denote $Q = \ell X(\cdot)$, which is an $(m \times n)-$matrix, let Q^* be its transpose, Q^+ its unique Moore-Penrose pseudoinverse $(n \times m)-$matrix [99, 100]. Denote by $\mathcal{P} \equiv \mathcal{P}_Q$ the orthoprojector $\mathbb{R}^n \to \operatorname{Ker}(Q)$ and by $\mathcal{P}^* \equiv \mathcal{P}_{Q^*}$ the orthoprojector $\mathbb{R}^m \to \operatorname{Ker}(Q^*)$.

For $n_1 = \operatorname{Rank} Q \le \min(m,n)$, let $r = n - n_1$, $d = m - n_1$. Denote by $\mathcal{P}_r$ an $(n \times r)-$matrix whose columns are r linearly independent columns of $\mathcal{P}$ and, similarly, let $\mathcal{P}_d^*$ be a $(d \times m)-$matrix whose rows are d linearly independent rows of $\mathcal{P}^*$.

Then the necessary and sufficient condition for solvability of the algebraic system (2.2.4.8) is

$$\mathcal{P}_d^* \left(\alpha - \ell \int_a^b K(\cdot,\tau) f(\tau)\, d\tau - \sum_{i=1}^{p} \ell K(\cdot, t_i + 0) a_i \right) = 0. \qquad (2.2.4.9)$$

If (2.2.4.9) is satisfied, system (2.2.4.8) has an $r-$parametric family of solutions

$$c = Q^* \left(\alpha - \ell \int_a^b K(\cdot,\tau) f(\tau)\, d\tau - \sum_{i=1}^{p} \ell K(\cdot, t_i + 0) a_i \right) + \mathcal{P}\tilde{c}, \quad \tilde{c} \in \mathbb{R}^n. \qquad (2.2.4.10)$$

If we substitute (2.2.4.10) into (2.2.4.7), we find an $r-$parametric family of solutions of BVP (2.2.4.3), (2.2.4.4) which can be represented in the form

$$x_0(t, c_r) = X_r(t) c_r + X(t) Q^+ \alpha + (\Gamma f)(t) + \sum_{i=1}^{p} \gamma_i(t) a_i, \qquad (2.2.4.11)$$

where $X_r(t) = X(t)\mathcal{P}_r$ is an $(n \times r)-$matrix whose columns make a complete system of r linearly independent solutions of (2.2.4.6) satisfying $\ell x = 0$, $c_r \in \mathbb{R}^r$ is an arbitrary vector;

$$\begin{aligned} (\Gamma f)(t) &= \int_a^b K(t,\tau) f(\tau)\, d\tau - X(t) Q^+ \ell \int_a^b K(\cdot,\tau) f(\tau)\, d\tau, \\ \gamma_i(t) &= K(t, t_i + 0) - X(t) Q^+ \ell K(\cdot, t_i + 0), \qquad i = \overline{1,p}. \end{aligned}$$

Then the following assertion is valid.

Lemma 2.2.4.1. ([37, Theorem 1]) *Let conditions* **A2.2.4.1, A2.2.4.5, A2.2.4.6** *hold and* $\operatorname{Rank} Q = n_1$. *Then system* (2.2.4.6) *has just* $r = n - n_1$ *linearly independent solutions satisfying* $\ell x = 0$. *The nonhomogeneous BVP* (2.2.4.3), (2.2.4.4) *has solutions if and only if the nonhomogeneities* $f(t)$, a_i, *and* α *satisfy condition* (2.2.4.9). *Then BVP* (2.2.4.3), (2.2.4.4) *has an* $r-$*parametric family of solutions of the form* (2.2.4.11).

Remark 2.2.4.3. Suppose that the linear functional ℓ satisfies the equality

$$\ell \int_a^b K(\cdot,\tau) f(\tau)\, d\tau = \int_a^b \ell K(\cdot,\tau) f(\tau)\, d\tau$$

for any $f \in C([a,b] \setminus \{t_i\})$. Then equality (2.2.4.11) can be written in the form

$$x_0(t, c_r) = X_r(t)c_r + X(t)Q^+\alpha + \int_a^b G(t,\tau)f(\tau)\,d\tau + \sum_{i=1}^{p} G(t, t_i + 0)a_i,$$

where $G(t,\tau)$ is the generalized Green's matrix

$$G(t,\tau) \equiv K(t,\tau) - X(t)Q^+\ell K(\cdot,\tau).$$

Further on we consider the so called *critical case* when $r > 0$. Let us find conditions for the existence of a solution $x(t,\varepsilon)$ of BVP (2.2.4.1) belonging to the space $C([a,b] \setminus \{t_i\})$ as a function of t, depending continuously on ε and such that, for some $c_r \in \mathbb{R}^r$, we have $x(t,0) = x_0(t, c_r)$. A necessary condition for the existence of such solutions is given by the following theorem.

Theorem 2.2.4.1. *Suppose that BVP* (2.2.4.1) *satisfies conditions* **A2.2.4.1** – **A2.2.4.7** *and* (2.2.4.9) *and it has a solution $x(t,\varepsilon)$ which for $\varepsilon = 0$ becomes a generating solution $x_0(t, c_r^*)$. Then the vector $c_r^* \in \mathbb{R}^r$ satisfies the equation*

$$\begin{aligned} F(c_r^*) \equiv \mathcal{P}_d^* \Big\{ & J(x_0(\cdot, c_r^*), 0) \\ & - \ell \int_a^b K(\cdot,\tau)\mathcal{H}(\tau, x_0(\tau, c_r^*))\left(A(\tau)x_0(\tau, c_r^*) + f(\tau)\right) d\tau \\ & - \ell K(\cdot, a)\mathbf{H}_0(x_0(a, c_r^*))\left(x_0(a, c_r^*) - \varphi(a)\right) \\ & - \sum_{i=1}^{p} \ell K(\cdot, t_i + 0)\left[\mathbf{J}_i(x_0(t_i, c_r^*))\left(A_i x_0(t_i, c_r^*) + f_i\right)\right. \\ & \left. + \mathbf{H}_i(x_0(t_i, c_r^*))\left(B_i x_0(t_i, c_r^*) + a_i\right)\right] \Big\} = 0, \end{aligned} \tag{2.2.4.12}$$

where $\mathcal{H}(\tau, x)$, $\mathbf{J}_i(x)$, $i = \overline{1,p}$, $\mathbf{H}_0(x)$ and $\mathbf{H}_i(x)$, $i = \overline{0,p}$, will be given below by formulae (2.2.4.18), (2.2.4.16), (2.2.4.26) *and* (2.2.4.23), *respectively.*

Proof. In (2.2.4.1) we change the variables according to the formula

$$x(t,\varepsilon) = x_0(t, c_r^*) + z(t,\varepsilon). \tag{2.2.4.13}$$

This leads to the problem of finding a solution $z = z(t, \varepsilon)$ for the impulsive system of differential equations with retarded argument

$$\begin{aligned} \dot{z} &= A(t)z + H(t, x(t,\varepsilon), y(t,\varepsilon)), \\ \Delta z(t_i) &= B_i z_i + I_i(x_i, y_i), \quad i = \overline{1,p}, \\ \ell z &= \varepsilon J(x, \varepsilon), \end{aligned} \tag{2.2.4.14}$$

with an initial condition $z(t) = \varphi(t) - x_0(t, c_r^*)$ for $t < a$, such that it would belong to the space $C([a,b] \setminus \{t_i\})$ as a function of t, depend continuously on ε, and $z(t,\varepsilon) \to 0$ as $\varepsilon \to 0$.

We can formally consider (2.2.4.14) as a nonhomogeneous system of the form (2.2.4.3), (2.2.4.4). Then the solvability condition (2.2.4.9) becomes

$$\begin{aligned} \mathcal{P}_d^* \Bigg\{ \varepsilon J(x(\cdot), \varepsilon) - \ell \int_a^b K(\cdot, \tau) H(\tau, x(\tau), y(\tau))\, d\tau \qquad (2.2.4.15) \\ - \sum_{i=1}^{p} \ell K(\cdot, t_i + 0) I_i(x_i, y_i) \Bigg\} = 0. \end{aligned}$$

For the sake of later convenience we shall denote by $\eta(\varepsilon, x)$ expressions tending to 0 as $\varepsilon \to 0$, and satisfying a Lipschitz condition with respect to x with a constant tending to 0 as $\varepsilon \to 0$. We shall sometimes write $\eta(\varepsilon)$ instead of $\eta(\varepsilon, x)$, and $x(t)$ instead of $x(t,\varepsilon)$, if this would not lead to misunderstanding. Thus, for instance, we sometimes write x_i instead of $x(t_i, \varepsilon)$, etc.

Since the left-hand side of equality (2.2.4.15) tends to 0 as $\varepsilon \to 0$, we first divide it by ε and then study its behaviour as $\varepsilon \to 0$. First we notice that

$$\begin{aligned} & (x_i - y_i^j)/\varepsilon = \left(x(t_i) - x(t_i - \varepsilon\omega^j(t_i))\right)/\varepsilon \\ = \; & \omega^j(t_i)\dot{x}(t_i) + \eta(\varepsilon) = \omega^j(t_i)\left(A_i x_i + f_i\right) + \eta(\varepsilon) \end{aligned}$$

since the interval $(t_i - \varepsilon\omega^j(t_i), t_i)$ contains no points of discontinuity of the function $x(t,\varepsilon)$ or its derivative. Thus,

$$I_i(x_i, y_i)/\varepsilon = \sum_{j=1}^{k} I_{ij}(x_i, y_i)(x_i - y_i^j)/\varepsilon = \sum_{j=1}^{k} I_{ij}(\underbrace{x_i, x_i, \ldots, x_i}_{k+1})\omega^j(t_i)(A_i x_i + f_i) + \eta(\varepsilon).$$

Let us denote

$$\mathbf{J}_i(x) = \sum_{j=1}^{k} I_{ij}(\underbrace{x, x, \ldots, x}_{k+1})\omega^j(t_i). \tag{2.2.4.16}$$

Then,

$$I_i(x_i, y_i)/\varepsilon = \mathbf{J}_i(x_i)(A_i x_i + f_i) + \eta(\varepsilon). \qquad (2.2.4.17)$$

We can represent the integral $\int_a^b$ in (2.2.4.15) as a sum of integrals over intervals containing no points of discontinuity of the integrand. It is obvious that for $\tau \in (t_i, t_i + \varepsilon)$ (more precisely, for $\tau = t_i^j(\varepsilon)$, the interval $(\tau - \varepsilon\omega^j(\tau), \tau)$ contains the point of discontinuity t_i, $i = \overline{0,p}$, while for τ inside the remaining intervals, the interval $(\tau - \varepsilon, \tau)$ contains no such points. We denote $\Delta_1^\varepsilon = \bigcup_{i=0}^{p}(t_i, t_i+\varepsilon)$, $\Delta_2^\varepsilon = [a,b] \setminus \Delta_1^\varepsilon$ and make use of the representation $\int_a^b = \int_{\Delta_1^\varepsilon} + \int_{\Delta_2^\varepsilon}$.

We first begin with the "good" set Δ_2^ε. We have

$$\begin{aligned} & \int_{\Delta_2^\varepsilon} K(t,\tau) H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon \\ = & \int_{\Delta_2^\varepsilon} K(t,\tau) \sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau)) \left(x(\tau) - x(\tau - \varepsilon\omega^j(\tau))\right)\, d\tau/\varepsilon \\ = & \int_{\Delta_2^\varepsilon} K(t,\tau) \sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau)) \omega^j(\tau) \dot{x}(\tau)\, d\tau + \eta(\varepsilon). \end{aligned}$$

Denote

$$\mathcal{H}(\tau, x) = \sum_{j=1}^{k} H_j(\tau, \underbrace{x, x, \ldots, x}_{k+1}) \omega^j(\tau). \qquad (2.2.4.18)$$

Thus

$$\sum_{j=1}^{k} H_j(\tau, x(\tau), y(\tau)) \omega^j(\tau) = \mathcal{H}(\tau, x(\tau)) + \eta(\varepsilon)$$

and

$$\begin{aligned} & \int_{\Delta_2^\varepsilon} K(t,\tau) H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon \qquad (2.2.4.19) \\ = & \int_{\Delta_2^\varepsilon} K(t,\tau) \mathcal{H}(\tau, x(\tau)) \left(A(\tau)x(\tau) + f(\tau)\right)\, d\tau + \eta(\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b K(t,\tau)\mathcal{H}(\tau,x(\tau))\left(A(\tau)x(\tau)+f(\tau)\right)\,d\tau \\
&- \int_{\Delta_1^\varepsilon} K(t,\tau)\mathcal{H}(\tau,x(\tau))\left(A(\tau)x(\tau)+f(\tau)\right)\,d\tau+\eta(\varepsilon) \\
&= \int_a^b K(t,\tau)\mathcal{H}(\tau,x(\tau))\left(A(\tau)x(\tau)+f(\tau)\right)\,d\tau+\eta(\varepsilon).
\end{aligned}
$$

Next we estimate any of the integrals

$$\int_{t_i}^{t_i+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon,$$

since $\int_{\Delta_1^\varepsilon}=\sum\limits_{i=0}^{p}\int_{t_i}^{t_i+\varepsilon}$. First let $i\in\{1,\dots,p\}$ and suppose that

$$t_i\le t_i^1<t_i^2<\cdots<t_i^k\le t_i+\varepsilon. \tag{2.2.4.20}$$

By definition, $t_i^j=t_i+\varepsilon\omega^j(t_i^j)$. By the Lipschitz continuity of $\omega^j(t)$ we have

$$|\omega^j(t_i^j)-\omega^j(t_i)|\le K|t_i^j-t_i|=\varepsilon K\omega^j(t_i^j)\le\varepsilon K,$$

thus

$$t_i^j=t_i+\varepsilon\omega^j(t_i)+\varepsilon\eta(\varepsilon). \tag{2.2.4.21}$$

It is easily seen that for $\varepsilon>0$ small enough, the strict inequalities

$$\omega^1(t_i)<\omega^2(t_i)<\cdots<\omega^k(t_i) \tag{2.2.4.22}$$

imply (2.2.4.20).

Now we have

$$\int_{t_i}^{t_i+\varepsilon}=\int_{t_i}^{t_i^1}+\sum_{j=1}^{k-1}\int_{t_i^j}^{t_i^{j+1}}+\int_{t_i^k}^{t_i+\varepsilon}$$

and consider successively the integrals in the right-hand side. First suppose that $t_i<t_i^1$. Then

$$\int_{t_i}^{t_i^1} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau=(t_i^1-t_i)K(t,t_i^{1*})H(t_i^{1*},x(t_i^{1*}),y(t_i^{1*})),$$

where $t_i<t_i^{1*}<t_i^1$. In view of (2.2.4.21) for $j=1$ we have

$$(t_i^1-t_i)/\varepsilon=\omega^1(t_i)+\eta(\varepsilon).$$

Next,

$$\begin{aligned} K(t,t_i^{1*}) &= K(t,t_i+0)+\eta(\varepsilon),\\ x(t_i^{1*}) &= x(t_i+0)+\eta(\varepsilon), \quad y^j(t_i^{1*}) = x_i+\eta(\varepsilon), \qquad j=\overline{1,k}. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{t_i}^{t_i^1} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon\\ =\ &\omega^1(t_i)K(t,t_i+0)\sum_{j=1}^{k} H_j(t_i+0,\underbrace{x(t_i+0)}_{1},\underbrace{x_i,\dots,x_i}_{k})\Delta x(t_i)+\eta(\varepsilon)\\ =\ &\omega^1(t_i)K(t,t_i+0)\sum_{j=1}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{1},\underbrace{x_i}_{k})(B_ix_i+a_i)+\eta(\varepsilon). \end{aligned}$$

It is easy to see that the last equality still holds for $t_i=t_i^1$. Then the integral on the left is 0, while equality (2.2.4.21) for $j=1$ implies $\omega^1(t_i^1)=0$, that is, $\omega^1(t_i^1)=0$ and the right-hand side of the equality is reduced to $\eta(\varepsilon)$.

For the next interval we have

$$\int_{t_i^1}^{t_i^2} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau=(t_i^2-t_i^1)K(t,t_i^{2*})H(t_i^{2*},x(t_i^{2*}),y(t_i^{2*})),$$

where $t_i^1<t_i^{2*}<t_i^2$. In view of (2.2.4.21) for $j=1,2$ we have

$$(t_i^2-t_i^1)/\varepsilon=\omega^2(t_i)-\omega^1(t_i)+\eta(\varepsilon).$$

Next,

$$\begin{aligned} K(t,t_i^{2*}) &= K(t,t_i+0)+\eta(\varepsilon),\\ x(t_i^{2*}) &= x(t_i+0)+\eta(\varepsilon), \quad y^1(t_i^{2*})=x(t_i+0)+\eta(\varepsilon),\\ y^j(t_i^{2*}) &= x_i+\eta(\varepsilon), \qquad j=\overline{2,k}. \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{t_i^1}^{t_i^2} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon \\
= \;& (\omega^2(t_i)-\omega^1(t_i))K(t,t_i+0)\sum_{j=2}^{k} H_j(t_i+0,\underbrace{x(t_i+0)}_{2},\underbrace{x_i}_{k-1})\Delta x(t_i)+\eta(\varepsilon) \\
= \;& (\omega^2(t_i)-\omega^1(t_i))K(t,t_i+0) \\
\times \;& \sum_{j=2}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{2},\underbrace{x_i}_{k-1})(B_ix_i+a_i)+\eta(\varepsilon).
\end{aligned}$$

Similarly we find

$$\begin{aligned}
& \int_{t_i^2}^{t_i^3} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon \\
= \;& (\omega^3(t_i)-\omega^2(t_i))K(t,t_i+0) \\
\times \;& \sum_{j=2}^{k} H_j(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{3},\underbrace{x_i}_{k-2})(B_ix_i+a_i)+\eta(\varepsilon), \\
& \vdots \\
& \int_{t_i^{k-1}}^{t_i^k} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon \\
= \;& (\omega^k(t_i)-\omega^{k-1}(t_i))K(t,t_i+0) \\
\times \;& H_k(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{k},\underbrace{x_i}_{1})(B_ix_i+a_i)+\eta(\varepsilon), \\
& \int_{t_i^k}^{t_i+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon \\
= \;& (1-\omega^k(t_i))K(t,t_i+0)H_k(t_i+0,\underbrace{(E+B_i)x_i+a_i}_{k+1})(B_ix_i+a_i)+\eta(\varepsilon).
\end{aligned}$$

In fact, the last inequality is obtained for $t_i^k < t_i+\varepsilon$. It still holds for $t_i^k = t_i+\varepsilon$. In this case, the integral on the left is 0, $\omega^k(t_i^k)=1$,

$$1-\omega^k(t_i) = |\omega^k(t_i^k)-\omega^k(t_i)| \le K|t_i^k-t_i| \le \varepsilon K$$

by virtue of assumption **A2.2.4.4** and equality (2.2.4.21) for $j = k$, thus the right-hand side of the equality is reduced to $\eta(\varepsilon)$.

Introduce the notation

$$\begin{aligned}
\mathbf{H}_i(x) &= \omega^1(t_i)\sum_{j=1}^{k} H_j(t_i+0, \underbrace{(E+B_i)x+a_i}_{1}, \underbrace{x}_{k}) \qquad (2.2.4.23)\\
&+ (\omega^2(t_i)-\omega^1(t_i))\sum_{j=2}^{k} H_j(t_i+0, \underbrace{(E+B_i)x+a_i}_{2}, \underbrace{x}_{k-1})\\
&+ (\omega^3(t_i)-\omega^2(t_i))\sum_{j=2}^{k} H_j(t_i+0, \underbrace{(E+B_i)x+a_i}_{3}, \underbrace{x}_{k-2}) + \cdots\\
&+ (\omega^k(t_i)-\omega^{k-1}(t_i))H_k(t_i+0, \underbrace{(E+B_i)x+a_i}_{k}, \underbrace{x}_{1})\\
&+ (1-\omega^k(t_i))H_k(t_i+0, \underbrace{(E+B_i)x+a_i}_{k+1}), \qquad i=\overline{1,p}.
\end{aligned}$$

Now

$$\int_{t_i}^{t_i+\varepsilon} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon = K(t,t_i+0)\mathbf{H}_i(x_i)(B_ix_i+a_i)+\eta(\varepsilon). \qquad (2.2.4.24)$$

Equality (2.2.4.24) was obtained under the assumption (2.2.4.20) which is implied by (2.2.4.22) for ε small enough. However, assumption **A2.2.4.4** implies just

$$\omega^1(t_i) \le \omega^2(t_i) \le \cdots \le \omega^k(t_i). \qquad (2.2.4.25)$$

Suppose that

$$\omega^j(t_i) = \omega^{j+1}(t_i)$$

for some $j \in \{1,2,\ldots,k-1\}$. Then the difference $t_i^{j+1} - t_i^j$ can have an arbitrary sign or vanish for any ε small enough. However, in view of $(t_i^{j+1} - t_i^j)/\varepsilon = \eta(\varepsilon)$ and, consequently,

$$\int_{t_i^J}^{t_i^{j+1}} K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau/\varepsilon = \eta(\varepsilon),$$

equality (2.2.4.24) still holds.

Now let $i = 0$. Under the assumption (2.2.4.25) for $i = 0$ ($t_0 = a$), as above, we obtain (2.2.4.24) with $i = 0$:

$$\int_a^{a+\varepsilon} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon = K(t,a)\mathbf{H}_0(x(a))(x(a) - \varphi(a)) + \eta(\varepsilon),$$

where

$$\begin{aligned} \mathbf{H}_0(x) &= \omega^1(a)\sum_{j=1}^{k} H_j(a, \underbrace{x}_{1}, \underbrace{\varphi(a)}_{k}) \qquad (2.2.4.26) \\ &+ (\omega^2(a) - \omega^1(a))\sum_{j=2}^{k} H_j(a, \underbrace{x}_{2}, \underbrace{\varphi(a)}_{k-1}) \\ &+ (\omega^3(a) - \omega^2(a))\sum_{j=2}^{k} H_j(a, \underbrace{x}_{3}, \underbrace{\varphi(a)}_{k-2}) + \cdots \\ &+ (\omega^k(a) - \omega^{k-1}(a))H_k(a, \underbrace{x}_{k}, \underbrace{\varphi(a)}_{1}) + (1 - \omega^k(a))H_k(a, \underbrace{x}_{k+1}). \end{aligned}$$

Summing up equalities (2.2.4.24), $i = \overline{0, p}$, we find

$$\begin{aligned} &\int_{\Delta_1^\varepsilon} K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon \\ = \;& K(t,a)\mathbf{H}_0(x(a))(x(a) - \varphi(a)) + \sum_{i=1}^{p} K(t, t_i + 0)\mathbf{H}_i(x_i)(B_i x_i + a_i) + \eta(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} &\int_a^b K(t,\tau)H(\tau, x(\tau), y(\tau))\, d\tau/\varepsilon \qquad (2.2.4.27) \\ = \;& \int_a^b K(t,\tau)\mathcal{H}(\tau, x(\tau))\left(A(\tau)x(\tau) + f(\tau)\right)\, d\tau + K(t,a)\mathbf{H}_0(x(a))(x(a) - \varphi(a)) \\ + \;& \sum_{i=1}^{p} K(t, t_i + 0)\mathbf{H}_i(x_i)(B_i x_i + a_i) + \eta(\varepsilon). \end{aligned}$$

Thus, dividing equality (2.2.4.15) by ε, in view of equalities (2.2.4.17) and

(2.2.4.27), we have

$$\mathcal{P}_d^*\left\{J(x(\cdot),0)-\ell\int_a^b K(\cdot,\tau)\mathcal{H}(\tau,x(\tau))\left(A(\tau)x(\tau)+f(\tau)\right)\,d\tau \right. \quad (2.2.4.28)$$
$$-\ell K(\cdot,a)\mathbf{H}_0(x(a))\left(x(a)-\varphi(a)\right)$$
$$\left.-\sum_{i=1}^{p}\ell K(\cdot,t_i+0)\left[\mathbf{J}_i(x_i)\left(A_i x_i+f_i\right)+\mathbf{H}_i(x_i)\left(B_i x_i+a_i\right)\right]+\eta(\varepsilon)\right\}=0.$$

Now we easily see that (2.2.4.12) is obtained from (2.2.4.28) by passing to the limit as $\varepsilon\to 0$. The theorem is proved. $\Box$.

Equation (2.2.4.12) can be called *equation for the generating amplitudes* (see, for instance, [63] or [28, 29, 34, 35, 37, 38]) for the BVP for the impulsive system with concentrated delay (2.2.4.1).

Now suppose that c_r^* is a solution of equation (2.2.4.12). Then the solution $z(t,\varepsilon)$ of system (2.2.4.14) such that $z(t,0)\equiv 0$ can be represented in the form

$$z(t,\varepsilon)=X_r(t)c+\varepsilon z^{(1)}(t,\varepsilon), \quad (2.2.4.29)$$

where the unknown constant vector $c=c(\varepsilon)\in\mathbb{R}^r$ must satisfy an equation derived below from (2.2.4.28), while the unknown vector-valued function $z^{(1)}(t,\varepsilon)$ can be represented as

$$z^{(1)}(t,\varepsilon)=X(t)Q^+J(x(\cdot),\varepsilon)+(\Gamma H)(t)/\varepsilon+\sum_{i=1}^{p}\gamma_i(t)I_i(x_i,y_i)/\varepsilon, \quad (2.2.4.30)$$

where

$$(\Gamma H)(t)=\int_a^b K(t,\tau)H(\tau,x(\tau),y(\tau))\,d\tau-X(t)Q^+\ell\int_a^b K(\cdot,\tau)H(\tau,x(\tau),y(\tau))\,d\tau.$$

In view of the above considerations we can write the solvability condition (2.2.4.15) in the form

$$\mathcal{P}_d^*\left\{J\left(x_0(\cdot,c_r^*)+z(\cdot,\varepsilon),\varepsilon\right)-\ell\int_a^b K(\cdot,\tau)h\left(\tau,x_0(\tau,c_r^*)+z(\tau,\varepsilon)\right)\,d\tau\right.$$
$$\left.-\sum_{i=0}^{p}\ell K(\cdot,t_i+0)\mathcal{J}_i\left(x_0(t_i,c_r^*)+z(t_i,\varepsilon)\right)+\eta(\varepsilon,x)\right\}=0, \quad (2.2.4.31)$$

where $t_0 = a = t_0+0$, $h(\tau,x) \equiv \mathcal{H}(\tau,x)\,(A(\tau)x + f(\tau))$, $\mathcal{J}_0(x) \equiv \mathbf{H}_0\,(x - \varphi(a))$, $\mathcal{J}_i(x) \equiv \mathbf{J}_i(x)\,(A_i x + f_i) + \mathbf{H}_i(x)\,(B_i x + a_i)$, $i = \overline{1,p}$, and the quantity $\eta(\varepsilon,x)$ tends to 0 as $\varepsilon \to 0$.

Similarly, equality (2.2.4.30) can be represented in the form

$$\begin{aligned} z^{(1)}(t,\varepsilon) &= X(t)Q^{+}J\left(x_0(\cdot,c_r^*) + z(\cdot,\varepsilon),\varepsilon\right) \\ &+ \left(\Gamma h(\cdot,x)\Big|_{x=x_0(\cdot,c_r^*)+z(\cdot,\varepsilon)}\right)(t) \\ &+ \sum_{i=0}^{p} \gamma_i(t)\mathcal{J}_i\left(x_0(t_i,c_r^*) + z(t_i,\varepsilon)\right) + \eta(\varepsilon, x(t,\varepsilon)), \end{aligned} \tag{2.2.4.32}$$

where $\gamma_0(t) \equiv K(t,a)$.

We expand the left-hand side of (2.2.4.31) about the point $z = 0$. We have

$$h\left(\tau, x_0(\tau,c_r^*) + z\right) = h_0(\tau) + h_1(\tau)z + h_2(\tau,z),$$

where

$$h_0(\tau) = h\left(\tau, x_0(\tau,c_r^*)\right), \qquad h_1(\tau) = \frac{\partial}{\partial x}h(\tau,x)\Big|_{x=x_0(\tau,c_r^*)},$$

$h_2(\tau,z)$ is such that

$$h_2(\tau,0) = 0, \qquad \frac{\partial}{\partial z}h_2(\tau,0) = 0.$$

Analogously we represent

$$\mathcal{J}_i\left(x_0(t_i,c_r^*) + z\right) = \mathcal{J}_{0i} + \mathcal{J}_{1i}z + \mathcal{J}_{2i}(z), \qquad i = \overline{0,p},$$

where $\mathcal{J}_{0i}$, $\mathcal{J}_{1i}$ are represented in a similar way, while $\mathcal{J}_{2i}(z)$ is such that

$$\mathcal{J}_{2i}(0) = 0, \qquad \frac{\partial}{\partial z}\mathcal{J}_{2i}(0) = 0,$$

and

$$J\left(x_0(\cdot,c_r^*) + z,\varepsilon\right) = J_0 + J_1 z + J_2(z,\varepsilon).$$

Now by virtue of the assumption $F(c_r^*) = 0$, equality (2.2.4.31) takes the form

$$\begin{aligned} \mathcal{P}_d^*\Bigg\{ & J_1 z(\cdot,\varepsilon) + J_2\left(z(\cdot,\varepsilon),\varepsilon\right) - \ell\int_a^b K(\cdot,\tau)\left(h_1(\tau)z(\tau,\varepsilon) + h_2(\tau,z(\tau,\varepsilon))\right)d\tau \\ & - \sum_{i=0}^{p} \ell K(\cdot,t_i+0)\left(\mathcal{J}_{1i}z(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon))\right) + \eta(\varepsilon,x)\Bigg\} = 0. \end{aligned} \tag{2.2.4.33}$$

In view of the representation (2.2.4.29) let us denote

$$\mathcal{B}_0 = \mathcal{P}_d^* \left(J_1 X_r(\cdot) - \ell \int_a^b K(\cdot,\tau) h_1(\tau) X_r(\tau)\, d\tau - \sum_{i=0}^{p} \ell K(\cdot, t_i+0) \mathcal{J}_{1i} X_r(t_i) \right),$$

which is a $(d \times r)-$matrix. Then we have

$$\begin{aligned} \mathcal{B}_0 c \;=\; & -\mathcal{P}_d^* \Big\{ \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + J_2\left(z(\cdot,\varepsilon),\varepsilon\right) \\ & -\ell \int_a^b K(\cdot,\tau)\left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon))\right) d\tau \\ & - \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon))\right) + \eta(\varepsilon, x) \Big\}. \end{aligned} \tag{2.2.4.34}$$

In the representation (2.2.4.32) we use the same expansions and obtain

$$\begin{aligned} z^{(1)}(t,\varepsilon) \;=\; & X(t) Q^+ \left(J_0 + J_1\left(X_r(\cdot)c + \varepsilon z^{(1)}(\cdot,\varepsilon)\right) + J_2(z(\cdot,\varepsilon),\varepsilon)\right) \\ + \; & (\Gamma h_0)(t) + \left(\Gamma\left[h_1(\cdot)\left(X_r(\cdot)c + \varepsilon z^{(1)}(\cdot,\varepsilon)\right)\right]\right)(t) + (\Gamma h_2(\cdot, z(\cdot,\varepsilon)))(t) \\ + \; & \sum_{i=0}^{p} \gamma_i(t)\left(\mathcal{J}_{0i} + \mathcal{J}_{1i}\left(X_r(t_i)c + \varepsilon z^{(1)}(t_i,\varepsilon)\right) + \mathcal{J}_{2i}(z(t_i,\varepsilon))\right) + \eta(\varepsilon, x(t,\varepsilon)). \end{aligned} \tag{2.2.4.35}$$

Thus we have reduced problem (2.2.4.1) to the equivalent operator system (2.2.4.13), (2.2.4.29), (2.2.4.34), (2.2.4.35).

Denote by $\mathcal{P}_0 \equiv \mathcal{P}_{\mathcal{B}_0}$ and $\mathcal{P}_0^* \equiv \mathcal{P}_{\mathcal{B}_0^*}$, respectively, the orthoprojectors $\mathcal{P}_0 : \mathbb{R}^r \to \mathrm{Ker}(\mathcal{B}_0)$ and $\mathcal{P}_0^* : \mathbb{R}^d \to \mathrm{Ker}(\mathcal{B}_0^*)$. Suppose that $\operatorname{rank} \mathcal{B}_0 = r$, *i.e.*, $\mathcal{P}_0 = 0$. This is the so called *critical case of first order*. In this case the inequality $d \geq r$ necessarily holds, which implies $m \geq n$. Then equation (2.2.4.34) can be solved with respect to c if and only if

$$\begin{aligned} & \mathcal{P}_0^* \mathcal{P}_d^* \Big\{ \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + J_2\left(z(\cdot,\varepsilon),\varepsilon\right) \\ & -\ell \int_a^b K(\cdot,\tau)\left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon))\right) d\tau \\ & - \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon))\right) + \eta(\varepsilon, x) \Big\} = 0. \end{aligned} \tag{2.2.4.36}$$

If $\mathcal{P}_0^* \mathcal{P}_d^* = 0$, then condition (2.2.4.36) is always fulfilled and equation (2.2.4.34) can be uniquely solved with respect to c:

$$
\begin{aligned}
c \;=\; & -\mathcal{B}_0^+ \mathcal{P}_d^* \Big\{ \varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + J_2\left(z(\cdot,\varepsilon),\varepsilon\right) \\
& -\ell \int_a^b K(\cdot,\tau)\left(\varepsilon h_1(\tau) z^{(1)}(\tau,\varepsilon) + h_2(\tau, z(\tau,\varepsilon))\right) d\tau \\
& - \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left(\varepsilon \mathcal{J}_{1i} z^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z(t_i,\varepsilon))\right) + \eta(\varepsilon, x) \Big\},
\end{aligned} \tag{2.2.4.37}
$$

where $\mathcal{B}_0^+$ is an $(r \times d)-$matrix which is the Moore-Penrose pseudoinverse to $\mathcal{B}_0$. Thus we obtain [63] an operator system (2.2.4.13), (2.2.4.29), (2.2.4.37), (2.2.4.35) to which a convergent simple iteration method can be applied.

Theorem 2.2.4.2. *For BVP* (2.2.4.1) *let conditions* **A2.2.4.1**–**A2.2.4.7** *and* (2.2.4.9) *hold and* $\operatorname{rank} Q = n_1$, $r = n - n_1 > 0$. *Then for any root* $c_r = c_r^* \in \mathbb{R}^r$ *of equation* (2.2.4.12) *such that* $\mathcal{P}_0 = 0$, $\mathcal{P}_0^* \mathcal{P}_d^* = 0$ *there exists a constant* $\varepsilon_* \in (0, \varepsilon_0)$ *such that for* $\varepsilon \in [0, \varepsilon_*]$ *BVP* (2.2.4.1) *has a unique solution* $x(t,\varepsilon) \in C([a,b] \setminus \{t_i\})$ *as a function of* t*, depending continuously on* ε*, and such that* $x(t,0) = x_0(t, c_r^*)$*. This solution is determined by means of a convergent for* $\varepsilon \in [0, \varepsilon_*]$ *simple iteration method*

$$
\begin{aligned}
c_\nu \;=\; & -\mathcal{B}_0^+ \mathcal{P}_d^* \Big\{ \varepsilon J_1 z_\nu^{(1)}(\cdot,\varepsilon) + J_2\left(z_\nu(\cdot,\varepsilon),\varepsilon\right) \\
& -\ell \int_a^b K(\cdot,\tau)\left(\varepsilon h_1(\tau) z_\nu^{(1)}(\tau,\varepsilon) + h_2(\tau, z_\nu(\tau,\varepsilon))\right) d\tau \\
& - \sum_{i=0}^{p} \ell K(\cdot, t_i+0)\left(\varepsilon \mathcal{J}_{1i} z_\nu^{(1)}(t_i,\varepsilon) + \mathcal{J}_{2i}(z_\nu(t_i,\varepsilon))\right) + \eta(\varepsilon, x_\nu) \Big\},
\end{aligned} \tag{2.2.4.38}
$$

$$
\begin{aligned}
z_{\nu+1}^{(1)}(t,\varepsilon) \;=\; & X(t) Q^+ \left(J_0 + J_1\left(X_r(\cdot) c_\nu + \varepsilon z_\nu^{(1)}(\cdot,\varepsilon)\right) + J_2(z_\nu(\cdot,\varepsilon),\varepsilon)\right) \\
+ \; & (\Gamma h_0)(t) + \left(\Gamma\left[h_1(\cdot)\left(X_r(\cdot) c_\nu + \varepsilon z_\nu^{(1)}(\cdot,\varepsilon)\right)\right]\right)(t) + (\Gamma h_2(\cdot, z_\nu(\cdot,\varepsilon)))(t) \\
+ \; & \sum_{i=0}^{p} \gamma_i(t)\left(\mathcal{J}_{0i} + \mathcal{J}_{1i}\left(X_r(t_i) c_\nu + \varepsilon z^{(1)}(t_i,\varepsilon)\right) + \mathcal{J}_{2i}(z_\nu(t_i,\varepsilon))\right) + \eta(\varepsilon, x_\nu(t,\varepsilon)), \\
z_{\nu+1}(t,\varepsilon) \;=\; & X_r(t) c_\nu + \varepsilon z_{\nu+1}^{(1)}(t,\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
x_\nu(t,\varepsilon) &= \begin{cases} x_0(t,c_r^*) + z_\nu(t,\varepsilon), & t\in[a,b],\\ \qquad\qquad \varphi(t), & t\in[a-\varepsilon_*,a), \end{cases}\\
\nu &= 0,1,2,\dots; \qquad z_0(t,\varepsilon) = z_0^{(1)}(t,\varepsilon) = 0.
\end{aligned}
$$

The convergence of the method can be proved by using Lyapunov's majorants technique as in [29, 63].

Let $m = n$ (the number of the boundary conditions equals the order of the system). Now the condition $\mathcal{P}_0 = 0$ implies $\mathcal{P}_0^* = 0$, thus the condition $\mathcal{P}_0^*\mathcal{P}_d^* = 0$ is automatically fulfilled. Moreover, $\mathcal{P}_0 = 0$ implies $\det \mathcal{B}_0 \neq 0$, and in equality (2.2.4.37) and in the first equality of the iteration procedure (2.2.4.38) we write $\mathcal{B}_0^{-1}$ instead of $\mathcal{B}_0^+$. It is easy to see [29, 38, 63] that this condition is equivalent to the simplicity of the root $c_r = c_r^*$ of the equation for the generating amplitudes:

$$
F(c_r^*) = 0, \qquad \det\left(\frac{\partial F(c_r)}{\partial c_r}\right)\Bigg|_{c_r=c_r^*} \neq 0.
$$

In particular, the periodic BVP with impulse effect for system (2.2.4.1) is of Fredholm type. It was considered in [38] (see also the monographs [29, 35]). The existence of a periodic solution of an impulsive system with a small constant delay ($\omega^j(t) \equiv 1$ in system (2.2.4.1)) was proved in §2.2.1 for the critical case of first order. Moreover, under some simplifying assumptions the result was extended to the case of a *nonlinear* generating system [9].

Now suppose that $\mathcal{P}_0 \neq 0$. Then the operator system (2.2.4.13), (2.2.4.29), (2.2.4.34), (2.2.4.35) does not belong to the class of systems to which the simple iteration method is applicable. Introducing an additional variable, system (2.2.4.13), (2.2.4.29), (2.2.4.34), (2.2.4.35) under condition (2.2.4.36) is regularized (reduced to an operator system of higher dimension to which the simple iteration method can be applied). Below we just sketch this regularization.

If condition (2.2.4.36) is satisfied, then from (2.2.4.34) we determine

$$
c = c^{(0)} + c^{(1)},
$$

where $c^{(0)}$ is given by the right-hand side of equality (2.2.4.37), $c^{(1)}$ is an arbitrary constant vector in $\mathrm{Ker}(\mathcal{B}_0)$, $c^{(1)} = \mathcal{P}_0 c$, $c^{(0)} = (\mathrm{Id} - \mathcal{P}_0)c$. Then equality (2.2.4.35) can be written in the form

$$
z^{(1)}(t,\varepsilon) = G_1(t)c^{(1)} + z^{(2)}(t,\varepsilon), \tag{2.2.4.39}
$$

where

$$G_1(t) \equiv X(t)Q^+ J_1 X_r(\cdot) + (\Gamma[h_1(\cdot)X_r(\cdot)])(t) + \sum_{i=0}^{p} \gamma_i(t)\mathcal{J}_{1i}X_r(t_i),$$

and $z^{(2)}(t,\varepsilon)$ is given by the right-hand side of equality (2.2.4.35), with c being replaced by $c^{(0)}$.

If $\mathcal{P}_0^*\mathcal{P}_d^* = 0$, then condition (2.2.4.36) is always fulfilled and BVP (2.2.4.1) has an $(r - \operatorname{Rank}\mathcal{B}_0)-$parametric family of solutions. If this is not the case, further computations need more precise expansions with respect to the "small parameter" ε, which require the existence of a piecewise continuous second derivative of the solution x, respectively piecewise continuous differentiability of the known functions in system (2.2.4.1) with respect to t, the existence of some continuous second derivatives of the functions H_j, I_{ij} and continuous differentiability of $J(x,\varepsilon)$ with respect to ε.

Thus the solvability condition (2.2.4.36) is represented in the form

$$\begin{aligned}\mathcal{P}_0^*\mathcal{P}_d^* \Big\{ &\varepsilon J_1 z^{(1)}(\cdot,\varepsilon) + \varepsilon J_2' z(\cdot,\varepsilon) + J_2''(z(\cdot,\varepsilon),\varepsilon) \\ &-\ell\int_a^b K(\cdot,\tau)\left(\varepsilon h_1(\tau)z^{(1)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)z(\tau,\varepsilon) + h_2''(\tau,z(\tau,\varepsilon))\right)d\tau \\ &-\sum_{i=0}^{p} \ell K(\cdot,t_i+0)\left(\varepsilon\mathcal{J}_{1i}z^{(1)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'z(t_i,\varepsilon) + \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right) + \eta(\varepsilon,x)\Big\} = 0.\end{aligned} \tag{2.2.4.40}$$

In view of (2.2.4.29) and (2.2.4.39) we obtain the system

$$\begin{aligned}\varepsilon\mathcal{B}_1 c^{(1)} = -\mathcal{P}_0^*\mathcal{P}_d^* \Big\{ &\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2'\left(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\right) \\ &+ J_2''(z(\cdot,\varepsilon),\varepsilon) \\ &-\ell\int_a^b K(\cdot,\tau)\left[\varepsilon h_1(\tau)z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)\left(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\right)\right. \\ &\left.+ h_2''(\tau,z(\tau,\varepsilon))\right]d\tau \\ &-\sum_{i=0}^{p} \ell K(\cdot,t_i+0)\left[\varepsilon\mathcal{J}_{1i}z^{(2)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'\left(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\right)\right. \\ &\left.+ \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right] + \eta(\varepsilon,x)\Big\},\end{aligned} \tag{2.2.4.41}$$

where

$$\begin{aligned}\mathcal{B}_1 &= \mathcal{P}_0^*\mathcal{P}_d^*\left\{J_1G_1(\cdot) + J_2'X_r(\cdot) - \ell\int_a^b K(\cdot,\tau)\left(h_1(\tau)G_1(\tau) + h_2'(\tau)X_r(\tau)\right)\,d\tau\right.\\ &- \left.\sum_{i=0}^{p}\ell K(\cdot,t_i+0)\left(\mathcal{J}_{1i}G_1(t_i) + \mathcal{J}_{2i}'X_r(t_i)\right)\right\}\mathcal{P}_0\end{aligned}$$

is a $(d\times r)-$matrix.

Denote by $\mathcal{P}_1 \equiv \mathcal{P}_{\mathcal{B}_1}$ and $\mathcal{P}_1^* \equiv \mathcal{P}_{\mathcal{B}_1^*}$, respectively, the orthoprojectors $\mathcal{P}_1 : \mathbb{R}^r \to \mathrm{Ker}(\mathcal{B}_1)$ and $\mathcal{P}_1^* : \mathbb{R}^d \to \mathrm{Ker}(\mathcal{B}_1^*)$. Then system (2.2.4.41) is solvable with respect to $\varepsilon c^{(1)} \in \mathrm{Ker}(\mathcal{B}_0)$ if and only if

$$\begin{aligned}&\mathcal{P}_1^*\mathcal{P}_0^*\mathcal{P}_d^*\left\{\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2'\left(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\right)\right.\\ &\qquad\qquad + J_2''\left(z(\cdot,\varepsilon),\varepsilon\right)\\ &-\ell\int_a^b K(\cdot,\tau)\left[\varepsilon h_1(\tau)z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)\left(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\right)\right.\\ &\qquad\qquad \left. + h_2''(\tau,z(\tau,\varepsilon))\right]d\tau\\ &-\sum_{i=0}^{p}\ell K(\cdot,t_i+0)\left[\varepsilon\mathcal{J}_{1i}z^{(2)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'\left(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\right)\right.\\ &\qquad\qquad \left.\left. + \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right] + \eta(\varepsilon,x)\right\} = 0,\end{aligned} \tag{2.2.4.42}$$

and

$$\begin{aligned}\varepsilon c^{(1)} &= -\mathcal{B}_1^+\mathcal{P}_0^*\mathcal{P}_d^*\left\{\varepsilon J_1 z^{(2)}(\cdot,\varepsilon) + \varepsilon J_2'\left(X_r(\cdot)c^{(0)} + \varepsilon z^{(1)}(\cdot,\varepsilon)\right)\right.\\ &\qquad\qquad + J_2''\left(z(\cdot,\varepsilon),\varepsilon\right)\\ &-\ell\int_a^b K(\cdot,\tau)\left[\varepsilon h_1(\tau)z^{(2)}(\tau,\varepsilon) + \varepsilon h_2'(\tau)\left(X_r(\tau)c^{(0)} + \varepsilon z^{(1)}(\tau,\varepsilon)\right)\right.\\ &\qquad\qquad \left. + h_2''(\tau,z(\tau,\varepsilon))\right]d\tau\\ &-\sum_{i=0}^{p}\ell K(\cdot,t_i+0)\left[\varepsilon\mathcal{J}_{1i}z^{(2)}(t_i,\varepsilon) + \varepsilon\mathcal{J}_{2i}'\left(X_r(t_i)c^{(0)} + \varepsilon z^{(1)}(t_i,\varepsilon)\right)\right.\\ &\qquad\qquad \left.\left. + \mathcal{J}_{2i}''(z(t_i,\varepsilon))\right] + \eta(\varepsilon,x)\right\} + c^{(2)},\end{aligned}$$

where $\mathcal{B}_1^+$ is the Moore-Penrose pseudoinverse matrix to $\mathcal{B}_1$, $c^{(2)}$ is an arbitrary vector in $\mathrm{Ker}(\mathcal{B}_0) \cap \mathrm{Ker}(\mathcal{B}_1)$.

Suppose that $\mathrm{Ker}(\mathcal{B}_0) \cap \mathrm{Ker}(\mathcal{B}_1) = 0$. Then (2.2.4.41) has a unique solution. A sufficient condition for (2.2.4.42) is $\mathcal{P}_1^* \mathcal{P}_0^* \mathcal{P}_d^* = 0$, *i.e.*,

$$\mathrm{Ker}(\mathcal{B}_0^*) \cap \mathrm{Ker}(\mathcal{B}_1^*) \cap \mathrm{Ker}(Q^*) = 0.$$

Thus, under the conditions

$$\mathcal{P}_0 \neq 0, \qquad \mathcal{P}_0 \mathcal{P}_1 = 0, \qquad \mathcal{P}_1^* \mathcal{P}_0^* \mathcal{P}_d^* = 0$$

system (2.2.4.13), (2.2.4.29), (2.2.4.34), (2.2.4.35) is reduced to a system to which a simple iteration method can be applied. As stated above, this requires additional smoothness assumptions and cumbersome computations. For the case of a periodic problem for an impulsive system with a small constant delay ($\omega^j(t) \equiv 1$), these computations were carried out in details in §2.2.2.

The results of the present subsection were reported at the 24-th Summer School “Applications of Mathematics in Engineering”, Sozopol, Bulgaria, 1998, and were published in a very concise form in its proceedings [30] and in details in [33].

2.3 Periodic and Almost Periodic Solutions of Impulsive Systems with Periodic Fluctuations of the Delay

An impulsive system with delay which differs from a constant by a small-amplitude periodic perturbation is considered. If the corresponding system with constant delay has an isolated ω–periodic solution, then under a nondegeneracy assumption it is proved that in any sufficiently small neighbourhood of this orbit the perturbed system also has a unique ω–periodic solution. If the period of the delay is rationally independent with ω, the perturbed system has a unique almost periodic solution. These results are extended to the case of a neutral impulsive system with a small delay of the argument of the derivative and another delay which differs from a constant by a small-amplitude periodic perturbation.

2.3.1 Periodic solutions of retarded systems

In the present subsection we study a system with impulses at fixed moments and delay fluctuating around a constant value which may be assumed 1 without loss of generality:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-1-\varepsilon\varphi(t))), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x(t_i), x(t_i-1-\varepsilon\varphi(t_i))), \quad i \in \mathbb{Z}, \\ \Delta x(t) &= 0 \quad \text{if} \quad t-1-\varepsilon\varphi(t) = t_i \quad \text{for some} \quad i \in \mathbb{Z}, \quad t \notin \{t_i\}_{i\in\mathbb{Z}}, \end{aligned} \tag{2.3.1.1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \mathbb{R}\times\Omega\times\Omega \to \mathbb{R}^n$, Ω is a domain in $\mathbb{R}^n$; $\varphi : \mathbb{R} \to [-1, 1]$; $\Delta x(t_i)$ are the impulses at moments t_i and $\{t_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{i\to\pm\infty} t_i = \pm\infty$; $I_i : \Omega \times \Omega \to \mathbb{R}^n$ ($i \in \mathbb{Z}$), $1 + \varepsilon\varphi(t)$ is the delay, $\varepsilon \in [0, \varepsilon_0)$ is a small parameter; ε_0 will be specified below.

It is clear that, in general, the derivatives $\dot{x}(t_i)$ do not exist. On the other hand, there do exist the limits $\dot{x}(t_i \pm 0)$. According to the convention of §1.1, we assume $\dot{x}(t_i) \equiv \dot{x}(t_i - 0)$.

Similarly, the derivative $\dot{x}(t)$ does not exist at the other points of discontinuity of the right-hand side $f(t, x(t), x(t - \varepsilon\varphi(t)))$, *i.e.*, at points t which are solutions of the equations

$$t - 1 - \varepsilon\varphi(t) = t_i, \quad i \in \mathbb{Z}. \tag{2.3.1.2}$$

We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_i.

For the sake of brevity we shall use the notation:

$$\bar{x}(t) = x(t-1), \quad x_i = x(t_i), \quad y^{\varepsilon}(t) = x(t-1-\varepsilon\varphi(t))$$

(thus, for instance, $y_i^0 = x(t_i - 1) = \bar{x}_i$).

In the sequel we require the fulfillment of the following assumptions:

A2.3.1.1. The function $f(t, x, y)$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω– periodic with respect to t, continuously differentiable with respect to $x, y \in \Omega$, with locally Lipschitz-continuous with respect to x, y first derivatives.

A2.3.1.2. The functions $I_i(x, y)$, $i \in \mathbb{Z}$, are continuously differentiable with respect to $x, y \in \Omega$, with locally Lipschitz-continuous with respect to x, y first derivatives.

A2.3.1.3. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, y) = I_i(x, y)$ for $i \in \mathbb{Z}$ and $x, y \in \Omega$.

A2.3.1.4. The function $\varphi(t)$ is ω–periodic and Lipschitz continuous:

$$|\varphi(t') - \varphi(t'')| \ \leq \ K|t' - t''|, \quad t', t'' \in \mathbb{R}.$$

If $\varepsilon_0 \leq \ \min\{1, 1/K\}$, then for $\varepsilon \in (0, \varepsilon_0)$ each equation (2.3.1.2) has a unique solution $t_i(\varepsilon)$. It obviously satisfies

$$|t_i(\varepsilon) - t_i - 1| \leq \varepsilon, \quad t_i(0) = t_i + 1.$$

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

For $\varepsilon = 0$, from (2.3.1.1) we obtain

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-1)), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x_i, \bar{x}_i), \quad i \in \mathbb{Z}, \\ \Delta x(t_i + 1) &= 0 \quad \text{if} \quad t_i + 1 \neq t_j \ \ \forall j, \end{aligned} \tag{2.3.1.3}$$

so called *generating system*.

A1.3.1.5. The generating system (2.3.1.3) has an $\omega-$periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)z(t-1), \quad t \neq t_i, & (2.3.1.4)\\ \Delta z(t_i) &= C_i z_i + D_i \bar{z}_i, \quad i \in \mathbb{Z}, & (2.3.1.5) \end{aligned}$$

where

$$A(t) = \frac{\partial}{\partial x} f(t, x, \psi(t-1))\Big|_{x=\psi(t)}, \quad B(t) = \frac{\partial}{\partial y} f(t, \psi(t), y)\Big|_{y=\psi(t-1)},$$

and

$$C_i = \frac{\partial}{\partial x} I_i(x, \bar{\psi}_i)\Big|_{x=\psi_i}, \quad D_i = \frac{\partial}{\partial y} I_i(\psi_i, y)\Big|_{y=\bar{\psi}_i}.$$

Let the $(n \times n)$–matrix $X(t)$ be the fundamental solution of the system (2.3.1.4) [20, 69] (*i.e.*, $X(t) = 0$ for $t < 0$, $X(0) = E$; $\dot{X}(t) = A(t)X(t) + B(t)X(t-1)$ for $t > 0$). Now we make two additional assumptions:

A2.3.1.6. The matrices $(E + C_i)X(t_i) + D_i X(t_i - 1)$ are nonsingular for $t_i \in (0, \omega)$.

A2.3.1.7. The only $\omega-$periodic solution of system (2.3.1.4), (2.3.1.5) is $z(t) \equiv 0$.

If the last two conditions hold, as in [36, 103] we can define *Green's function* $G(t, \tau)$ of the periodic problem for the nonhomogeneous system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)z(t-1) + g(t), \quad t \neq t_i, & (2.3.1.6)\\ \Delta z(t_i) &= C_i z_i + D_i \bar{z}_i + a_i, \quad i \in \mathbb{Z}, \end{aligned}$$

corresponding to (2.3.1.4), (2.3.1.5), where $g(\cdot) \in \tilde{C}_{\omega,n}$ and $a_{i+m} = a_i,\ i \in \mathbb{Z}$, *i.e.*, system (2.3.1.6) has a unique $\omega-$periodic solution given by the formula

$$z(t) = \int_0^\omega G(t, \tau) g(\tau)\, d\tau + \sum_{i=1}^{m} G(t, t_i + 0) a_i. \quad (2.3.1.7)$$

Our result in the present subsection is the following

Theorem 2.3.1.1. *Let conditions* **A2.3.1.1**–**A2.3.1.7** *hold. Then there exists a number* $\varepsilon_* > 0$ *such that for* $\varepsilon \in (0, \varepsilon_*)$ *system* (2.3.1.1) *has a unique* $\omega-$*periodic solution* $x(t, \varepsilon)$ *depending continuously on* ε *and such that* $x(t, \varepsilon) \to \psi(t)$ *as* $\varepsilon \to 0$.

Proof. In system (2.3.1.1) we change the variables according to the formula

$$x = \psi(t) + z \tag{2.3.1.8}$$

and obtain the system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)z(t-1) + Q(t, z(t), z(t-1)) + \delta f(t, x(t), y^{\varepsilon}(t)), \ t \neq t_i, \\ \Delta z(t_i) &= C_i z_i + D_i \bar{z}_i + J_i(z_i, \bar{z}_i) + \delta I_i(x_i, y_i^{\varepsilon}), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.1.9}$$

where

$$\begin{aligned} Q(t, z, \bar{z}) &\equiv f(t, \psi(t) + z, \psi(t-1) + \bar{z}) - f(t, \psi(t), \psi(t-1)) - A(t)z - B(t)\bar{z}, \\ J_i(z_i, \bar{z}_i) &\equiv I_i(\psi_i + z_i, \bar{\psi}_i + \bar{z}_i) - I_i(\psi_i, \bar{\psi}_i) - C_i z_i - D_i \bar{z}_i \end{aligned}$$

are nonlinearities inherent to the generating system (2.3.1.3) and therefore independent of the fluctuation of the delay $\varepsilon\varphi(t)$, while

$$\begin{aligned} \delta f(t, x(t), y^{\varepsilon}(t)) &\equiv f(t, x(t), y^{\varepsilon}(t)) - f(t, x(t), y^0(t)), \\ \delta I_i(x_i, y_i^{\varepsilon}) &\equiv I_i(x_i, y_i^{\varepsilon}) - I_i(x_i, y_i^0) \end{aligned}$$

are increments due to this fluctuation.

We can formally consider (2.3.1.9) as a nonhomogeneous system of the form (2.3.1.6). Then its unique $\omega-$periodic solution $z(t)$ must satisfy an equality of the form (2.3.1.7) which in this case is the operator equation

$$z = \mathcal{U}_{\varepsilon} z, \tag{2.3.1.10}$$

where

$$\begin{aligned} \mathcal{U}_{\varepsilon} z(t) &\equiv \int_0^{\omega} G(t, \tau) Q(\tau, z(\tau), z(\tau - 1))) \, d\tau + \int_0^{\omega} G(t, \tau) \delta f(\tau, x(\tau), y^{\varepsilon}(\tau)) \, d\tau \\ &+ \sum_{0 < t_i < \omega} G(t, t_i + 0) J_i(z_i, \bar{z}_i) + \sum_{0 < t_i < \omega} G(t, t_i + 0) \delta I_i(x_i, y_i^{\varepsilon}) \\ &\equiv \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t). \end{aligned}$$

For the sake of brevity we still write x instead of $\psi(t)+z$ in $\delta f(t, x(t), y^\varepsilon(t))$, $\delta I_i(x_i, y_i^\varepsilon)$ as well as in $\mathcal{I}_2 z$ and $\mathcal{S}_2 z$. Moreover, we will further transform the expressions $\mathcal{I}_2 z(t)$ and $\mathcal{S}_2 z(t)$ under the assumption that $x(t)$ is a solution of system (2.3.1.1). This will considerably simplify some estimates henceforth.

An $\omega-$periodic solution $x(t) = x(t, \varepsilon)$ of system (2.3.1.1) corresponds to a fixed point z of the operator $\mathcal{U}_\varepsilon$ in a suitable set of ω–periodic functions. To this end we shall prove that $\mathcal{U}_\varepsilon$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. For $x, y \in \Omega_1$ the functions $f(t, x, y)$ $(t \in [0, \omega])$ and $I_i(x, y)$ $(i = \overline{1, m})$ are bounded, together with their first derivatives with respect to x, y. We shall not denote explicitly the respective upper bounds.

Let L_1 and L_2 be respectively the greatest Lipschitz constants for the first derivatives of $f(t, x, y),\ t \in [0, \omega],\ x, y \in \Omega_1$, and of $I_i(x, y),\ i = \overline{1, m},\ x, y \in \Omega_1$, whose existence is provided by conditions **A2.3.1.1**, **A2.3.1.2** and the compactness of the set Ω_1.

For $a, b \in \mathbb{R}$ denote

$$]a, b[= \begin{cases} (a, b) & \text{if } a < b, \\ (b, a) & \text{if } a > b, \\ \emptyset & \text{if } a = b. \end{cases}$$

We may note that

$$\tau \in]t_i(\varepsilon), t_i + 1[\quad \Longleftrightarrow \quad t_i \in]\tau - 1, \tau - 1 - \varepsilon\varphi(\tau)[.$$

Define the "bad" set $\Delta_1^\varepsilon = \bigcup_{i=1}^{m}]t_i(\varepsilon), t_i + 1[$. If $\varepsilon > 0$ is small enough, then Δ_1^ε is a disjoint union of intervals contained in $(1, \omega + 1)$, and

$$\text{meas } \Delta_1^\varepsilon \le m\varepsilon. \tag{2.3.1.11}$$

We further define the "good" set $\Delta_2^\varepsilon = [1, \omega + 1] \setminus \Delta_1^\varepsilon$.

For the sake of convenience we assume that for $i = \overline{1, m}$ $\quad t_i + 1 \neq t_j\ \forall j \in \mathbb{Z}$. Then for $\varepsilon > 0$ small enough the "bad" set Δ_1^ε contains none of the points $t_i,\ i \in \mathbb{Z}$.

Let $\varepsilon_0 > 0$ be so small that all the above assumptions are valid for $\varepsilon \in (0, \varepsilon_0)$.

For $\mu \in (0, \mu_0]$ define a set of functions

$$\mathcal{T}_\mu = \{z \in \tilde{C}_{\omega,n} : \quad \|z\| \le \mu\}.$$

We shall find a dependence between ε and μ so that the operator $\mathcal{U}_\varepsilon$ in (2.3.1.10) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_\varepsilon$. Let $z \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_\varepsilon z(t)|$ using the representation

$$\mathcal{U}_\varepsilon z = \mathcal{I}_1 z + \mathcal{I}_2 z + \mathcal{S}_1 z(t) + \mathcal{S}_2 z$$

and system (2.3.1.1).

First we have

$$\begin{aligned} J_i(z_i, \bar{z}_i) &= \int_0^1 \left[\partial_x I_i(\psi_i + s z_i, \bar{\psi}_i + s\bar{z}_i) - \partial_x I_i(\psi_i, \bar{\psi}_i)\right] \, ds \cdot z_i \\ &+ \int_0^1 \left[\partial_y I_i(\psi_i + s z_i, \bar{\psi}_i + s\bar{z}_i) - \partial_y I_i(\psi_i, \bar{\psi}_i)\right] \, ds \cdot \bar{z}_i, \end{aligned}$$

thus

$$\begin{aligned} |J_i(z_i, \bar{z}_i)| &\le \int_0^1 L_2 s(|z_i| + |\bar{z}_i|) \, ds \cdot |z_i| + \int_0^1 L_2 s(|z_i| + |\bar{z}_i|) \, ds \cdot |\bar{z}_i| \\ &= L_2(|z_i| + |\bar{z}_i|)^2/2 \end{aligned}$$

and

$$\mathcal{S}_1 z(t) \equiv \sum_{0<t_i<\omega} G(t, t_i + 0) J_i(z_i, \bar{z}_i) = O(\mu^2).$$

Similarly, for $\tau \ne t_i$, $\tau \ne t_i + 1$ we have

$$\begin{aligned} & Q(\tau, z(\tau), z(\tau - 1)) \\ = & \int_0^1 \left[\partial_x f(\tau, \psi(\tau) + s z(\tau), \bar{\psi}(\tau) + s\bar{z}(\tau)) - \partial_x f(\tau, \psi(\tau), \bar{\psi}(\tau))\right] \, ds \cdot z(\tau) \\ + & \int_0^1 \left[\partial_y f(\tau, \psi(\tau) + s z(\tau), \bar{\psi}(\tau) + s\bar{z}(\tau)) - \partial_y f(\tau, \psi(\tau), \bar{\psi}(\tau))\right] \, ds \cdot \bar{z}(\tau), \end{aligned}$$

thus

$$
\begin{aligned}
& |Q(\tau, z(\tau), z(\tau-1))| \\
\leq \; & \int_0^1 L_1 s \left(|z(\tau)| + |\bar{z}(\tau)|\right) \, ds \cdot |z(\tau)| + \int_0^1 L_1 s \left(|z(\tau)| + |\bar{z}(\tau)|\right) \, ds \cdot |\bar{z}(\tau)| \\
& = L_1 \left(|z(\tau)| + |\bar{z}(\tau)|\right)^2 / 2
\end{aligned}
$$

and

$$\mathcal{I}_1 z(t) \equiv \int_0^\omega G(t,\tau) Q(\tau, z(\tau), z(\tau-1)) = O(\mu^2).$$

Now we can choose $\tilde{\mu}_0 \in (0, \mu_0]$ so that for any $\mu \in (0, \tilde{\mu}_0]$ we have

$$|\mathcal{I}_1 z(t) + \mathcal{S}_1 z(t)| \leq \mu/2. \qquad (2.3.1.12)$$

Further on, since the intervals $]t_i(\varepsilon), t_i + 1[$ contain none of the points t_j, we have

$$
\begin{aligned}
\delta I_i(x_i, y_i^\varepsilon) &= \int_0^1 \frac{\partial}{\partial s} I_i(x_i, y_i^{s\varepsilon}) \, ds = \int_0^1 \partial_y I_i(x_i, y_i^{s\varepsilon}) \frac{\partial}{\partial s} x(t_i - 1 - s\varepsilon\varphi(t_i)) \, ds \\
&= - \; \varepsilon\varphi(t_i) \int_0^1 \partial_y \, I_i(x_i, y_i^{s\varepsilon}) \dot{x}(t_i - 1 - s\varepsilon\varphi(t_i)) \, ds \qquad (2.3.1.13) \\
&= - \; \varepsilon\varphi(t_i) \int_0^1 \partial_y \, I_i(x_i, y_i^{s\varepsilon}) f(t_i - 1 - s\varepsilon\varphi(t_i), y_i^{s\varepsilon}, y^\varepsilon(t_i - 1 - s\varepsilon\varphi(t_i))) \, ds,
\end{aligned}
$$

thus $\delta I_i(x_i, y_i^\varepsilon) = O(\varepsilon)$ and

$$\mathcal{S}_2 z(t) \equiv \sum_{0 < t_i < \omega} G(t, t_i + 0) \delta I_i(x_i, y_i^\varepsilon) = O(\varepsilon). \qquad (2.3.1.14)$$

If $\tau \in \Delta_2^\varepsilon \setminus \{t_i\}_{i=1}^m$, as above we have

$$
\begin{aligned}
\delta f(\tau, x(\tau), y^\varepsilon(\tau)) &= -\varepsilon\varphi(\tau) \int_0^1 \partial_y f(\tau, x(\tau), y^{s\varepsilon}(\tau)) \dot{x}(\tau - 1 - s\varepsilon\varphi(\tau)) \, ds \qquad (2.3.1.15) \\
= - \; & \varepsilon\varphi(\tau) \int_0^1 \partial_y f(\tau, x(\tau), y^{s\varepsilon}(\tau)) f(\tau - 1 - s\varepsilon\varphi(\tau), y^{s\varepsilon}(\tau), y^\varepsilon(\tau - 1 - s\varepsilon\varphi(\tau))) \, ds
\end{aligned}
$$

and

$$\delta f(\tau, x(\tau), y^\varepsilon(\tau)) = O(\varepsilon). \qquad (2.3.1.16)$$

Let $\tau \in]t_i(\varepsilon), t_i + 1[$ for some $i \in \mathbb{Z}$. This means that the interval $]\tau - 1, \tau - 1 - \varepsilon\varphi(\tau)[$ contains just one discontinuity point t_i. Now

$$\begin{aligned} & \delta f(\tau, x(\tau), y^\varepsilon(\tau)) \qquad (2.3.1.17) \\ = & \int_0^1 \partial_y f(\tau, x(\tau), sy^\varepsilon(\tau) + (1-s)y^0(\tau))\, ds \cdot (x(\tau - 1 - \varepsilon\varphi(\tau)) - x(\tau - 1)), \end{aligned}$$

where

$$\begin{aligned} & x(\tau - 1 - \varepsilon\varphi(\tau)) - x(\tau - 1) \\ = & [x(\tau - 1 - \varepsilon\varphi(\tau)) - x(t_i - 0 \cdot \operatorname{sgn} \varphi(\tau))] \\ + & [x(t_i + 0 \cdot \operatorname{sgn} \varphi(\tau)) - x(\tau - 1)] - \Delta x(t_i) \cdot \operatorname{sgn} \varphi(\tau) \\ = & \int_0^1 f(\tau_\sigma^1, x(\tau_\sigma^1), y^\varepsilon(\tau_\sigma^1))\, d\sigma \cdot (\tau - 1 - \varepsilon\varphi(\tau) - t_i) \\ + & \int_0^1 f(\tau_\sigma^2, x(\tau_\sigma^2), y^\varepsilon(\tau_\sigma^2))\, d\sigma \cdot (t_i - \tau + 1) - I_i(x_i, y_i^\varepsilon) \cdot \operatorname{sgn} \varphi(\tau) \end{aligned}$$

and $\tau_\sigma^1 \equiv \sigma(\tau - 1 - \varepsilon\varphi(\tau)) + (1-\sigma)t_i \in]\tau - 1 - \varepsilon\varphi(\tau), t_i[$, $\tau_\sigma^2 \equiv \sigma t_i + (1 - \sigma)(\tau - 1) \in]t_i, \tau - 1[$. We shall use this representation in the proof of the contraction property of the operator $\mathcal{U}_\varepsilon$. For the moment it suffices to note that $\delta f(\tau, x(\tau), y^\varepsilon(\tau))$ is bounded.

Because of the ω–periodicity of the integrand in $\mathcal{I}_2 z(t)$ we have

$$\begin{aligned} \mathcal{I}_2 z(t) &= \int_1^{\omega+1} G(t,\tau)\delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau \\ &= \int_{\Delta_1^\varepsilon} G(t,\tau)\delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau \;+\; \int_{\Delta_2^\varepsilon} G(t,\tau)\delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau. \end{aligned}$$

By virtue of (2.3.1.16) and (2.3.1.11) both integrals can be estimated by $O(\varepsilon)$, thus

$$\mathcal{I}_2 z(t) = O(\varepsilon). \qquad (2.3.1.18)$$

From (2.3.1.14) and (2.3.1.18) it follows that we can choose $\tilde{\varepsilon}(\mu) \in (0, \varepsilon_0]$ so that for any $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$ we have

$$|\mathcal{I}_2 z(t) + \mathcal{S}_2 z(t)| \le \mu/2. \qquad (2.3.1.19)$$

Finally, by virtue of the estimates (2.3.1.12) and (2.3.1.19) we obtain

$$|\mathcal{U}_\varepsilon z(t)| \;\le\; \mu,$$

$i.e.$, the operator $\mathcal{U}_\varepsilon$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$.

Contraction property of the operator $\mathcal{U}_\varepsilon$. Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned} \mathcal{U}_\varepsilon z'(t) - \mathcal{U}_\varepsilon z''(t) &= (\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)) \\ +(\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)) &+ (\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)) + (\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)). \end{aligned}$$

First we consider

$$\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t) \equiv \sum_{i=1}^{m} G(t, t_i + 0)(J_i(z'_i, \bar{z}'_i) - J_i(z''_i, \bar{z}''_i)).$$

We have

$$\begin{aligned} & J_i(z'_i, \bar{z}'_i) - J_i(z''_i, \bar{z}''_i) \\ = \;& \big(I_i(\psi_i + z'_i, \bar{\psi}_i + \bar{z}'_i) - I_i(\psi_i + z''_i, \bar{\psi}_i + \bar{z}''_i)\big) - C_i(z'_i - z''_i) - D_i(\bar{z}'_i - \bar{z}''_i) \\ = \;& \int_0^1 \big(\partial_x I_i(\psi_i + s z'_i + (1-s) z''_i, \bar{\psi}_i + s\bar{z}'_i + (1-s)\bar{z}''_i) - \partial_x I_i(\psi_i, \bar{\psi}_i)\big)\, ds \cdot (z'_i - z''_i) \end{aligned}$$

$$+ \int_0^1 \big(\partial_y I_i(\psi_i + s z'_i + (1-s) z''_i, \bar{\psi}_i + s\bar{z}'_i + (1-s)\bar{z}''_i) - \partial_y I_i(\psi_i, \bar{\psi}_i)\big)\, ds \cdot (\bar{z}'_i - \bar{z}''_i),$$

thus

$$\begin{aligned} & |J_i(z'_i, \bar{z}'_i) - J_i(z''_i, \bar{z}''_i)| \\ \le \;& \int_0^1 L_2 \left[s(|z'_i| + |\bar{z}'_i|) + (1-s)(|z''_i| + |\bar{z}''_i|) \right] ds \cdot |z'_i - z''_i| \\ + \;& \int_0^1 L_2 \left[s(|z'_i| + |\bar{z}'_i|) + (1-s)(|z''_i| + |\bar{z}''_i|) \right] ds \cdot |\bar{z}'_i - \bar{z}''_i| \\ & \le 4\mu L_2 \|z' - z''\| \end{aligned}$$

and

$$|\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)| \le O(\mu) \|z' - z''\|. \tag{2.3.1.20}$$

Next,

$$\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t) = \int_0^\omega G(t, \tau) \left(Q(\tau, z'(\tau), \bar{z}'(\tau)) - Q(\tau, z''(\tau), \bar{z}''(\tau)) \right) d\tau.$$

For $\tau \neq t_i,\ \tau \neq t_i + 1$ we have

$$
\begin{aligned}
& Q(\tau, z'(\tau), \bar{z}'(\tau)) - Q(\tau, z''(\tau), \bar{z}''(\tau)) \\
= & \int_0^1 \big[\partial_x f \left(\tau, \psi(\tau) + s z'(\tau) + (1-s) z''(\tau), \bar{\psi}(\tau) + s\bar{z}'(\tau) + (1-s)\bar{z}''(\tau)\right) \\
& \quad - \partial_x f(\tau, \psi(\tau), \bar{\psi}(\tau))\big] \, ds \cdot (z'(\tau) - z''(\tau)) \\
+ & \int_0^1 \big[\partial_y f \left(\tau, \psi(\tau) + s z'(\tau) + (1-s) z''(\tau), \bar{\psi}(\tau) + s\bar{z}'(\tau) + (1-s)\bar{z}''(\tau)\right) \\
& \quad - \partial_y f(\tau, \psi(\tau), \bar{\psi}(\tau))\big] \, ds \cdot (\bar{z}'(\tau) - \bar{z}''(\tau)) \,,
\end{aligned}
$$

thus

$$
|Q(\tau, z'(\tau), \bar{z}'(\tau)) - Q(\tau, z''(\tau), \bar{z}''(\tau))| \leq 4\mu L_1 \, \|z' - z''\|
$$

and

$$
|\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)| \leq O(\mu) \|z' \, - \, z''\|. \tag{2.3.1.21}
$$

In order to estimate $\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)$ we use the representation (2.3.1.13). Let $x' = \psi(t) + z',\ x'' = \psi(t) + z'',\ y'^{\varepsilon}(t) = x'(t - 1 - \varepsilon\varphi(t))$, etc. Now

$$
\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t) = \sum_{i=1}^{m} G(t, t_i + 0) \left(\delta I_i(x'_i, y'^{\varepsilon}_i) \, - \, \delta I_i(x''_i, y''^{\varepsilon}_i)\right)
$$

and

$$
\begin{aligned}
& \delta I_i(x'_i, y'^{\varepsilon}_i) - \delta I_i(x''_i, y''^{\varepsilon}_i) \\
= \; - \; & \varepsilon\varphi(t_i) \int_0^1 [\partial_y \, I_i(x'_i, y'^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
- \; & \partial_y \, I_i(x''_i, y''^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))] \, ds.
\end{aligned}
$$

Further on,

$$
\begin{aligned}
& |\partial_y \, I_i(x'_i, y'^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
& \quad - \partial_y \, I_i(x''_i, y''^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \\
\leq \; & |\partial_y \, I_i(x'_i, y'^{s\varepsilon}_i) - \partial_y \, I_i(x''_i, y''^{s\varepsilon}_i)| \cdot |f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \\
+ \; & |\partial_y \, I_i(x''_i, y''^{s\varepsilon}_i)| \cdot |f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
& \quad - f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \; \leq \; O(1)\|z' - z''\|
\end{aligned}
$$

and thus

$$|\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \le O(\varepsilon)\|z' - z''\|. \qquad (2.3.1.22)$$

Similarly, in order to estimate $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ we use the representations (2.3.1.15) and (2.3.1.17). If $\tau \in \Delta_2^{\varepsilon} \setminus \{t_i\}_{i=1}^m$, then

$$|\delta f(\tau, x'(\tau), y'^{\varepsilon}(\tau)) - \delta f(\tau, x''(\tau), y''^{\varepsilon}(\tau))| \le O(\varepsilon)\|z' - z''\|.$$

If, however, $\tau \in]t_i(\varepsilon), t_i + 1[$ for some $i \in \mathbb{Z}$, we have

$$|\delta f(\tau, x'(\tau), y'^{\varepsilon}(\tau)) - \delta f(\tau, x''(\tau), y''^{\varepsilon}(\tau))| \le O(1)\|z' - z''\|$$

and as above we obtain

$$|\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)| \le O(\varepsilon)\|z' - z''\|. \qquad (2.3.1.23)$$

Choose an arbitrary number $q \in (0, 1)$. By virtue of (2.3.1.21) and (2.3.1.20) we can find $\mu_1 \in (0, \tilde{\mu}_0]$ so that for any $\mu \in (0, \mu_1]$ we have

$$|\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)| + |\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)| \le \frac{q}{2}\|z' - z''\|.$$

Next by virtue of (2.3.1.23) and (2.3.1.22) we find $\varepsilon_1 \in (0, \tilde{\varepsilon}(\mu_1)]$ so that for any $\varepsilon \in (0, \varepsilon_1]$ we have

$$|\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)| + |\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \le \frac{q}{2}\|z' - z''\|.$$

Then for any $\mu \in (0, \mu_1]$ and $\varepsilon \in [0, \varepsilon_1]$ the estimate

$$\|\mathcal{U}_\varepsilon z' - \mathcal{U}_\varepsilon z''\| \le q\|z' - z''\|, \qquad q \in (0, 1),$$

is valid for any $z', z'' \in \mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_\varepsilon$ has a unique fixed point in $\mathcal{T}_\mu$, which is an ω–periodic solution $z(t, \varepsilon)$ of system (2.3.1.9). Since $z(t) \equiv 0$ is the unique ω−periodic solution of system (2.3.1.9) for $\varepsilon = 0$, then $z(t, 0) \equiv 0$. Now $x(t, \varepsilon) = \psi(t) + z(t, \varepsilon)$ is the unique ω–periodic solution of system (2.3.1.1) and $x(t, 0) = \psi(t)$. This completes the proof of Theorem 2.3.1.1. □

The results of the present subsection were reported at the Conference on Biomathematics-Bioinformatics and Applications of Functional Differential-Difference Equations, Alanya, Turkey, 1999, and appeared in its proceedings [10].

2.3.2 Almost periodic solutions of retarded systems

Consider again system (2.3.1.1) satisfying the assumptions **A2.3.1.1–A2.3.1.3** and

A2.3.2.4. The function $\varphi(t)$ is ω_1–periodic, where ω_1/ω is irrational, and Lipschitz continuous:

$$|\varphi(t') - \varphi(t'')| \ \leq \ K|t' - t''|, \quad t', t'' \in \mathbb{R}.$$

If $\varepsilon_0 \leq \ \min \{1, \ 1/K\}$, then for $\varepsilon \in (0, \varepsilon_0)$ equation (2.3.1.2) has a unique solution $t_i(\varepsilon)$ for each $i \in \mathbb{Z}$. It obviously satisfies

$$|t_i(\varepsilon) - t_i - 1| \leq \varepsilon, \quad t_i(0) = t_i + 1.$$

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and

$$0 < t_1 < t_2 < \cdots < t_n < \omega.$$

We make two more assumptions:

A2.3.2.5. The generating system (2.3.1.3) has an $\omega-$periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

A2.3.2.6. $\left.\dfrac{\partial}{\partial y} f(t, \psi(t), y)\right|_{y=\psi(t-1)} = 0, \quad \left.\dfrac{\partial}{\partial y} I_i(\psi_i, y)\right|_{y=\bar{\psi}_i} = 0$ — zero matrices of dimension $(n \times n)$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned} \dot{z}(t) &= A(t)z(t), \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.2.1}$$

where

$$A(t) = \left.\frac{\partial}{\partial x} f(t, x, \psi(t-1))\right|_{x=\psi(t)}, \quad B_i = \left.\frac{\partial}{\partial x} I_i(x, \bar{\psi}_i)\right|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t, s)$ be the Cauchy matrix of (2.3.2.1), $X(t) = X(t, 0)$ be its fundamental solution [103]. Denote

$$\Lambda = \frac{1}{\omega} \ln X(\omega), \qquad \Phi(t) = X(t)e^{-\Lambda t}.$$

$\Phi(t)$ is an $\omega-$periodic piecewise continuous nondegenerate matrix-valued function, with points of discontinuity of the first kind at $t_i,\ i \in \mathbb{Z}$. Now we make two additional assumptions:

A2.3.2.7. The matrices $E + B_i,\ i \in \mathbb{Z}$, are nonsingular.

A2.3.2.8. The matrix Λ has no eigenvalues with real part zero.

Together with (2.3.2.1) we consider the nonhomogeneous system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + g(t), \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.2.2}$$

where $g(\cdot) \in AP_n(\{t_i\}_{i\in\mathbb{Z}})$ and $\{a_i\}_{i\in\mathbb{Z}} \in ap_n$. Under these assumptions system (2.3.2.2) has a unique almost periodic solution (see §1.3) given by

$$z_0(t) = \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)g(\tau)\,d\tau + \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)a_i. \tag{2.3.2.3}$$

Moreover, the estimates (1.3.1), (1.3.3) and (1.3.4) are valid.

Our result in the present subsection is the following

Theorem 2.3.2.1. *Let conditions* **A2.3.1.1 – A2.3.1.3** *and* **A2.3.2.4** – **A2.3.2.8** *hold. Then there exists a number* $\varepsilon_* > 0$ *such that for* $\varepsilon \in (0, \varepsilon_*)$ *system* (2.3.1.1) *has a unique almost periodic solution* $x(t,\varepsilon)$ *depending continuously on* ε *and such that* $x(t,\varepsilon) \to \psi(t)$ *as* $\varepsilon \to 0$.

Remark 2.3.2.1. Condition **A2.3.2.6** is of technical character. It enables us to apply Floquet's theory adapted for impulsive systems in [103]. Otherwise we would have to adapt the spectral decompositions given in [69] for impulsive systems and apply them to our case.

Proof of the main result. In system (2.3.1.1) we change the variables according to the formula

$$x = \psi(t) + z \tag{2.3.2.4}$$

and obtain the system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + Q(t, z(t), z(t-1)) + \delta f(t, x(t), y^{\varepsilon}(t)), \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i + J_i(z_i, \bar{z}_i) + \delta I_i(x_i, y_i^{\varepsilon}), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.2.5}$$

where

$$\begin{aligned} Q(t,z,\bar{z}) &\equiv f(t,\psi(t)+z,\psi(t-1)+\bar{z}) - f(t,\psi(t),\psi(t-1)) - A(t)z, \\ J_i(z_i,\bar{z}_i) &\equiv I_i(\psi_i+z_i,\bar{\psi}_i+\bar{z}_i) - I_i(\psi_i,\bar{\psi}_i) - B_i z_i \end{aligned}$$

are nonlinearities inherent to the generating system (2.3.1.3) and therefore independent of the fluctuation of the delay $\varepsilon\varphi(t)$, while

$$\begin{aligned} \delta f(t,x(t),y^\varepsilon(t)) &\equiv f(t,x(t),y^\varepsilon(t)) - f(t,x(t),y^0(t)), \\ \delta I_i(x_i,y_i^\varepsilon) &\equiv I_i(x_i,y_i^\varepsilon) - I_i(x_i,y_i^0) \end{aligned}$$

are increments due to this fluctuation.

We can formally consider (2.3.2.5) as a nonhomogeneous system of the form (2.3.2.2). Since ω_1/ω is irrational, the nonhomogeneities are almost periodic if $z(t)$ is almost periodic. Then its unique almost periodic solution $z(t)$ must satisfy an equality of the form (2.3.2.3) which in this case is the operator equation

$$z = \mathcal{U}_\varepsilon z, \tag{2.3.2.6}$$

where

$$\begin{aligned} \mathcal{U}_\varepsilon z(t) &\equiv \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)Q(\tau,z(\tau),z(\tau-1))\,d\tau \\ &+ \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)\delta f(\tau,x(\tau),y^\varepsilon(\tau))\,d\tau \\ &+ \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)J_i(z_i,\bar{z}_i) \\ &+ \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\delta I_i(x_i,y_i^\varepsilon) \\ &\equiv \mathcal{I}_1 z(t) \;+\; \mathcal{I}_2 z(t) \;+\; \mathcal{S}_1 z(t) \;+\; \mathcal{S}_2 z(t). \end{aligned}$$

For the sake of brevity we still write x instead of $\psi(t)+z$ in $\delta f(t,x(t),y^\varepsilon(t))$, $\delta I_i(x_i,y_i^\varepsilon)$ as well as in $\mathcal{I}_2 z$ and $\mathcal{S}_2 z$. Moreover, in we will further transform the expressions $\mathcal{I}_2 z(t)$ and $\mathcal{S}_2 z(t)$ under the assumption that $x(t)$ is a solution of system (2.3.1.1). This will considerably simplify some estimates henceforth.

An almost periodic solution $x(t) = x(t,\varepsilon)$ of system (2.3.1.1) corresponds to a fixed point z of the operator $\mathcal{U}_\varepsilon$ in a suitable set of almost periodic functions. To this end we shall prove that $\mathcal{U}_\varepsilon$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. Let us denote

$$\begin{aligned}
M_1 &= \sup\left\{|f(t,x,y)| :\ t \in [0,\omega],\ x,y \in \Omega_1\right\},\\
M_2 &= \sup\left\{|I_i(x,y)| :\ i = \overline{1,m},\ x,y \in \Omega_1\right\},\\
M_3 &= \sup\left\{|\partial_x f(t,x,y)| :\ t \in [0,\omega],\ x,y \in \Omega_1\right\},\\
M_4 &= \sup\left\{|\partial_y f(t,x,y)| :\ t \in [0,\omega],\ x,y \in \Omega_1\right\},\\
M_5 &= \sup\left\{|\partial_x I_i(x,y)| :\ i = \overline{1,m},\ x,y \in \Omega_1\right\},\\
M_6 &= \sup\left\{|\partial_y I_i(x,y)| :\ i = \overline{1,m},\ x,y \in \Omega_1\right\};\\
\mathcal{M} &= \sup\left\{\|\Phi(t)\|\|\Phi^{-1}(\tau)\| :\ t,\tau \in [0,\omega]\right\}.
\end{aligned}$$

Let L_1 and L_2 be respectively the greatest Lipschitz constants for the first derivatives of $f(t,x,y),\ t \in [0,\omega],\ x,y \in \Omega_1$, and of $I_i(x,y),\ i = \overline{1,m},\ x,y \in \Omega_1$, whose existence is provided by conditions **A1.3.2.1**, **A1.3.2.2** and the compactness of the set Ω_1.

For $a,b \in \mathbb{R}$ denote $]a,b[$ as in §2.3.1.

Define the "bad" set $\Delta_1^\varepsilon = \bigcup_{i\in\mathbb{Z}}]t_i(\varepsilon), t_i + 1[$. If $\varepsilon > 0$ is small enough, then Δ_1^ε is a disjoint union of intervals. We further define the "good" set $\Delta_2^\varepsilon = \mathbb{R} \setminus \Delta_1^\varepsilon$.

For the sake of convenience we assume that for $i = \overline{1,m}\quad t_i + 1 \neq t_j\ \forall j \in \mathbb{Z}$. Then for $\varepsilon > 0$ small enough the "bad" set Δ_1^ε contains none of the points $t_i,\ i \in \mathbb{Z}$.

Let $\varepsilon_0 > 0$ be so small that all the above assumptions are valid for $\varepsilon \in (0, \varepsilon_0)$.

For $\mu \in (0, \mu_0]$ define a set of functions

$$\mathcal{T}_\mu = \{\ z \in AP_n(\{t_i\}_{i\in\mathbb{Z}}) :\quad \|z\| \le \mu\ \}.$$

We shall find a dependence between ε and μ so that the operator $\mathcal{U}_\varepsilon$ in (2.3.2.6) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_\varepsilon$. Let $z \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_\varepsilon z(t)|$ using the representation

$$\mathcal{U}_\varepsilon z \;=\; \mathcal{I}_1 z \,+\, \mathcal{I}_2 z \,+\, \mathcal{S}_1 z \,+\, \mathcal{S}_2 z,$$

system (2.3.1.1), results from §1.3 and calculations from §2.3.1 when appropriate.

First we have

$$|J_i(z_i, \bar{z}_i)| \leq L_2(|z_i| + |\bar{z}_i|)^2/2$$

and

$$|\mathcal{S}_1 z(t)| \leq L_2 \mathcal{M} C \sum_{i\in\mathbb{Z}} e^{-\alpha|t-t_i|}(|z_i| + |\bar{z}_i|)^2/2 \leq \frac{4L_2\mathcal{M}\mathcal{C}\mu^2}{1-e^{-\alpha\theta}}. \qquad (2.3.2.7)$$

Similarly, for $\tau \neq t_i,\ \tau \neq t_i + 1$ we have

$$|Q(\tau, z(\tau), z(\tau-1))| \leq L_1(|z(\tau)| + |\bar{z}(\tau)|)^2/2$$

and

$$|\mathcal{I}_1 z(t)| \leq L_1 \mathcal{M} \int_{-\infty}^{\infty} \|G(t-\tau)\|(|z(\tau)| + |\bar{z}(\tau)|)^2\, d\tau/2 \leq 4L_1\mathcal{M}C\mu^2/\alpha. \qquad (2.3.2.8)$$

Further on, $\delta I_i(x_i, y_i^\varepsilon)$ admits the representation (2.3.1.13), thus $|\delta I_i(x_i, y_i^\varepsilon)| \leq \varepsilon M_1 M_6$ and

$$|\mathcal{S}_2 z(t)| \leq \varepsilon \mathcal{M} M_1 M_6 \frac{2C}{1-e^{-\alpha\theta}}. \qquad (2.3.2.9)$$

If $\tau \in \Delta_2^\varepsilon \setminus \{t_i\}_{i\in\mathbb{Z}}$, then $\delta f(\tau, x(\tau), y^\varepsilon(\tau))$ satisfies (2.3.1.15) and

$$|\delta f(\tau, x(\tau), y^\varepsilon(\tau))| \leq \varepsilon M_1 M_4. \qquad (2.3.2.10)$$

Let $\tau \in]t_i(\varepsilon), t_i + 1[$ for some $i \in \mathbb{Z}$. This means that the interval $]\tau-1, \tau - 1 - \varepsilon\varphi(\tau)[$ contains just one discontinuity point t_i. Now

$$\delta f(\tau, x(\tau), y^\varepsilon(\tau)) \qquad (2.3.2.11)$$

$$= \int_0^1 \partial_y f(\tau, x(\tau), s y^\varepsilon(\tau) + (1-s) y^0(\tau))\, ds \cdot (x(\tau - 1 - \varepsilon\varphi(\tau)) - x(\tau-1))$$

$$
\begin{aligned}
&= \int_0^1 \partial_y f(\tau, x(\tau), sy^\varepsilon(\tau) + (1-s)y^0(\tau))\, ds \\
&\times \left(\int_0^1 f(\tau_\sigma^1, x(\tau_\sigma^1), y^\varepsilon(\tau_\sigma^1))\, d\sigma \cdot (\tau - 1 - \varepsilon\varphi(\tau) - t_i) \right. \\
&+ \left. \int_0^1 f(\tau_\sigma^2, x(\tau_\sigma^2), y^\varepsilon(\tau_\sigma^2))\, d\sigma \cdot (t_i - \tau + 1) - I_i(x_i, y_i^\varepsilon) \cdot \operatorname{sgn} \varphi(\tau) \right),
\end{aligned}
$$

where $\tau_\sigma^1 \equiv \sigma(\tau - 1 - \varepsilon\varphi(\tau)) + (1-\sigma)t_i \in]\tau - 1 - \varepsilon\varphi(\tau), t_i[$ and $\tau_\sigma^2 \equiv \sigma t_i + (1-\sigma)(\tau - 1) \in]t_i, \tau - 1[$. Thus

$$
\begin{aligned}
|\delta f(\tau, x(\tau), y^\varepsilon(\tau))| &\leq M_4 \left(\varepsilon \max_{t \in]\tau-1, \tau-1-\varepsilon\varphi(\tau)[} |f(t, x(t), y^\varepsilon(t))| + |I_i(x_i, y_i^\varepsilon|\right) \\
&\leq M_4(\varepsilon M_1 + M_2). \qquad (2.3.2.12)
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathcal{I}_2 z(t) &= \int_{-\infty}^{\infty} \Phi(t) G(t-\tau) \Phi^{-1}(\tau) \delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau \\
&= \int_{\Delta_1^\varepsilon} \Phi(t) G(t-\tau) \Phi^{-1}(\tau) \delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau \\
&+ \int_{\Delta_2^\varepsilon} \Phi(t) G(t-\tau) \Phi^{-1}(\tau) \delta f(\tau, x(\tau), y^\varepsilon(\tau))\, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{I}_2 z(t)| \qquad (2.3.2.13) \\
&\leq C\mathcal{M} \left\{ \int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|} |\delta f(\tau, x(\tau), y^\varepsilon(\tau))|\, d\tau + \int_{\Delta_2^\varepsilon} e^{-\alpha|t-\tau|} |\delta f(\tau, x(\tau), y^\varepsilon(\tau))|\, d\tau \right\} \\
&\leq C\mathcal{M} \left\{ \int_{\Delta_1^\varepsilon} M_4(\varepsilon M_1 + M_2) e^{-\alpha|t-\tau|}\, d\tau + \int_{\Delta_2^\varepsilon} M_4 \varepsilon M_1 e^{-\alpha|t-\tau|}\, d\tau \right\} \\
&= C\mathcal{M} M_4 \left\{ \varepsilon M_1 \int_{-\infty}^{\infty} e^{-\alpha|t-\tau|}\, d\tau + M_2 \int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\, d\tau \right\}.
\end{aligned}
$$

But as in §1.3

$$
\int_{-\infty}^{\infty} e^{-\alpha|t-\tau|}\, d\tau = 2/\alpha,
$$

while

$$\int_{\Delta_1^\varepsilon} e^{-\alpha|t-\tau|}\,d\tau = \sum_{i\in\mathbb{Z}} \int\limits_{]t_i(\varepsilon),t_i+1[} e^{-\alpha|t-\tau|}\,d\tau \equiv \Sigma_\varepsilon(t).$$

To estimate $\Sigma_\varepsilon(t)$ we shall use that

$$\text{meas}\,]t_i(\varepsilon),t_i+1[\le\varepsilon \quad \text{and} \quad]t_i(\varepsilon),t_i+1[\subset[t_i+1-\varepsilon,t_i+1+\varepsilon]\ \forall i\in\mathbb{Z}.$$

For $t\in\mathbb{R}$ we shall consider two possibilities:

a) t belongs to none of the segments $[t_i+1-\varepsilon,t_i+1+\varepsilon],\ i\in\mathbb{Z}$. Then we may assume that $t_0+1+\varepsilon\le t\le t_1+1-\varepsilon$. Now for $i\in\mathbb{N}$ we have

$$\int\limits_{]t_i(\varepsilon),t_i+1[} e^{-\alpha|t-\tau|}\,d\tau \le \varepsilon e^{-\alpha[(t_i+1-\varepsilon)-(t_1+1-\varepsilon)]} = \varepsilon e^{-\alpha(t_i-t_1)} \le \varepsilon e^{-\alpha\theta(i-1)},$$

while for $i\notin\mathbb{N}$

$$\int\limits_{]t_i(\varepsilon),t_i+1[} e^{-\alpha|t-\tau|}\,d\tau \le \varepsilon e^{-\alpha[(t_0+1+\varepsilon)-(t_i+1+\varepsilon)]} = \varepsilon e^{-\alpha(t_0-t_i)} \le \varepsilon e^{\alpha\theta i},$$

and as in §1.3 we conclude that

$$\Sigma_\varepsilon(t) \le \frac{2\varepsilon}{1-e^{-\alpha\theta}}. \tag{2.3.2.14}$$

b) t belongs to one of these segments, say, $t_0+1-\varepsilon\le t\le t_0+1+\varepsilon$. Now for $i\in\mathbb{N}$ we have

$$\int\limits_{]t_i(\varepsilon),t_i+1[} e^{-\alpha|t-\tau|}\,d\tau \le \varepsilon e^{-\alpha[(t_i+1-\varepsilon)-(t_0+1+\varepsilon)]} = \varepsilon e^{-\alpha(t_i-t_0-2\varepsilon)} \le \varepsilon e^{2\alpha\varepsilon} e^{-\alpha\theta i},$$

while for $-i\in\mathbb{N}$

$$\int\limits_{]t_i(\varepsilon),t_i+1[} e^{-\alpha|t-\tau|}\,d\tau \le \varepsilon e^{-\alpha[(t_0+1-\varepsilon)-(t_i+1+\varepsilon)]} = \varepsilon e^{-\alpha(t_0-t_i-2\varepsilon)} \le \varepsilon e^{2\alpha\varepsilon} e^{\alpha\theta i}.$$

Finally,

$$\int\limits_{]t_0(\varepsilon),t_0+1[} e^{-\alpha|t-\tau|}\,d\tau \le \varepsilon$$

and

$$\Sigma_\varepsilon(t) \le \varepsilon \left(1 + 2e^{2\alpha\varepsilon} \sum_{i=1}^{\infty} e^{-i\alpha\theta}\right) = \varepsilon \left(1 + \frac{2e^{\alpha(2\varepsilon-\theta)}}{1-e^{-\alpha\theta}}\right). \qquad (2.3.2.15)$$

Combining the estimates (2.3.2.14) and (2.3.2.15), we have

$$\Sigma_\varepsilon(t) \le \frac{\varepsilon}{1-e^{-\alpha\theta}} \max\left\{2, 1 - e^{-\alpha\theta} + 2e^{\alpha(2\varepsilon-\theta)}\right\}.$$

For ε small enough, namely, for

$$\varepsilon \le \varepsilon_1 = \frac{\ln\left(1+e^{\alpha\theta}\right)}{2\alpha},$$

we have

$$\Sigma_\varepsilon(t) \le \frac{2\varepsilon}{1-e^{-\alpha\theta}} \quad \forall t \in \mathbb{R}.$$

Substituting this into (2.3.2.13), we obtain

$$|\mathcal{I}_2 z(t)| \le 2C\mathcal{M}M_4\varepsilon \left(\frac{M_1}{\alpha} + \frac{M_2}{1-e^{-\alpha\theta}}\right). \qquad (2.3.2.16)$$

Adding together the estimates (2.3.2.7), (2.3.2.8), (2.3.2.9) and (2.3.2.16), we obtain

$$|\mathcal{U}_\varepsilon z(t)| \le 2C\mathcal{M}(2K_1\mu^2 + K_2\varepsilon), \qquad (2.3.2.17)$$

where

$$K_1 = \frac{L_1}{\alpha} + \frac{L_2}{1-e^{-\alpha\theta}} \quad \text{and} \quad K_2 = \frac{M_1M_4}{\alpha} + \frac{M_2M_4 + M_1M_6}{1-e^{-\alpha\theta}}.$$

To provide the validity of the inequality $|\mathcal{U}_\varepsilon z(t)| \le \mu$, we first choose

$$\tilde{\mu}_0 = \min\left\{\mu_0,\ \frac{1}{8\mathcal{M}CK_1}\right\}.$$

Then for any $\mu \in (0, \tilde{\mu}_0]$ we have $4\mathcal{M}CK_1\mu^2 \le \mu/2$ and inequality (2.3.2.17) takes on the form

$$|\mathcal{U}_\varepsilon z(t)| \le \mu/2 + 2\mathcal{M}CK_2\varepsilon.$$

If we choose

$$\tilde{\varepsilon}(\mu) = \min\left\{\varepsilon_0,\ \varepsilon_1,\ \frac{\mu}{4\mathcal{M}CK_2}\right\},$$

then for any $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$ we have $2\mathcal{M}CK_2\varepsilon \le \mu/2$ and thus

$$|\mathcal{U}_\varepsilon z(t)| \le \mu,$$

i.e., the operator $\mathcal{U}_\varepsilon$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $\varepsilon \in (0, \tilde{\varepsilon}(\mu)]$.

Contraction property of the operator $\mathcal{U}_\varepsilon$. Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned} \mathcal{U}_\varepsilon z'(t) - \mathcal{U}_\varepsilon z''(t) &= (\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)) + (\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)) \\ &+ (\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)) + (\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)). \end{aligned}$$

First we consider

$$\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t) = \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\left(J_i(z_i', \bar{z}_i') - J_i(z_i'', \bar{z}_i'')\right).$$

We have

$$|J_i(z_i', \bar{z}_i') - J_i(z_i'', \bar{z}_i'')| \le 4\mu L_2 \|z' - z''\|$$

and

$$|\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)| \le \frac{8\mu\mathcal{M}CL_2}{1-e^{-\alpha\theta}}\|z' - z''\|. \qquad (2.3.2.18)$$

Next,

$$\begin{aligned} &\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t) \\ = &\int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)\left(Q(\tau, z'(\tau), \bar{z}'(\tau)) - Q(\tau, z''(\tau), \bar{z}''(\tau))\right) d\tau. \end{aligned}$$

For $\tau \ne t_i,\ \tau \ne t_i + 1$ we have

$$|Q(\tau, z'(\tau), \bar{z}'(\tau)) - Q(\tau, z''(\tau), \bar{z}''(\tau))| \le 4\mu L_1 \|z' - z''\|$$

and

$$|\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)| \le \frac{8\mu\mathcal{M}CL_1}{\alpha}\|z' - z''\|. \qquad (2.3.2.19)$$

In order to estimate $\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)$ we use the representation (2.3.1.13). Let $x' = \psi(t) + z',\ x'' = \psi(t) + z'',\ y'^{\varepsilon}(t) = x'(t-1-\varepsilon\varphi(t))$, etc. Now

$$\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t) = \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\left(\delta I_i(x_i', y_i'^{\varepsilon}) - \delta I_i(x_i'', y_i''^{\varepsilon})\right)$$

and

$$
\begin{aligned}
& \delta I_i(x'_i, y'^{\varepsilon}_i) - \delta I_i(x''_i, y''^{\varepsilon}_i) \\
= \;& -\varepsilon\varphi(t_i)\int_0^1 (\partial_y I_i(x'_i, y'^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
& - \partial_y I_i(x''_i, y''^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))) \, ds.
\end{aligned}
$$

Further on,

$$
\begin{aligned}
& |\partial_y I_i(x'_i, y'^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
& - \; \partial_y I_i(x''_i, y''^{s\varepsilon}_i) f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \\
\leq \;& |\partial_y I_i(x'_i, y'^{s\varepsilon}_i) - \partial_y I_i(x''_i, y''^{s\varepsilon}_i)| \cdot |f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \\
+ \;& |\partial_y I_i(x''_i, y''^{s\varepsilon}_i)| \cdot |f(t_i - 1 - s\varepsilon\varphi(t_i), y'^{s\varepsilon}_i, y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))) \\
& - \; f(t_i - 1 - s\varepsilon\varphi(t_i), y''^{s\varepsilon}_i, y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)))| \\
\leq \;& M_1 L_2(|x'_i - x''_i| \; + \; |y'^{s\varepsilon}_i - y''^{s\varepsilon}_i|) + M_6 \, (M_3 |y'^{s\varepsilon}_i - y''^{s\varepsilon}_i| \\
+ \;& M_4 \, |y'^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i)) - y''^{\varepsilon}(t_i - 1 - s\varepsilon\varphi(t_i))|) \\
& \leq \; [2M_1 L_2 + M_6(M_3 + M_4)] \|z' \; - \; z''\|
\end{aligned}
$$

and thus

$$
|\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \leq \frac{2\mathcal{M}C\varepsilon}{1 - e^{-\alpha\theta}} [2M_1 L_2 + M_6(M_3 \; + \; M_4)] \|z' - z''\|. \quad (2.3.2.20)
$$

Similarly, in order to estimate $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ we use the representations (2.3.1.15) and (2.3.2.11). If $\tau \in \Delta_2^{\varepsilon} \setminus \{t_i\}_{i \in \mathbb{Z}}$, then

$$
\begin{aligned}
& |\delta f(\tau, x'(\tau), y'^{\varepsilon}(\tau)) \; - \; \delta f(\tau, x''(\tau), y''^{\varepsilon}(\tau))| \\
& \leq \varepsilon \, [2L_1 M_1 \; + \; M_4(M_3 + M_4)] \, \|z' - z''\|.
\end{aligned}
$$

If, however, $\tau \in]t_i(\varepsilon), t_i + 1[$ for some $i \in \mathbb{Z}$, we have

$$
\begin{aligned}
& |\delta f(\tau, x'(\tau), y'^{\varepsilon}(\tau)) \; - \; \delta f(\tau, x''(\tau), y''^{\varepsilon}(\tau))| \\
\leq \;& \{\varepsilon \, [2L_1 M_1 + M_4(M_3 + M_4)] + 2L_1 M_2 + M_4(M_5 + M_6)\} \, \|z' - z''\|.
\end{aligned}
$$

As above we obtain

$$
|\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)| \quad (2.3.2.21)
$$

$$\leq \mathcal{M}C\left\{\int_{\Delta_1^\varepsilon}\{\varepsilon[2L_1M_1+M_4(M_3+M_4)]+2L_1M_2+M_4(M_5+M_6)\}\,e^{-\alpha|t-\tau|}\,d\tau\right.$$

$$\left.+\int_{\Delta_2^\varepsilon}\varepsilon[2L_1M_1\ +\ M_4(M_3+M_4)]e^{-\alpha|t-\tau|}\,d\tau\right\}\|z'-z''\|$$

$$\leq 2\varepsilon\mathcal{M}C\left\{\frac{2L_1M_1+M_4(M_3+M_4)}{\alpha}+\frac{2L_1M_2+M_4(M_5+M_6)}{1-e^{-\alpha\theta}}\right\}\|z'-z''\|.$$

Adding together the estimates (2.3.2.18), (2.3.2.19), (2.3.2.20) and (2.3.2.21), we obtain

$$\|\mathcal{U}_\varepsilon z'-\mathcal{U}_\varepsilon z''\|\leq 2\mathcal{M}C(4K_1\mu+K_3\varepsilon)\|z'-z''\|,$$

where

$$K_3=\frac{2L_1M_1+M_4(M_3+M_4)}{\alpha}+\frac{2(L_1M_2+L_2M_1)+M_3M_6+M_4M_5+2M_4M_6}{1-e^{-\alpha\theta}}.$$

Choose an arbitrary number $q\in(0,1)$ and denote $\mu_1=\min\left\{\mu_0,\ \dfrac{q}{16\mathcal{M}CK_1}\right\}$ (then obviously $\mu_1\leq\tilde{\mu}_0$) and $\varepsilon_1 = \min\left\{\tilde{\varepsilon}(\mu_1),\ \dfrac{q}{4\mathcal{M}CK_3}\right\}$. Then for any $\mu\in(0,\mu_1]$ and $\varepsilon\in[0,\varepsilon_1]$ we have

$$\|\mathcal{U}_\varepsilon z'-\mathcal{U}_\varepsilon z''\|\leq q\|z'-z''\|,\qquad q\in(0,1),$$

for any $z',z''\in\mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_\varepsilon$ has a unique fixed point in $\mathcal{T}_\mu$, which is an almost periodic solution $z(t,\varepsilon)$ of system (2.3.2.5). Since $z(t)\equiv 0$ is the unique almost periodic solution of system (2.3.2.5) for $\varepsilon=0$, then $z(t,0)\equiv 0$. Now $x(t,\varepsilon)=\psi(t)+z(t,\varepsilon)$ is the unique almost periodic solution of system (2.3.1.1) and $x(t,0)=\psi(t)$. This completes the proof of Theorem 2.3.2.1. □

The results of the present subsection were reported at the 27-th Summer School "Applications of Mathematics in Engineering and Economics", Sozopol, Bulgaria, 2001, and published in a short form in its proceedings [52] and in an extended form in [44].

2.3.3 Periodic solutions of neutral systems

In the present subsection we study a neutral system with impulses at fixed moments and a small delay of the argument of the derivative and another delay fluctuating around a constant value which may be assumed 1 without loss of generality:

$$\begin{aligned} \dot{x}(t) &= D(t)\dot{x}(t-h) + f(t, x(t), x(t-h), x(t-1-h\varphi(t))), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x(t_i), x(t_i - h)), \quad i \in \mathbb{Z}, \end{aligned} \qquad (2.3.3.1)$$

where $x \in \Omega \subset \mathbb{R}^n$, $D : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \times \Omega \times \Omega \times \Omega \to \mathbb{R}^n$, Ω is a domain in $\mathbb{R}^n$; $\varphi : \mathbb{R} \mapsto [-1, 1]$; $\Delta x(t_i)$ are the impulses at moments t_i and $\{t_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim_{i\to\pm\infty} t_i = \pm\infty$; $I_i : \Omega\times\Omega \to \mathbb{R}^n$ $(i \in \mathbb{Z})$, h and $1 + h\varphi(t)$ are the delays, $h \in [0, h_0)$ is a small parameter; h_0 will be specified below.

It is clear that, in general, the derivative $\dot{x}$ does not exist at the points of discontinuity of the right-hand side $f(t, x(t), x(t - h), x(t - 1 - h\varphi(t)))$, *i.e.*, at the points $t_i + kh$, $k \in \mathbb{N}$, and at points t which are solutions of the equations

$$t - 1 - h\varphi(t) = t_i, \qquad (2.3.3.2)$$

$i \in \mathbb{Z}$. We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_i.

For the sake of brevity we shall use the notation:

$$x_i = x(t_i), \quad \bar{x}(t) = x(t - h), \quad \tilde{x}(t) = x(t - 1), \quad y^h(t) = x(t - 1 - h\varphi(t))$$

(thus, for instance, $y^0_i = x(t_i - 1) = \tilde{x}_i$).

In the sequel we require the fulfillment of the following assumptions:

A2.3.3.1. The function $f(t, x, \bar{x}, y)$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω–periodic with respect to t, twice continuously differentiable with respect to $x, \bar{x}, y \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}, y$ second derivatives.

A2.3.3.2. The matrix $D(t)$ is ω– periodic, $\sup_{t\in[0,\omega]} |D(t)| = \eta < 1$, its first derivative is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and its second derivative is bounded on each interval of continuity.

A2.3.3.3. The functions $I_i(x, \bar{x}),\ i \in \mathbb{Z}$, are twice continuously differentiable with respect to $x, \bar{x} \in \Omega$, with locally Lipschitz continuous with respect to $x, \bar{x}$ second derivatives.

A2.3.3.4. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, \bar{x}) = I_i(x, \bar{x})$ for $i \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

A2.3.3.5. The function $\varphi(t)$ is ω–periodic and Lipschitz continuous:

$$|\varphi(t') - \varphi(t'')| \ \leq \ K|t' - t''|, \quad t', t'' \in \mathbb{R}.$$

We may note that the invertibility of the matrix $E - D(t)$ follows from the inequality $\eta < 1$ (condition **A2.3.3.2**). Moreover, $\sup\limits_{t \in [0,\omega]} |(E - D(t))^{-1}| \leq (1 - \eta)^{-1}$.

If $h_0 \leq \ \min\ \{1,\ 1/K\}$, then for $h \in (0, h_0)$ equation (2.3.3.2) has a unique solution $t_i(h)$ for each $i \in \mathbb{Z}$. It obviously satisfies

$$|t_i(h) - t_i - 1| \ \leq \ h, \quad t_i(0) = t_i + 1.$$

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and $t_i \neq 0\ \forall i \in \mathbb{Z}$.

For $h = 0$, from (2.3.3.1) we obtain

$$\begin{aligned} \dot{x}(t) &= (E - D(t))^{-1} f(t, x(t), x(t), x(t-1)), \quad t \neq t_i, \\ \Delta x(t_i) &= I_i(x_i, x_i), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.3.3}$$

so called *generating system*, and suppose that

A2.3.3.6. The generating system (2.3.3.3) has an ω–periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

Now define the linearized system with respect to $\psi(t)$:

$$\dot{z}(t) = (E - D(t))^{-1}(A(t)z(t) + B(t)z(t-1)),\ t \neq t_i, \tag{2.3.3.4}$$

$$\Delta z(t_i) = C_i z_i, \quad i \in \mathbb{Z}, \tag{2.3.3.5}$$

where

$$A(t) = \frac{\partial}{\partial x} f(t, x, x, \psi(t-1))\bigg|_{x=\psi(t)}, \quad B(t) = \frac{\partial}{\partial y} f(t, \psi(t), \psi(t), y)\bigg|_{y=\psi(t-1)},$$

and

$$C_i = \frac{\partial}{\partial x} I_i(x,x)\Big|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t)$ be the fundamental solution of the system (2.3.3.4) [20, 69] (*i.e.*, $X(t) = 0$ for $t < 0$, $X(0) = E$;

$$\dot{X}(t) = (E - D(t))^{-1}\big(A(t)X(t) + B(t)X(t-1)\big) \text{ for } t > 0).$$

Now we make two additional assumptions:

A2.3.3.7. The matrices $E + C_i,\ i \in \mathbb{Z}$, are nonsingular.

A2.3.3.8. The only ω–periodic solution of system (2.3.3.4), (2.3.3.5) is $z(t) \equiv 0$.

If the last two conditions hold, as in [36, 103] we can define *Green's function* $G(t,\tau)$ *of the periodic problem for the nonhomogeneous system*

$$\begin{aligned} \dot{z}(t) &= (E - D(t))^{-1}(A(t)z(t) + B(t)z(t-1)) + g(t), \quad t \neq t_i, \\ \Delta z(t_i) &= C_i z_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \qquad (2.3.3.6)$$

corresponding to (2.3.3.4), (2.3.3.5), where $g(\cdot) \in \tilde{C}_{\omega,n}$ and $a_{i+m} = a_i,\ i \in \mathbb{Z}$, *i.e.*, system (2.3.3.6) has a unique ω–periodic solution given by the formula

$$z(t) = \int_0^\omega G(t,\tau)g(\tau)\,d\tau + \sum_{0<t_i<\omega} G(t, t_i + 0)a_i. \qquad (2.3.3.7)$$

Denote also

$$\mathcal{M} = \sup\left\{|G(t,\tau)| :\ t,\tau \in [0,\omega]\right\}, \qquad \beta = \sum_{i=1}^{m} |C_i|.$$

Our result in the present subsection is the following

Theorem 2.3.3.1. *Let conditions* **A2.3.3.1**–**A2.3.3.8** *hold. If*

$$\eta(3 + 2\beta\mathcal{M}) < 1, \qquad (2.3.3.8)$$

then there exists a number $h_* > 0$ such that for $h \in (0, h_*)$ system (2.3.3.1) *has a unique ω–periodic solution $x(t,h)$ depending continuously on h and such that $x(t,h) \to \psi(t)$ as $h \to 0$.*

Proof. In system (2.3.3.1) we change the variables according to the formula

$$x = \psi(t) + z \tag{2.3.3.9}$$

and obtain the system

$$\begin{aligned}
\dot{z}(t) &= (E - D(t))^{-1}\big\{A(t)z(t) + B(t)z(t-1) \\
&+ Q(t, z(t), z(t-1)) + \delta f(t, x(t), x(t-h), y^h(t)) \\
&- D(t)(\dot{x}(t) - \dot{x}(t-h))\big\},\ t \neq t_i, \qquad (2.3.3.10) \\
\Delta z(t_i) &= C_i z_i + J_i(z_i) + \delta I_i(x_i, \bar{x}_i), \quad i \in \mathbb{Z},
\end{aligned}$$

where

$$\begin{aligned}
Q(t, z, \tilde{z}) &\equiv f(t, \psi(t) + z, \psi(t) + z, \psi(t-1) + \tilde{z}) \\
&- f(t, \psi(t), \psi(t), \psi(t-1)) - A(t)z - B(t)\tilde{z}, \\
J_i(z_i) &\equiv I_i(\psi_i + z_i, \psi_i + z_i) - I_i(\psi_i, \psi_i) - C_i z_i
\end{aligned}$$

are nonlinearities inherent to the generating system (2.3.3.3) and therefore independent of the small parameter h, while

$$\begin{aligned}
\delta f(t, x(t), \bar{x}(t), y^h(t)) &\equiv f(t, x(t), \bar{x}(t), y^h(t)) - f(t, x(t), x(t), y^0(t)), \\
\delta I_i(x_i, \bar{x}_i) &\equiv I_i(x_i, \bar{x}_i) - I_i(x_i, x_i)
\end{aligned}$$

are increments due to the presence of the small parameter.

We can formally consider (2.3.3.10) as a nonhomogeneous system of the form (2.3.3.6). Then its unique ω–periodic solution $z(t)$ must satisfy an equality of the form (2.3.3.7) which in this case is the operator equation

$$z = \mathcal{U}_h z - \mathcal{V}_h z, \tag{2.3.3.11}$$

where

$$\begin{aligned}
\mathcal{U}_h z(t) &\equiv \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} Q(\tau, z(\tau), z(\tau-1))\, d\tau \\
&+ \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} \delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))\, d\tau \\
&+ \sum_{0<t_i<\omega} G(t, t_i + 0) J_i(z_i) + \sum_{0<t_i<\omega} G(t, t_i + 0)\delta I_i(x_i, \bar{x}_i) \\
&\equiv \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t),
\end{aligned}$$

$$\mathcal{V}_h z(t) \equiv \int_0^\omega G(t,\tau)(E - D(\tau))^{-1} D(\tau)\big(\dot{x}(\tau) - \dot{x}(\tau - h)\big)\, d\tau. \qquad (2.3.3.12)$$

For brevity we still write x instead of $\psi(t) + z$ in $\delta f(t, x(t), \bar{x}(t), y^h(t))$, $\delta I_i(x_i, \bar{x}_i)$ as well as in $\mathcal{I}_2 z$, $\mathcal{S}_2 z$ and in $\mathcal{V}_h z$. Moreover, we will further estimate the expressions $\mathcal{I}_2 z(t)$ and $\mathcal{S}_2 z(t)$ under the assumption that $x(t)$ is a solution of system (2.3.3.1).

An ω–periodic solution $x(t) = x(t,h)$ of system (2.3.3.1) corresponds to a fixed point z of the operator $\mathcal{U}_h - \mathcal{V}_h$ in a suitable set of ω–periodic functions. To this end we shall prove that $\mathcal{U}_h - \mathcal{V}_h$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. Suppose, for the sake of definiteness, that

$$0 < t_1 < t_2 < \cdots < t_m < \omega.$$

There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. Let us denote

$$\begin{aligned} M_0 &= \max\Big\{ \sup\big\{|f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\}, \\ &\qquad \sup\big\{|I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}\Big\}, \\ M_1 &= \max\Big\{ \sup\big\{|\partial_x f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_y f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_x I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}, \\ &\qquad \sup\big\{|\partial_{\bar{x}} I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}\Big\} \end{aligned}$$

and, similarly, let M_2 be the maximum of the suprema of the matrices of the second derivatives of $f(t,x,\bar{x},y)$ with respect to $x, \bar{x}, y$ for $t \in [0,\omega],\ x,\bar{x},y \in \Omega_1$ and of the second derivatives of $I_i(x,\bar{x})$ for $i = \overline{1,m},\ x,y \in \Omega_1$. We shall not explicitly denote the Lipschitz constants for the second derivatives of $f(t,x,\bar{x},y)$ and $I_i(x,\bar{x})$.

We define the "bad" sets

$$\Delta_1^h = \bigcup_{i=1}^m (t_i, t_i + h), \qquad \Delta_2^h = \bigcup_{-1<t_i<\omega-1}]t_i(h), t_i + 1[.$$

We further define the "good" set $\Delta_3^h = [0,\omega] \setminus (\Delta_1^h \cup \Delta_2^h)$.

For the sake of convenience we assume that for $i = \overline{1,m} \quad t_i + 1 \neq t_j \ \forall j \in \mathbb{Z}$. Then for h small enough the "bad" set $\Delta_1^h \cup \Delta_2^h$ is a disjoint union of $2m$ intervals.

Let $h_0 > 0$ be so small that all the above assumptions are valid for $h \in (0, h_0)$.

For $\mu \in (0, \mu_0]$ define a set of functions

$$\mathcal{T}_\mu \ = \ \{ \ z \in \tilde{C}_{\omega,n} : \quad \|z\| \leq \mu \ \}.$$

We shall find a dependence between h and μ so that the operator $\mathcal{U}_h - \mathcal{V}_h$ in (2.3.3.11) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h - \mathcal{V}_h$. Let $z \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_h z(t)|$ using the representation

$$\mathcal{U}_h z(t) \ = \ \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t)$$

and system (2.3.3.1).

First we have

$$\begin{aligned} J_i(z_i) \ &= \ \left\{ \int_0^1 (\partial_x I_i(\psi_i + sz_i, \psi_i + sz_i) \ - \ \partial_x I_i(\psi_i, \psi_i)) \, ds \right. \\ &+ \ \left. \int_0^1 (\partial_{\bar{x}} I_i(\psi_i + sz_i, \psi_i + sz_i) \ - \ \partial_{\bar{x}} I_i(\psi_i, \psi_i)) \, ds \right\} z_i, \end{aligned}$$

thus

$$|J_i(z_i)| \leq 2 \int_0^1 2M_2 s|z_i| \, ds \cdot |z_i| = 2M_2 |z_i|^2$$

and

$$|\mathcal{S}_1 z(t)| \ \leq \ 2M_2 \mathcal{M} \sum_{i=1}^m |z_i|^2 \ = \ O(\mu^2). \qquad (2.3.3.13)$$

Similarly, we have

$$\begin{aligned} &Q(\tau, z(\tau), z(\tau-1)) \\ = \ &\left\{ \int_0^1 \big[\partial_x f(\tau, \psi(\tau) + sz(\tau), \psi(\tau) + sz(\tau), \tilde{\psi}(\tau) + s\tilde{z}(\tau)) \right. \\ &- \partial_x f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau)) \big] \, ds \end{aligned}$$

$$+\int_0^1 \big[\partial_{\bar{x}} f(\tau, \psi(\tau) + sz(\tau), \psi(\tau) + sz(\tau), \tilde{\psi}(\tau) + s\tilde{z}(\tau)) \\ - \partial_{\bar{x}} f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\big]\, ds \Big\}\, z(\tau)$$

$$+\int_0^1 \big[\partial_y f(\tau, \psi(\tau) + sz(\tau), \psi(\tau) + sz(\tau), \tilde{\psi}(\tau) + s\tilde{z}(\tau)) \\ - \partial_y f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\big]\, ds \cdot \tilde{z}(\tau),$$

thus

$$\begin{aligned} |Q(\tau, z(\tau), z(\tau-1))| &\le 2\int_0^1 M_2 s\big(2|z(\tau)| + |\tilde{z}(\tau)|\big)\, ds \cdot |z(\tau)| \\ &+ \int_0^1 M_2 s\big(2|z(\tau)| + |\tilde{z}(\tau)|\big)\, ds \cdot |\tilde{z}(\tau)| \\ &= M_2\big(2|z(\tau)| + |\tilde{z}(\tau)|\big)^2/2 \end{aligned}$$

and

$$|\mathcal{I}_1 z(t)| \le M_2 \mathcal{M}(1-\eta)^{-1} \int_0^\omega \big(2|z(\tau)| + |\tilde{z}(\tau)|\big)^2\, d\tau/2 = O(\mu^2). \quad (2.3.3.14)$$

Now let us estimate $|\dot{x}(t)|$, where $x(t)$ is a solution of (2.3.3.1). We have

$$|\dot{x}(t)| \le |D(t)||\dot{x}(t-h)| + |f(t, x(t), x(t-h), y^h(t))| \le \eta \sup|\dot{x}(t)| + M_0.$$

Thus

$$\sup|\dot{x}(t)| \le \eta \sup|\dot{x}(t)| + M_0$$

and, finally,

$$\sup|\dot{x}(t)| \le M_0(1-\eta)^{-1}.$$

Further on, since the intervals $(t_i - h, t_i)$ contain none of the points t_j, we have

$$\begin{aligned} \delta I_i(x_i, \bar{x}_i) &= \int_0^1 \frac{\partial}{\partial s} I_i(x_i, x(t_i - sh))\, ds \qquad (2.3.3.15) \\ &= \int_0^1 \partial_{\bar{x}} I_i(x_i, x(t_i - sh))\, \frac{\partial}{\partial s} x(t_i - sh)\, ds \\ &= -h\int_0^1 \partial_{\bar{x}}\, I_i(x_i, x(t_i - sh))\dot{x}(t_i - sh)\, ds, \end{aligned}$$

thus $|\delta I_i(x_i, \bar{x}_i)| \le hM_1M_0(1-\eta)^{-1}$ and

$$|\mathcal{S}_2 z(t)| \ \le \ hm\mathcal{M}M_1M_0(1-\eta)^{-1} \ = \ O(h). \tag{2.3.3.16}$$

Then we estimate the difference $x(t) - x(t-h)$. If t is not in Δ_1^h, then $x(t)$ is continuous in $[t-h, t]$ and $\dot{x}$ exists in this segment, with the possible exception of finitely many points. Then we have

$$|x(t) - x(t-h)| \le hM_0(1-\eta)^{-1}.$$

Now we shall obtain an analogous estimate for Δ_1^h. Then the interval $(t-h, t)$ contains just one point of discontinuity t_i of $x(t)$, thus

$$|x(t) - x(t-h)| \le |x(t) - x(t_i+0)| + |x(t_i+0) - x(t_i)| + |x(t_i) - x(t-h)|$$

$$\le M_0(1-\eta)^{-1}(t-t_i) + M_0 + M_0(1-\eta)^{-1}(t_i - t + h) \ = \ M_0(1 + h(1-\eta)^{-1}).$$

Next we estimate the difference $x(t-1) - x(t-1-h\varphi(t))$. If $t \notin \Delta_2^h$, then $x(t)$ is continuous in the interval $]t_i(h), t_i+1[$ and

$$|x(t-1) - x(t-1-h\varphi(t))| \le M_0 h(1-\eta)^{-1}.$$

Let $t \in \Delta_2^h$, *i.e.*, $t \in]t_i(h), t_i+1[$ for some $i \in \mathbb{Z}$. This means that the interval $]t-1, t-1-h\varphi(t)[$ contains just one discontinuity point t_i. Then we have

$$\begin{aligned}
& |x(t-1) - x(t-1-h\varphi(t))| \le |x(t-1) - x(t_i + \operatorname{sgn}\varphi(t))| \\
+ \ & |x(t_i + \operatorname{sgn}\varphi(t)) - x(t_i - \operatorname{sgn}\varphi(t))| + |x(t_i - \operatorname{sgn}\varphi(t)) - x(t-1-h\varphi(t))| \\
\le \ & \sup|\dot{x}(t)|(t-1-t_i)\operatorname{sgn}\varphi(t) + |I_i(x_i, \bar{x}_i)| \\
+ \ & \sup|\dot{x}(t)|(t_i - t + 1 + h\varphi(t))\operatorname{sgn}\varphi(t) \\
= \ & h\sup|\dot{x}(t)||\varphi(t)| + |I_i(x_i, \bar{x}_i)| \ \le \ M_0(1 + h(1-\eta)^{-1}).
\end{aligned}$$

Using these estimates, we evaluate $\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))$. If $\tau \in \Delta_3^h$, we have

$$\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \ = \ \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau))\, ds$$

$$= \ -h\int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau))\dot{x}(\tau - sh)\, ds$$

$$-h\varphi(\tau)\int_0^1 \partial_y f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau))\dot{x}(\tau - 1 - sh\varphi(\tau))\, ds$$

and

$$|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| \;\leq\; 2hM_0M_1(1-\eta)^{-1}. \qquad (2.3.3.17)$$

Next, if $\tau \in \Delta_1^h$, we have

$$\begin{aligned}
& \delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \\
= & \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), sx(\tau - h) + (1-s)x(\tau), y^{sh}(\tau))\, ds \\
= & \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), sx(\tau - h) + (1-s)x(\tau), y^{sh}(\tau))\, ds \cdot (x(\tau - h) - x(\tau)) \\
- & \; h\varphi(\tau) \int_0^1 \partial_y f(\tau, x(\tau), sx(\tau-h)+(1-s)x(\tau), y^{sh}(\tau))\dot{x}(\tau-1-sh\varphi(\tau))\, ds
\end{aligned}$$

and

$$\begin{aligned}
|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| &\leq M_1\big\{M_0(1 + h(1-\eta)^{-1}) + hM_0(1-\eta)^{-1}\big\} \\
&= M_0M_1(1 + 2h(1-\eta)^{-1}). \qquad (2.3.3.18)
\end{aligned}$$

Finally, for $\tau \in \Delta_2^h$

$$\begin{aligned}
& \delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \\
= & \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau))\, ds \\
= & -h \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau)) \cdot \dot{x}(\tau - sh)\, ds \\
+ & \int_0^1 \partial_y f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau))\, ds \\
& \quad \times \big(x(\tau - 1 - h\varphi(\tau)) - x(\tau - 1)\big)
\end{aligned}$$

and

$$\begin{aligned}
|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| &\leq M_1\big\{hM_0(1-\eta)^{-1} + M_0(1 + h(1-\eta)^{-1})\big\} \\
&\leq M_0M_1(1 + 2h(1-\eta)^{-1}). \qquad (2.3.3.19)
\end{aligned}$$

Making use of the estimates (2.3.3.17), (2.3.3.18) and (2.3.3.19), we find

$$|\mathcal{I}_2 z(t)| \;\leq\; \int_{\Delta_3^h} 2h\mathcal{M}M_0M_1(1-\eta)^{-2}\,d\tau \tag{2.3.3.20}$$

$$+\;\int_{\Delta_1^h} \mathcal{M}(1-\eta)^{-1}M_0M_1(1+2h(1-\eta)^{-1})\,d\tau$$

$$+\;\int_{\Delta_2^h} \mathcal{M}(1-\eta)^{-1}M_0M_1(1+2h(1-\eta)^{-1})\,d\tau$$

$$=\;\int_0^\omega 2h\mathcal{M}M_0M_1(1-\eta)^{-2}\,d\tau \;+\; \int_{\Delta_1^h\cup\Delta_2^h} \mathcal{M}M_0M_1(1-\eta)^{-1}\,d\tau$$

$$\leq\; 2\omega h\mathcal{M}M_0M_1(1-\eta)^{-2} + 2hm\mathcal{M}M_0M_1(1-\eta)^{-1} = O(h).$$

Adding together the estimates (2.3.3.13), (2.3.3.14), (2.3.3.16) and (2.3.3.20), we obtain

$$|\mathcal{U}_h z(t)| \;=\; O(\mu^2) + O(h). \tag{2.3.3.21}$$

Further on, as in §1.1.2 we obtain

$$|-\mathcal{V}_h z(t)| \leq ChM_0(1-\eta)^{-1},$$

where the constant C can be chosen independent of h, and

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| = O(\mu^2) + O(h),$$

i.e.,

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq K_1\mu^2 + K_2 h \tag{2.3.3.22}$$

for some positive constants K_1 and K_2.

To provide the validity of the inequality $|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq \mu$, we first choose

$$\tilde{\mu}_0 \;=\; \min\{\;\mu_0,\; \frac{1}{2K_1}\;\}.$$

Then for any $\mu \in (0, \tilde{\mu}_0]$ we have $K_1\mu^2 \;\leq\; \mu/2$ and inequality (2.3.3.22) takes on the form

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq \mu/2 \;+\; K_2 h.$$

If we choose

$$\tilde{h}(\mu) \; = \; \min \{ \; h_0, \; \frac{\mu}{2K_2} \; \},$$

then for any $h \in (0, \tilde{h}(\mu)]$ we have $K_2 h \; \leq \; \mu/2$ and thus

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \; \leq \; \mu,$$

i.e., the operator $\mathcal{U}_h - \mathcal{V}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h \in (0, \tilde{h}(\mu)]$.

Contraction property of the operator $\mathcal{U}_h - \mathcal{V}_h$. Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned} \mathcal{U}_h z'(t) - \mathcal{U}_h z''(t) \;\; &= \;\; (\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)) \; + \; (\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)) \\ &+ \;\; (\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)) \; + \; (\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)). \end{aligned}$$

First we consider

$$\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t) \; = \; \sum_{i=1}^{m} G(t, t_i + 0) \big(J_i(z'_i) \; - \; J_i(z''_i) \big).$$

We have

$$\begin{aligned} & J_i(z'_i) \; - \; J_i(z''_i) \\ = \;\; & \big(I_i(\psi_i + z'_i, \psi_i + z'_i) \; - \; I_i(\psi_i + z''_i, \psi_i + z''_i) \big) \; - \; C_i(z'_i - z''_i) \\ = \;\; & \Big\{ \int_0^1 \big(\partial_x I_i(\psi_i + s z'_i + (1-s) z''_i, \psi_i + s z'_i + (1-s) z''_i) \; - \; \partial_x I_i(\psi_i, \psi_i) \big) \, ds \\ + \;\; & \int_0^1 \big(\partial_{\bar{x}} I_i(\psi_i + s z'_i + (1-s) z''_i, \psi_i + s z'_i + (1-s) z''_i) - \partial_{\bar{x}} I_i(\psi_i, \psi_i) \big) \, ds \Big\} \\ & \times (z'_i - z''_i), \end{aligned}$$

thus

$$\begin{aligned} |J_i(z'_i) \; - \; J_i(z''_i)| \;\; &\leq \;\; 4M_2 \int_0^1 \big[s|z'_i| + (1-s)|z''_i| \big] \, ds \cdot |z'_i - z''_i| \\ &\leq \;\; 2M_2 \big(|z'_i| + |z''_i| \big) |z'_i - z''_i| \; \leq \; 4\mu M_2 |z'_i - z''_i| \end{aligned}$$

and

$$|\mathcal{S}_1 z'(t) \; - \; \mathcal{S}_1 z''(t)| \; \leq \; 4\mu \mathcal{M} m M_2 \|z' \; - \; z''\|. \tag{2.3.3.23}$$

Next,

$$\begin{aligned} & \mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t) \\ = \; & \int_0^\omega G(t,\tau)(E - D(\tau))^{-1}\big(Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))\big)\, d\tau. \end{aligned}$$

We have

$$\begin{aligned} & Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau)) \\ = \; & \Big\{ \int_0^1 \Big[\partial_x f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_x f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds \\ + \; & \int_0^1 \Big[\partial_{\tilde{x}} f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_{\tilde{x}} f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds \Big\} \\ & \times \big(z'(\tau) - z''(\tau)\big) \\ + \; & \int_0^1 \Big[\partial_y f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_y f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds \\ & \times \big(\tilde{z}''(\tau) - \tilde{z}''(\tau)\big), \end{aligned}$$

where $x_s(\tau) = \psi(\tau) + s z'(\tau) + (1 - s) z''(\tau)$. Thus

$$\begin{aligned} & |Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))| \\ \leq \; & 2M_2 \int_0^1 \big[s\big(2|z'(\tau)| + |\tilde{z}'(\tau)|\big) + (1 - s)\big(2|z''(\tau)| + |\tilde{z}''(\tau)|\big)\big]\, ds \\ & \times |z'(\tau) - z''(\tau)| \\ + \; & M_2 \int_0^1 \big[s\big(2|z'(\tau)| + |\tilde{z}'(\tau)|\big) + (1 - s)\big(2|z''(\tau)| + |\tilde{z}''(\tau)|\big)\big]\, ds \\ & \times |\tilde{z}'(\tau) - \tilde{z}''(\tau)| \\ \leq \; & 2M_2 \frac{1}{2} \cdot 3\big(\|z'\| + \|z''\|\big) \cdot \|z' - z''\| + M_2 \frac{1}{2} \cdot 3\big(\|z'\| + \|z''\|\big) \cdot \|z' - z''\| \\ & \leq 9\mu M_2 \, \|z' - z''\| \end{aligned}$$

and

$$|\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)| \leq 9\mu \mathcal{M} \omega M_2 (1 - \eta)^{-1} \, \|z' - z''\|. \tag{2.3.3.24}$$

For the estimation of $\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)$ and $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ we denote $x' = \psi(t) + z',\ x'' = \psi(t) + z'',\ y'^h(t) = x'(t - 1 - h\varphi(t))$, etc. Now

$$\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t) \ = \ \sum_{i=1}^{m} G(t, t_i + 0)(\delta I_i(x_i', \bar{x}_i') \ - \ \delta I_i(x_i'', \bar{x}_i''))$$

and

$$\begin{aligned}
& \delta I_i(x_i', \bar{x}_i') \ - \ \delta I_i(x_i'', \bar{x}_i'') \\
= \ & \big(I_i(x_i', \bar{x}_i') - I_i(x_i', x_i')\big) \ - \ \big(I_i(x_i'', \bar{x}_i'') - I_i(x_i'', x_i'')\big) \\
= \ & \big(I_i(x_i', \bar{x}_i') - I_i(\psi_i, \psi_i)\big) \ - \ \big(I_i(x_i', x_i') - I_i(\psi_i, \psi_i)\big) \\
- \ & \big(I_i(x_i'', \bar{x}_i'') - I_i(\psi_i, \psi_i)\big) \ + \ \big(I_i(x_i'', x_i'') - I_i(\psi_i, \psi_i)\big) \\
= \ - \ & \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s z_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i'))\, d(1-s) \\
+ \ & \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s z_i', \psi_i + s z_i')\, d(1-s) \\
+ \ & \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s z_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i''))\, d(1-s) \\
- \ & \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + s z_i'', \psi_i + s z_i'')\, d(1-s).
\end{aligned}$$

Making use of the continuity of the second derivatives of $I_i(x, \bar{x})$ (condition **A2.3.3.3**), we integrate by parts and rearrange the terms to obtain

$$\begin{aligned}
& \delta I_i(x_i', \bar{x}_i') \ - \ \delta I_i(x_i'', \bar{x}_i'') \\
= \ & \left\{ \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} I_i(x, x)|_{x = \psi_i + s z_i''}\, ds \cdot z_i'', z_i'' \right\rangle \right. \\
- \ & \left. \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} I_i(x, x)|_{x = \psi_i + s z_i'}\, ds \cdot z_i', z_i' \right\rangle \right\} \\
+ \ & \left\{ \left\langle \int_0^1 (1-s) \partial_{xx}^2 I_i(\psi_i + s z_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i'))\, ds \cdot z_i', z_i' \right\rangle \right. \\
- \ & \left. \left\langle \int_0^1 (1-s) \partial_{xx}^2 I_i(\psi_i + s z_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i''))\, ds \cdot z_i'', z_i'' \right\rangle \right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\Bigg\{\Bigg\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} I_i(\psi_i + s z_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i'))\, ds \cdot z_i', \bar{\psi}_i - \psi_i + \bar{z}_i' \Bigg\rangle \\
&- \quad \Bigg\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} I_i(\psi_i + s z_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i''))\, ds \cdot z_i'', \bar{\psi}_i - \psi_i + \bar{z}_i'' \Bigg\rangle \Bigg\} \\
&+ \quad \Bigg\{\Bigg\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} I_i(\psi_i + s z_i', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i'))\, ds \cdot (\bar{\psi}_i - \psi_i + \bar{z}_i'), \\
&\qquad\qquad \bar{\psi}_i - \psi_i + \bar{z}_i' \Bigg\rangle \\
&- \quad \Bigg\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} I_i(\psi_i + s z_i'', \psi_i + s(\bar{\psi}_i - \psi_i + \bar{z}_i''))\, ds \cdot (\bar{\psi}_i - \psi_i + \bar{z}_i''), \\
&\qquad\qquad \bar{\psi}_i - \psi_i + \bar{z}_i'' \Bigg\rangle \Bigg\}.
\end{aligned}
$$

Now we estimate separately the four addends in the braces making use also of the Lipschitz continuity of the second derivatives of $I_i(x_i, \bar{x}_i)$ according to condition **A2.3.3.3**. It is easy to see that the first two addends are estimated by $O(\mu)\|z' - z''\|$, while the other two terms are estimated by $(O(\mu) + O(h))\|z' - z''\|$. Thus we obtain

$$
|\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \;\le\; (O(\mu) + O(h))\|z' - z''\|. \tag{2.3.3.25}
$$

We estimate $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ in a similar way, using condition **A2.3.3.1**. Now we have (the argument τ is dropped for brevity)

$$
\begin{aligned}
&\delta f(\cdot, x', \bar{x}', y'^h) \;-\; \delta f(\cdot, x'', \bar{x}'', y''^h) \\
&= \quad \Bigg\{\Bigg\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x^2} f(\cdot, x, x, \tilde{\psi} + s\tilde{z}'')|_{x=\psi+sz''}\, ds \cdot z'', z'' \Bigg\rangle \\
&- \quad \Bigg\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x^2} f(\cdot, x, x, \tilde{\psi} + s\tilde{z}')|_{x=\psi+sz'}\, ds \cdot z', z' \Bigg\rangle \Bigg\} \\
&+ \quad 2\Bigg\{\Bigg\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x \partial y} f(\cdot, x, x, y)\Bigg|_{\substack{x=\psi+sz'' \\ y=\tilde{\psi}+s\tilde{z}''}} ds \cdot z'', \tilde{z}'' \Bigg\rangle \\
&- \quad \Bigg\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x \partial y} f(\cdot, x, x, y)\Bigg|_{\substack{x=\psi+sz' \\ y=\tilde{\psi}+s\tilde{z}'}} ds \cdot z', \tilde{z}' \Bigg\rangle \Bigg\}
\end{aligned}
$$

$$+ \quad \left\{ \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz'', \psi + sz'', \tilde{\psi} + s\tilde{z}'')\, ds \cdot z'', z'' \right\rangle \right.$$

$$\left. - \quad \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz', \psi + sz', \tilde{\psi} + s\tilde{z}')\, ds \cdot z', z' \right\rangle \right\}$$

$$+ \quad \left\{ \left\langle \int_0^1 (1-s)\partial^2_{xx} f(\cdot, \psi + sz', \psi + s(\bar{\psi} - \psi + \bar{z}'), \right.\right.$$

$$\left. \tilde{\psi} + s(\psi^h - \psi^0 + z'^h))\, ds \cdot z', z' \right\rangle$$

$$- \quad \left\langle \int_0^1 (1-s)\partial^2_{xx} f(\cdot, \psi + sz'', \psi + s(\bar{\psi} - \psi + \bar{z}''), \right.$$

$$\left.\left. \tilde{\psi} + s(\psi^h - \psi^0 + z''^h))\, ds \cdot z'', z'' \right\rangle \right\}$$

$$+ \quad 2 \left\{ \left\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} f(\cdot, \psi + sz', \psi + s(\bar{\psi} - \psi + \bar{z}'), \right.\right.$$

$$\left. \tilde{\psi} + s(\psi^h - \psi^0 + z'^h))\, ds \cdot z', \bar{\psi} - \psi + \bar{z}' \right\rangle$$

$$- \quad \left\langle \int_0^1 (1-s)\partial^2_{x\bar{x}} f(\cdot, \psi + sz'', \psi + s(\bar{\psi} - \psi + \bar{z}''), \right.$$

$$\left.\left. \tilde{\psi} + s(\psi^h - \psi^0 + z''^h))\, ds \cdot z'', \bar{\psi} - \psi + \bar{z}'' \right\rangle \right\}$$

$$+ \quad \left\{ \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz', \psi + s(\bar{\psi} - \psi + \bar{z}'), \right.\right.$$

$$\left. \tilde{\psi} + s(\psi^h - \psi^0 + z'^h))\, ds \cdot (\bar{\psi} - \psi + \bar{z}'), \bar{\psi} - \psi + \bar{z}' \right\rangle$$

$$- \quad \left\langle \int_0^1 (1-s)\partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz'', \psi + s(\bar{\psi} - \psi + \bar{z}''), \right.$$

$$\left.\left. \tilde{\psi} + s(\psi^h - \psi^0 + z''^h))\, ds \cdot (\bar{\psi} - \psi + \bar{z}''), \bar{\psi} - \psi + \bar{z}'' \right\rangle \right\}$$

$$
\begin{aligned}
&+ \quad 2\Bigg\{\Bigg\langle\int_0^1 (1-s)\partial^2_{xy} f(\cdot, \psi + sz', \psi + s(\bar\psi - \psi + \bar z'),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z'^h))\, ds \cdot z', \psi^h - \psi^0 + z'^h\Bigg\rangle\\
&- \quad \Bigg\langle\int_0^1 (1-s)\partial^2_{xy} f(\cdot, \psi + sz'', \psi + s(\bar\psi - \psi + \bar z''),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z''^h))\, ds \cdot z'', \psi^h - \psi^0 + z''^h\Bigg\rangle\Bigg\}\\
&+ \quad 2\Bigg\{\Bigg\langle\int_0^1 (1-s)\partial^2_{\bar x y} f(\cdot, \psi + sz', \psi + s(\bar\psi - \psi + \bar z'),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z'^h))\, ds \cdot (\bar\psi - \psi + \bar z'), \psi^h - \psi^0 + z'^h\Bigg\rangle\\
&- \quad \Bigg\langle\int_0^1 (1-s)\partial^2_{\bar x y} f(\cdot, \psi + sz'', \psi + s(\bar\psi - \psi + \bar z''),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z''^h))\, ds \cdot (\bar\psi - \psi + \bar z''), \psi^h - \psi^0 + z''^h\Bigg\rangle\Bigg\}\\
&+ \quad \Bigg\{\Bigg\langle\int_0^1 (1-s)\partial^2_{yy} f(\cdot, \psi + sz', \psi + s(\bar\psi - \psi + \bar z'),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z'^h))\, ds \cdot (\psi^h - \psi^0 + z'^h), \psi^h - \psi^0 + z'^h\Bigg\rangle\\
&- \quad \Bigg\langle\int_0^1 (1-s)\partial^2_{yy} f(\cdot, \psi + sz'', \psi + s(\bar\psi - \psi + \bar z''),\\
&\qquad\qquad \tilde\psi + s(\psi^h - \psi^0 + z''^h))\, ds \cdot (\psi^h - \psi^0 + z''^h), \psi^h - \psi^0 + z''^h\Bigg\rangle\Bigg\}.
\end{aligned}
$$

The first four expressions in the braces are estimated by $O(\mu)\|z' - z''\|$ for all $\tau \in [0, \omega]$. The fifth and sixth expressions are estimated by

$$
\begin{cases} (O(\mu) + O(h))\|z' - z''\| & \text{for} \quad \tau \notin \Delta_1^h, \\ (O(\mu) + O(1))\|z' - z''\| & \text{for} \quad \tau \in \Delta_1^h. \end{cases}
$$

The seventh and ninth expressions are estimated by

$$
\begin{cases} (O(\mu) + O(h))\|z' - z''\| & \text{for} \quad \tau \notin \Delta_2^h, \\ (O(\mu) + O(1))\|z' - z''\| & \text{for} \quad \tau \in \Delta_2^h. \end{cases}
$$

Finally, the eighth expression is estimated by

$$\begin{cases} (O(\mu)+O(h))\|z'-z''\| & \text{for} \quad \tau\in\Delta_3^h, \\ (O(\mu)+O(1))\|z'-z''\| & \text{for} \quad \tau\in(\Delta_1^h\cup\Delta_2^h). \end{cases}$$

Using these estimates, by arguments similar to those in the proof of Lemma 2.1.2.1 we find

$$|\mathcal{I}_2 z'(t)-\mathcal{I}_2 z''(t)|\le (O(\mu)+O(h))\|z' - z''\|. \tag{2.3.3.26}$$

Now by virtue of the estimates (2.3.3.23), (2.3.3.24), (2.3.3.25) and (2.3.3.26) we obtain

$$\|\mathcal{U}_h z' - \mathcal{U}_h z''\| \le (O(\mu)+O(h))\|z' - z''\|.$$

In order to estimate $\mathcal{V}_h z' - \mathcal{V}_h z''$, we integrate by parts the expression for $\mathcal{V}_h z(t)$ taking into account that the function $G(t,\tau)$ is discontinuous at $\tau=t_1,\dots,t_m$ and $\tau=t$ while $x(\tau)$ is discontinuous at $t_1,\dots,t_m$ making use of the equalities

$$\frac{\partial G(t,\tau)}{\partial\tau}=-G(t,\tau)(E-D(\tau))^{-1}[A(\tau)+B(\tau)X(\tau-1)X^{-1}(\tau)]$$

and $G(t,t_i)=G(t,t_i+0)(E+C_i)$. We obtain

$$\begin{aligned} \mathcal{V}_h z(t) \ =\ & (E-D(t))^{-1}D(t)\big(x(t)-x(t-h)\big) \\ +\ & \sum_{i=1}^m\Big\{G(t,t_i+h)(E-D(t_i+h))^{-1}D(t_i+h) \\ & \quad -G(t,t_i+0)(E-D(t_i))^{-1}D(t_i)\Big\}I_i(x_i,\bar{x}_i) \\ +\ & \sum_{i=1}^m G(t,t_i+0)C_i(E-D(t_i))^{-1}D(t_i)(x_i-\bar{x}_i) \\ -\ & \int_0^\omega\Big\{G(t,\tau)(E-D(\tau))^{-1}\big[A(\tau)+B(\tau)X(\tau-1)X^{-1}(\tau)\big] \\ & \quad\times(E-D(\tau))^{-1}D(\tau)\ -\ G(t,\tau-h)(E-D(\tau-h))^{-1} \\ & \quad\times\big[A(\tau-h)+B(\tau-h)X(\tau-h-1)X^{-1}(\tau-h)\big] \\ & \qquad\times(E-D(\tau-h))^{-1}D(\tau-h)\Big\}x(\tau-h)\,d\tau \\ +\ & \int_0^\omega\Big\{G(t,\tau)(E-D(\tau))^{-1}\dot{D}(\tau)(E-D(\tau))^{-1} \\ & \quad -G(t,\tau-h)(E-D(\tau-h))^{-1}\dot{D}(\tau-h)(E-D(\tau-h))^{-1}\Big\}x(\tau-h)\,d\tau. \end{aligned}$$

In view of Lemma 2.1.2.1 the two integral terms are estimated by $O(h)\|x\|$. The sum of the first and third terms can be estimated by $2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}\|x\|$. The difference

$$G(t,t_i+h)(E-D(t_i+h))^{-1}D(t_i+h)-G(t,t_i+0)(E-D(t_i))^{-1}D(t_i)$$

is estimated by $O(h)$ for $t\notin\Delta_1^h$, and by $\eta(1-\eta)^{-1}+O(h)$ otherwise. At last, similarly to estimate (2.3.3.25), we note that

$$|I_i(x_i',\bar{x}_i')-I_i(x_i'',\bar{x}_i'')|\le\big(O(\mu)+O(h)\big)\|z'-z''\|,$$

thus we have

$$|\mathcal{V}_h z'(t)-\mathcal{V}_h z''(t)|\le\big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}+O(\mu)+O(h)\big)\|z'-z''\|$$

and

$$\begin{aligned}&|\big(\mathcal{U}_h z'(t)-\mathcal{V}_h z'(t)\big)-\big(\mathcal{U}_h z''(t)-\mathcal{V}_h z''(t)\big)|\\ \le\;&\big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}+\gamma_1\mu+\gamma_2 h\big)\|z'-z''\|,\end{aligned}$$

where γ_1 and γ_2 are some positive constants.

By condition (2.3.3.8) we have

$$\tilde{\eta}\equiv 2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}<1.$$

Choose a number $q\in(\tilde{\eta},1)$ and denote $r=q-\tilde{\eta}$, and $\mu_1=\min\{\tilde{\mu}_0,\,\frac{r}{2\gamma_1}\}$ and $h_1=\min\{\tilde{h}(\mu_1),\,\frac{r}{2\gamma_2}\}$. Then for any $\mu\in(0,\mu_1]$ and $h\in[0,h_1]$ we have

$$\|\big(\mathcal{U}_h z'-\mathcal{V}_h z'\big)-\big(\mathcal{U}_h z''-\mathcal{V}_h z''\big)\|\le q\|z'-z''\|,\qquad q\in(0,1),$$

for any $z',z''\in\mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_h-\mathcal{V}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an ω–periodic solution $z(t,h)$ of system (2.3.3.10). Since $z(t)\equiv 0$ is the unique ω–periodic solution of system (2.3.3.10) for $h=0$, then $z(t,0)\equiv 0$. Now $x(t,h)=\psi(t)+z(t,h)$ is the unique ω–periodic solution of system (2.3.3.1) and $x(t,0)=\psi(t)$. This completes the proof of Theorem 2.3.3.1. □

The results of the present subsection were reported at the International Conference on Functional Differential Equations and Applications, Beer-Sheva, Israel, 2002, and published in [46].

2.3.4 Almost periodic solutions of neutral systems

Consider again system (2.3.3.1) satisfying the assumptions **A2.3.3.1**–**A2.3.3.4**, **A2.3.2.4**, **A2.3.3.6** and

A2.3.4.1 $\left.\dfrac{\partial}{\partial y} f(t, \psi(t), \psi(t), y)\right|_{y=\psi(t-1)} = 0$ — the zero matrix of dimension $(n \times n)$.

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and

$$0 < t_1 < t_2 < \cdots < t_n < \omega.$$

Now define the linearized system with respect to $\psi(t)$:

$$\begin{aligned} \dot{z}(t) &= (E - D(t))^{-1} A(t) z(t), \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.4.1}$$

where

$$A(t) = \left.\frac{\partial}{\partial x} f(t, x, x, \psi(t-1))\right|_{x=\psi(t)}, \quad \text{and} \quad B_i = \left.\frac{\partial}{\partial x} I_i(x, x)\right|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t, s)$ be the Cauchy matrix of (2.3.4.1), and let $X(t) = X(t, 0)$ be its fundamental solution [103]. Denote

$$\Lambda = \frac{1}{\omega} \ln X(\omega), \qquad \Phi(t) = X(t) e^{-\Lambda t}.$$

$\Phi(t)$ is an ω–periodic piecewise-continuous nondegenerate matrix-valued function, with points of discontinuity of the first kind at $t_i,\ i \in \mathbb{Z}$. Now we make two additional assumptions **A2.3.2.7** and **A2.3.2.8**.

Together with (2.3.4.1) we consider the nonhomogeneous system

$$\begin{aligned} \dot{z}(t) &= (E - D(t))^{-1} \left(A(t) z(t) + g(t) \right), \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i + a_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.4.2}$$

where $g(\cdot) \in AP_n(\{t_i\}_{i \in \mathbb{Z}})$ and $\{a_i\}_{i \in \mathbb{Z}} \in ap_n$. Under these assumptions system (2.3.4.2) has a unique almost periodic solution (see §1.3) given by

$$z_0(t) = \int_{-\infty}^{\infty} \Phi(t) G(t-\tau) \Phi^{-1}(\tau) (E - D(\tau))^{-1} g(\tau)\, d\tau + \sum_{i \in \mathbb{Z}} \Phi(t) G(t - t_i) \Phi^{-1}(t_i) a_i. \tag{2.3.4.3}$$

Moreover, the estimates (1.3.1), (1.3.3) and (1.3.4) are valid.

Denote also

$$\mathcal{M} = \sup \left\{ \|\Phi(t)\| \|\Phi^{-1}(\tau)\| : \ t, \tau \in [0, \omega] \right\}, \qquad \beta = \sup_{t\in\mathbb{R}} \sum_{i\in\mathbb{Z}} \|G(t-t_i)\| \, |B_i|.$$

In fact, the last supremum does not exceed

$$\frac{2C}{1-e^{-\alpha\theta}} \sup_{i\in\mathbb{Z}} |B_i|.$$

Our result in the present subsection is the following

Theorem 2.3.4.1. *Let conditions* **A2.3.3.1**–**A2.3.3.4**, **A2.3.2.4**, **A2.3.3.6**, **A2.3.4.1**, **A2.3.2.7** *and* **A2.3.2.8** *hold. If*

$$\eta(3 + 2\beta\mathcal{M}) < 1, \tag{2.3.4.4}$$

then there exists a number $h_* > 0$ such that for $h \in (0, h_*)$ system (2.3.3.1) *has a unique almost-periodic solution $x(t,h)$ depending continuously on h and such that $x(t,h) \to \psi(t)$ as $h \to 0$.*

Remark 2.3.4.1. Condition **A2.3.4.1** is of technical character. It enables us to apply Floquet's theory adapted for impulsive systems in [103]. Otherwise we would have to adapt the spectral decompositions given in [69] for impulsive systems and apply them to our case.

Proof of the main result. In system (2.3.3.1) we change the variables according to the formula

$$x = \psi(t) + z \tag{2.3.4.5}$$

and obtain the system

$$\begin{aligned} \dot{z}(t) &= (E - D(t))^{-1}\big\{A(t)z(t) + Q(t, z(t), z(t-1)) \\ &+ \delta f(t, x(t), x(t-h), y^h(t)) - D(t)(\dot{x}(t) - \dot{x}(t-h))\big\}, \quad t \neq t_i, \\ \Delta z(t_i) &= B_i z_i + J_i(z_i) + \delta I_i(x_i, \bar{x}_i), \quad i \in \mathbb{Z}, \end{aligned} \tag{2.3.4.6}$$

where

$$\begin{aligned} Q(t, z, \tilde{z}) &\equiv f(t, \psi(t) + z, \psi(t) + z, \psi(t-1) + \tilde{z}) \\ &- f(t, \psi(t), \psi(t), \psi(t-1)) - A(t)z, \\ J_i(z_i) &\equiv I_i(\psi_i + z_i, \psi_i + z_i) - I_i(\psi_i, \psi_i) - B_i z_i \end{aligned}$$

are nonlinearities inherent to the generating system (2.3.3.3) and therefore independent of the small parameter h, while

$$\begin{aligned} \delta f(t,x(t),\bar{x}(t),y^h(t)) &\equiv f(t,x(t),\bar{x}(t),y^h(t)) - f(t,x(t),x(t),y^0(t)), \\ \delta I_i(x_i,\bar{x}_i) &\equiv I_i(x_i,\bar{x}_i) - I_i(x_i,x_i) \end{aligned}$$

are increments due to the presence of the small parameter.

We can formally consider (2.3.4.6) as a nonhomogeneous system of the form (2.3.4.2). Then its unique almost periodic solution $z(t)$ must satisfy an equality of the form (2.3.4.3) which in this case is the operator equation

$$z = \mathcal{U}_h z - \mathcal{V}_h z, \tag{2.3.4.7}$$

where

$$\begin{aligned} \mathcal{U}_h z(t) &\equiv \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}Q(\tau,z(\tau),z(\tau-1))\,d\tau \\ &+ \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}\delta f(\tau,x(\tau),\bar{x}(\tau),y^h(\tau))\,d\tau \\ &+ \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)J_i(z_i) + \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\delta I_i(x_i,\bar{x}_i) \\ &\equiv \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t), \end{aligned}$$

$$\mathcal{V}_h z(t) \equiv \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}D(\tau)\big(\dot{x}(\tau)-\dot{x}(\tau-h)\big)\,d\tau. \tag{2.3.4.8}$$

For the sake of brevity we still write x instead of $\psi(t)+z$ in $\delta f(t,x(t),\bar{x}(\tau),y^h(t))$, $\delta I_i(x_i,\bar{x}_i)$ as well as in $\mathcal{I}_2 z$, $\mathcal{S}_2 z$ and in $\mathcal{V}_h z$. Moreover, we will further transform the expressions $\mathcal{I}_2 z(t)$ and $\mathcal{S}_2 z(t)$ under the assumption that $x(t)$ is a solution of system (2.3.3.1). This will considerably simplify some estimates henceforth.

An almost periodic solution $x(t) = x(t,h)$ of system (2.3.3.1) corresponds to a fixed point z of the operator $\mathcal{U}_h - \mathcal{V}_h$ in a suitable set of almost periodic functions. To this end we shall prove that $\mathcal{U}_h - \mathcal{V}_h$ maps a suitably chosen set into itself as a contraction.

We first need to introduce some notation. There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit

$\{x=\psi(t);\ t\in\mathbb{R}\}$. Let us denote

$$\begin{aligned} M_0 &= \max\Big\{\sup\big\{|f(t,x,\bar{x},y)|:\ t\in[0,\omega],\ x,\bar{x},y\in\Omega_1\big\},\\ &\qquad \sup\big\{|I_i(x,\bar{x})|:\ i=\overline{1,m},\ x,\bar{x}\in\Omega_1\big\}\Big\},\\ M_1 &= \max\Big\{\sup\big\{|\partial_x f(t,x,\bar{x},y)|:\ t\in[0,\omega],\ x,\bar{x},y\in\Omega_1\big\},\\ &\qquad \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x},y)|:\ t\in[0,\omega],\ x,\bar{x},y\in\Omega_1\big\},\\ &\qquad \sup\big\{|\partial_y f(t,x,\bar{x},y)|:\ t\in[0,\omega],\ x,\bar{x},y\in\Omega_1\big\},\\ &\qquad \sup\big\{|\partial_x I_i(x,\bar{x})|:\ i=\overline{1,m},\ x,\bar{x}\in\Omega_1\big\},\\ &\qquad \sup\big\{|\partial_{\bar{x}} I_i(x,\bar{x})|:\ i=\overline{1,m},\ x,y\in\Omega_1\big\}\Big\} \end{aligned}$$

and, similarly, let M_2 be the maximum of the suprema of the matrices of the second derivatives of $f(t,x,\bar{x},y)$ with respect to $x,\bar{x},y$ for $t\in[0,\omega],\ x,\bar{x},y\in\Omega_1$ and of the second derivatives of $I_i(x,\bar{x})$ for $i=\overline{1,m},\ x,y\in\Omega_1$. We shall not explicitly denote the Lipschitz constants for the second derivatives of $f(t,x,\bar{x},y)$ and $I_i(x,\bar{x})$.

We define the "bad" sets

$$\Delta_1^h=\bigcup_{i\in\mathbb{Z}}(t_i,t_i+h), \qquad \Delta_2^h=\bigcup_{i\in\mathbb{Z}}]t_i(h),t_i+1[.$$

We further define the "good" set $\Delta_3^h=[0,\omega]\setminus(\Delta_1^h\cup\Delta_2^h)$.

For the sake of convenience we assume that for $i=\overline{1,m}\quad t_i+1\neq t_j\ \forall j\in\mathbb{Z}$. Then for h small enough the "bad" set $\Delta_1^h\cup\Delta_2^h$ is a disjoint union of intervals.

Let $h_0>0$ be so small that all the above assumptions are valid for $h\in(0,h_0)$.

For $\mu\in(0,\mu_0]$ define a set of functions

$$\mathcal{T}_\mu\ =\ \{\ z\in AP_n:\quad \|z\|\le\mu\ \}.$$

We shall find a dependence between h and μ so that the operator $\mathcal{U}_h-\mathcal{V}_h$ in (2.3.4.7) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h-\mathcal{V}_h$. Let $z\in\mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_h z(t)|$ using the representation

$$\mathcal{U}_h z(t)\ =\ \mathcal{I}_1 z(t)+\mathcal{I}_2 z(t)+\mathcal{S}_1 z(t)+\mathcal{S}_2 z(t),$$

system (2.3.3.1), estimates from §1.3 and calculations from §2.3.3 when appropriate.

First, as in §2.3.3 we have

$$|J_i(z_i)| \leq 2M_2|z_i|^2$$

and

$$\begin{aligned} |\mathcal{S}_1 z(t)| &\leq \sum_{i\in\mathbb{Z}} \|\Phi(t)G(t-t_i)\Phi^{-1}(t_i)\|\,\|J_i(z_i)\| \\ &\leq 2M_2\mathcal{M}\sum_{i\in\mathbb{Z}} \|G(t-t_i)\||z_i|^2 = O(\mu^2). \end{aligned} \tag{2.3.4.9}$$

Similarly, we have

$$|Q(\tau, z(\tau), z(\tau-1))| \leq M_2\big(2|z(\tau)| + |\tilde{z}(\tau)|\big)^2/2$$

and

$$\begin{aligned} &\mathcal{I}_1 z(t) \\ = &\int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}Q(\tau, z(\tau), z(\tau-1))\,d\tau = O(\mu^2). \end{aligned} \tag{2.3.4.10}$$

If $x(t)$ is a solution of (2.3.3.1), we have

$$\sup|\dot{x}(t)| \leq M_0(1-\eta)^{-1}.$$

Further on, $\delta I_i(x_i, \bar{x}_i)$ admits the representation (2.3.3.15), thus $|\delta I_i(x_i, \bar{x}_i)| \leq hM_1M_0(1-\eta)^{-1}$ and

$$\mathcal{S}_2 z(t) \equiv \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\delta I_i(x_i, \bar{x}_i) = O(h). \tag{2.3.4.11}$$

Recall that if t is not in Δ_1^h, then

$$|x(t) - x(t-h)| \leq hM_0(1-\eta)^{-1},$$

while for $t \in \Delta_1^h$ we have

$$|x(t) - x(t-h)| \leq M_0(1 + h(1-\eta)^{-1}).$$

Similarly, if $t \notin \Delta_2^h$, then

$$|x(t-1) - x(t-1-h\varphi(t))| \le hM_0(1-\eta)^{-1},$$

while for $t \in \Delta_2^h$ we have

$$|x(t-1) - x(t-1-h\varphi(t))| \le M_0(1 + h(1-\eta)^{-1}).$$

Using these estimates, we evaluate $\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))$. If $\tau \in \Delta_3^h$, we have estimate (2.3.3.17). Next, if $\tau \in \Delta_1^h$, we have (2.3.3.18). Finally, for $\tau \in \Delta_2^h$ estimate (2.3.3.19) is valid.

Next we use the representation

$$\begin{aligned} \mathcal{I}_2(t) &= \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1}\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))\, d\tau \\ &= \int_{\Delta_1^h} + \int_{\Delta_2^h} + \int_{\Delta_3^h} . \end{aligned}$$

Making use of the estimates (2.3.3.17), (2.3.3.18) and (2.3.3.19), we find

$$\begin{aligned} |\mathcal{I}_2 z(t)| &\le 2h\mathcal{M}M_0M_1(1-\eta)^{-2} \int_{\Delta_3^h} \|G(t-\tau)\|\, d\tau \qquad (2.3.4.12) \\ &+ \mathcal{M}M_0M_1(1-\eta)^{-1}\big(1 + 2h(1-\eta)^{-1}\big) \int_{\Delta_1^h \cup \Delta_2^h} \|G(t-\tau)\|\, d\tau \\ &= 2h\mathcal{M}M_0M_1(1-\eta)^{-2} \int_{-\infty}^{\infty} \|G(t-\tau)\|\, d\tau \\ &+ \mathcal{M}M_0M_1(1-\eta)^{-1} \int_{\Delta_1^h \cup \Delta_2^h} \|G(t-\tau)\|\, d\tau. \end{aligned}$$

Now we use the estimate (1.3.3) and we need estimates for

$$\int_{\Delta_j^h} \|G(t-\tau)\|\, d\tau, \quad j = 1, 2.$$

We will estimate the integral for $j = 1$. The arguments are similar to those used for deriving the estimate (1.3.4). We have

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \le C \sum_{i\in\mathbb{Z}} \int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau.$$

For $t \in \mathbb{R}$ we shall consider two possibilities:

a) t belongs to none of the segments $[t_i, t_i + h]$, $i \in \mathbb{Z}$. Then we may assume that $t_0 + h < t < t_1$. Now for $i \in \mathbb{N}$ we have

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \leq he^{-\alpha(t_i - t_1)} \leq he^{-\alpha\theta(i-1)},$$

while for $i \notin \mathbb{N}$

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \leq he^{-\alpha(t_0 - t_i)} \leq he^{\alpha\theta i},$$

and as above we conclude that

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \leq \frac{2Ch}{1 - e^{-\alpha\theta}}. \tag{2.3.4.13}$$

b) t belongs to one of these segments, say, $t_0 \leq t \leq t_0 + h$. Now for $i \in \mathbb{N}$ we have

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \leq he^{-\alpha(t_i - t_0 - h)} \leq he^{\alpha h} e^{-\alpha\theta i},$$

while for $-i \in \mathbb{N}$

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \leq he^{-\alpha(t_0 - t_i - h)} \leq he^{\alpha h} e^{\alpha\theta i}.$$

Finally,

$$\int_{t_0}^{t_0+h} e^{-\alpha|t-\tau|}\, d\tau \leq h$$

and

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \leq Ch\left(1 + 2e^{\alpha h}\sum_{i=1}^{\infty} e^{-i\alpha\theta}\right) = Ch\left(1 + \frac{2e^{\alpha(h-\theta)}}{1 - e^{-\alpha\theta}}\right). \tag{2.3.4.14}$$

Combining the estimates (2.3.4.13) and (2.3.4.14), we have

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \leq \frac{Ch}{1 - e^{-\alpha\theta}} \max\left\{2,\ 1 - e^{-\alpha\theta} + 2e^{\alpha(h-\theta)}\right\}.$$

For h small enough, namely, for

$$h \leq h_1 = \frac{\ln(1 + e^{\alpha\theta}) - \ln 2}{\alpha},$$

estimate (2.3.4.13) holds for any $t \in \mathbb{R}$.

In a similar way we derive the estimate

$$\int_{\Delta_2^h} \|G(t-\tau)\| \, d\tau \leq \frac{2Ch}{1 - e^{-\alpha\theta}} \tag{2.3.4.15}$$

for any $t \in \mathbb{R}$ and $h \leq h_1/2$. We omit the calculations which can be found in §2.3.2.

Substituting the estimates (1.3.3), (2.3.4.13) and (2.3.4.15) into (2.3.4.12), we find

$$|\mathcal{I}_2 z(t)| = O(h). \tag{2.3.4.16}$$

Adding together the estimates (2.3.4.9), (2.3.4.10), (2.3.4.11) and (2.3.4.16), we obtain

$$|\mathcal{U}_h z(t)| = O(\mu^2) + O(h). \tag{2.3.4.17}$$

In order to estimate $-\mathcal{V}_h z(t)$, we represent the integral in (2.3.4.8) as a difference of two integrals and change the integration variable in the first one to obtain

$$\begin{aligned} -\mathcal{V}_h z(t) &= \int_{-\infty}^{\infty} \Phi(t)\big\{G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau) \\ &- G(t-\tau+h)\Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\big\}\bar{\dot{x}}(\tau)\, d\tau. \end{aligned} \tag{2.3.4.18}$$

In order to apply Lemma 2.1.2.1, we carry out one more transformation to obtain

$$\begin{aligned} -\mathcal{V}_h z(t) &= \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\big\{\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau) \\ &\qquad - \Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\big\}\bar{\dot{x}}(\tau)\, d\tau \\ &+ \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\big\{\Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\dot{x}(\tau-h) \\ &\qquad - \Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau)\dot{x}(\tau)\big\}\, d\tau. \end{aligned}$$

We apply to the first integral Lemma 2.1.2.1 with $\Phi(t)G(t-\tau)\bar{\dot{x}}(\tau)$ considered as a function of τ for any fixed t instead of χ, and $\Phi^{-1}(\tau)(E -$

$D(\tau))^{-1}D(\tau)$ instead of y (with points of discontinuity $t_i,\ i \in \mathbb{Z}$). Similarly, we can apply the lemma to the second integral with $\Phi(t)G(t-\tau)$ considered as a function of τ for any fixed t instead of χ, and $\Phi^{-1}(\tau)(E-D(\tau))^{-1}D(\tau)\dot{x}(\tau)$ instead of y. Thus both integrals are estimated by $O(h)$,

$$-\mathcal{V}_h z(t) = O(h)$$

and

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| = O(\mu^2) + O(h),$$

i.e.,

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq K_1\mu^2 + K_2 h \tag{2.3.4.19}$$

for some positive constants K_1 and K_2.

To provide the validity of the inequality $|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq \mu$, we first choose

$$\tilde{\mu}_0 = \min\left\{\mu_0, \frac{1}{2K_1}\right\}.$$

Then for any $\mu \in (0, \tilde{\mu}_0]$ we have $K_1\mu^2 \leq \mu/2$ and inequality (2.3.4.19) takes on the form

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq \mu/2 + K_2 h.$$

If we choose

$$\tilde{h}(\mu) = \min\left\{h_0, \frac{\mu}{2K_2}\right\},$$

then for any $h \in (0, \tilde{h}(\mu)]$ we have $K_2 h \leq \mu/2$ and thus

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \leq \mu,$$

i.e., the operator $\mathcal{U}_h - \mathcal{V}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h \in (0, \tilde{h}(\mu)]$.

Contraction property of the operator $\mathcal{U}_h - \mathcal{V}_h$. Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_h z'(t) - \mathcal{U}_h z''(t) &= (\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)) + (\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)) \\ &+ (\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)) + (\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)).\end{aligned}$$

First we consider

$$\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t) = \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\big(J_i(z_i') - J_i(z_i'')\big).$$

We have

$$|J_i(z_i') - J_i(z_i'')| \le 4\mu M_2 |z_i' - z_i''|$$

and

$$|\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t)| \le O(\mu)\|z' - z''\|. \qquad (2.3.4.20)$$

Next,

$$\begin{aligned}\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t) &= \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1} \\ &\qquad \times \big(Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))\big)d\tau.\end{aligned}$$

We have

$$|Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))| \le 9\mu M_2 \|z' - z''\|$$

and

$$|\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)| \le O(\mu)\|z' - z''\|. \qquad (2.3.4.21)$$

For the estimation of $\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)$ and $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ we denote $x' = \psi(t) + z',\ x'' = \psi(t) + z''$, etc. Now

$$\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t) = \sum_{i \in \mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\big(\delta I_i(x_i', \bar{x}_i') - \delta I_i(x_i'', \bar{x}_i'')\big)$$

and as in §2.3.3 we obtain

$$|\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \le (O(\mu) + O(h))\|z' - z''\| \qquad (2.3.4.22)$$

and, similarly,

$$|\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)| \le (O(\mu) + O(h))\|z' - z''\|. \qquad (2.3.4.23)$$

Now by virtue of the estimates (2.3.4.20), (2.3.4.21), (2.3.4.22) and (2.3.4.23) we obtain

$$\|\mathcal{U}_h z' - \mathcal{U}_h z''\| \le (O(\mu) + O(h))\|z' - z''\|.$$

In order to estimate $\mathcal{V}_h z' - \mathcal{V}_h z''$, we integrate by parts the expression (2.3.4.18) for $\mathcal{V}_h z(t)$ taking into account that the function $G(t-\tau)$ is discontinuous at $\tau = t$ while $\Phi(\tau)$ and $x(\tau)$ are discontinuous at $t_1, \ldots, t_m$ making use of the equalities

$$\frac{\partial}{\partial \tau}\big[\Phi(t)G(t-\tau)\Phi^{-1}(\tau)\big] = -\Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1}A(\tau)$$

and $\Phi(t_i+0)=(E+B_i)\Phi(t_i),\ G(+0)-G(-0)=E$. We obtain

$$\mathcal{V}_h z(t)=(E-D(t))^{-1}D(t)\big(x(t)-x(t-h)\big)$$
$$+\quad \sum_{i\in\mathbb{Z}}\Phi(t)\Big\{G(t-t_i-h)\Phi^{-1}(t_i+h)(E-D(t_i+h))^{-1}D(t_i+h)$$
$$-\,G(t-t_i)\Phi^{-1}(t_i+0)(E-D(t_i))^{-1}D(t_i)\Big\}I_i(x_i,\bar{x}_i)$$
$$+\quad \sum_{i\in\mathbb{Z}}\Phi(t)G(t-t_i)\Phi^{-1}(t_i+0)B_i(E-D(t_i))^{-1}D(t_i)(x_i-\bar{x}_i)$$

$$-\quad \int_{-\infty}^{\infty}\Phi(t)\Big\{G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}A(\tau)(E-D(\tau))^{-1}D(\tau)$$
$$-\quad G(t-\tau+h)\Phi^{-1}(\tau-h)(E-D(\tau-h))^{-1}A(\tau-h)(E-D(\tau-h))^{-1}D(\tau-h)\Big\}x(\tau-h)\,d\tau$$
$$+\quad \int_{-\infty}^{\infty}\Phi(t)\Big\{G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}\dot{D}(\tau)(E-D(\tau))^{-1}$$
$$-\quad G(t-\tau+h)\Phi^{-1}(\tau-h)(E-D(\tau-h))^{-1}\dot{D}(\tau-h)(E-D(\tau-h))^{-1}\Big\}x(\tau-h)\,d\tau.$$

Further transforming the two integral terms and applying Lemma 2.1.2.1 or arguments of its proof, we see that they are estimated by $O(h)\|x\|$. The sum of the first and third terms can be estimated by $2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}\|x\|$.

The difference

$$G(t-t_i-h)\Phi^{-1}(t_i+h)(E-D(t_i+h))^{-1}D(t_i+h)-G(t-t_i)\Phi^{-1}(t_i+0)(E-D(t_i))^{-1}D(t_i)$$

is estimated by $O(h)$ for $t\notin\Delta_1^h$, and by $O(h)+\eta(1-\eta)^{-1}$ otherwise. At last, similarly to (2.3.4.22), we note that

$$|I_i(x_i',\bar{x}_i')-I_i(x_i'',\bar{x}_i'')|\le\big(O(\mu)+O(h)\big)\|z'-z''\|,$$

thus we have

$$|\mathcal{V}_h z'(t)-\mathcal{V}_h z''(t)|\le\big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}+O(\mu)+O(h)\big)\|z'-z''\|$$

and

$$|\big(\mathcal{U}_h z'(t)-\mathcal{V}_h z'(t)\big)-\big(\mathcal{U}_h z''(t)-\mathcal{V}_h z''(t)\big)|$$
$$\le\ \big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}+\gamma_1\mu+\gamma_2 h\big)\|z'-z''\|,$$

where γ_1 and γ_2 are some positive constants.

By condition (2.3.4.4) we have

$$\tilde{\eta} \equiv 2(1 + \mathcal{M}\beta)\eta(1 - \eta)^{-1} < 1.$$

Choose a number $q \in (\tilde{\eta}, 1)$ and denote $r = q - \tilde{\eta}$, and

$$\mu_1 = \min\left\{\tilde{\mu}_0, \frac{r}{2\gamma_1}\right\} \qquad \text{and} \qquad h_1 = \min\left\{\tilde{h}(\mu_1), \frac{r}{2\gamma_2}\right\}.$$

Then for any $\mu \in (0, \mu_1]$ and $h \in [0, h_1]$ we have

$$\|\big(\mathcal{U}_h z' - \mathcal{V}_h z'\big) - \big(\mathcal{U}_h z'' - \mathcal{V}_h z''\big)\| \le q\|z' - z''\|, \qquad q \in (0, 1),$$

for any $z', z'' \in \mathcal{T}_\mu$.

Thus the operator $\mathcal{U}_h - \mathcal{V}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an almost periodic solution $z(t, h)$ of system (2.3.4.6). Since $z(t) \equiv 0$ is the unique almost periodic solution of system (2.3.4.6) for $h = 0$, then $z(t, 0) \equiv 0$. Now $x(t, h) = \psi(t) + z(t, h)$ is the unique almost periodic solution of system (2.3.3.1) and $x(t, 0) = \psi(t)$. This completes the proof of Theorem 2.3.4.1. □

The results of the present subsection were published in [50].

Chapter 3

STABILITY OF EQUILIBRIUM POINTS AND PERIODIC SOLUTIONS OF NEURAL NETWORKS WITH DELAYS AND IMPULSES

Anyone can see that the human brain is superior to a digital computer at many tasks. For example, from the processing of visual information point of view; a one-year-old baby is much better and faster at recognizing objects, faces, and so on than even the most advanced fastest supercomputer systems. The following reasons are the real motivation for studying neural computation [70]. It is an alternative computational paradigm to the usual one (based on a programmed instruction sequence), which was introduced by von Neumann [110] and has been used as the basis of almost all machine computation to date. The brain has many other features that would be desirable in artificial systems:

- It is robust and fault tolerant. Nerve cells in the brain die every day without affecting its performance significantly.
- It is flexible. It can easily adjust to a new environment by "learning", no need to be programmed in Pascal, Fortran or C+ and so on.
- It can deal with information that is fuzzy, probabilistic, noisy, or in-

consistent.

- It is highly parallel.
- It is small, compact, and dissipates very little power.

Artificial neural networks are computational paradigms which implement simplified models of their biological counterparts, biological neural networks. Biological neural networks are the local assemblages of neurons and their dendritic connections that form the (human) brain. Accordingly, artificial neural networks are characterized by

- local processing in artificial neurons (or processing elements),
- massively parallel processing, implemented by rich connection pattern between processing elements,
- the ability to acquire knowledge via learning from experience,
- knowledge storage in distributed memory, the synaptic processing element connections.

Neural network simulations appear to be a recent development. However, this field was established before the advent of computers, and has survived at least one major setback and several eras. Many important advances have been boosted by the use of inexpensive computer emulations. Following an initial period of enthusiasm, the field survived a period of frustration and disrepute.

The first artificial neuron was produced in 1943 by the neurophysiologist Warren McCulloch and the logician Walter Pitts [87]. But the technology available at that time did not allow them to do too much. Neural networks process information in a similar way the human brain does. The network is composed of a large number of highly interconnected processing elements (neurons) working in parallel to solve a specific problem. Neural networks learn by example. Much is still unknown about how the brain trains itself to process information, so theories abound.

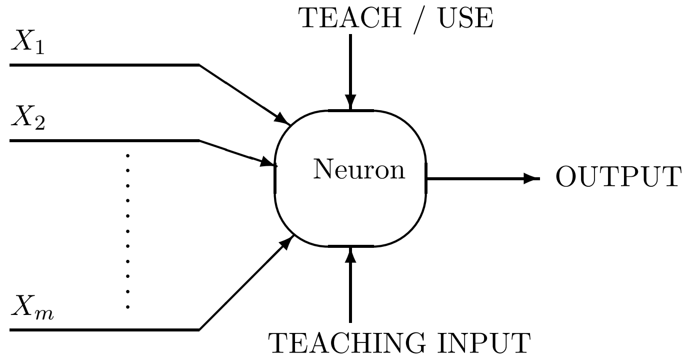

Figure 1: A simple neuron

An artificial neuron is a device with many inputs and one output (Figure 1). The neuron has two modes of operation; the training mode and the using mode. In the training mode, the neuron can be trained to fire (or not), for particular input patterns. In the using mode, when a taught input pattern is detected at the input, its associated output becomes the current output. If the input pattern does not belong in the taught list of input patterns, the firing rule is used to determine whether to fire or not.

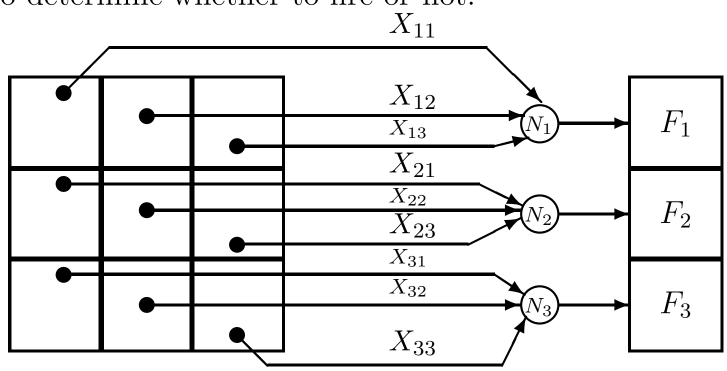

Figure 2: A feed-forward neural network

An important application of neural networks is pattern recognition. Pattern recognition can be implemented by using a feed-forward (Figure 2) neural network that has been trained accordingly. During training, the network is trained to associate outputs with input patterns. When the network is used, it identifies the input pattern and tries to output the associated output pattern. The power of neural networks comes to life when a pattern that has no output associated with it, is given as an input. In this case, the network gives the output that corresponds to a taught input pattern that is least different from the given pattern.

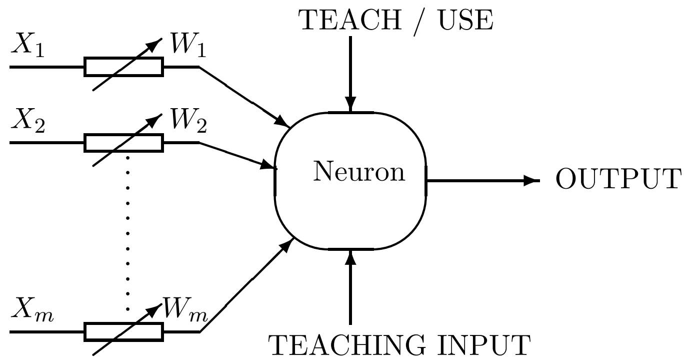

Figure 3: An MCP neuron

The above neuron does not do anything that conventional computers do not already do. A more sophisticated neuron (Figure 3) is the McCulloch and Pitts model (MCP). The difference from the previous model is that the inputs are 'weighted', the effect that each input has at decision making is dependent on the weight of the particular input. The weight of an input is a number which when multiplied with the input gives the weighted input. These weighted inputs are then added together and if they exceed a pre-set threshold value, the neuron fires. In any other case the neuron does not fire. In mathematical terms, the neuron fires if and only if

$$\sum_{i=1}^{m} X_i W_i > T,$$

where $W_i,\ i = \overline{1,m}$, are weights, $X_i,\ i = \overline{1,m}$, inputs, and T a threshold. The addition of input weights and of the threshold makes this neuron a very flexible and powerful one. The MCP neuron has the ability to adapt to a particular situation by changing its weights and/or threshold. Various algorithms exist that cause the neuron to 'adapt'; the most used ones are the Delta rule and the back error propagation. The former is used in feed-forward networks and the latter in feedback networks.

The attempt of implementing neural networks for brain-like computations like patterns recognition, decisions making, motory control and many others is made possible by the advent of large scale computers in the late 1950's. Indeed, artificial neural networks can be viewed as a major new approach to computational methodology since the introduction of digital computers.

Although the initial intent of artificial neural networks was to explore and reproduce human information processing tasks such as speech, vision, and knowledge processing, artificial neural networks also demonstrated their superior capability for classification and function approximation problems. This has great potential for solving complex problems such as systems control, data compression, optimization problems, pattern recognition, and system identification.

Neural networks have wide applicability to real world business problems. In fact, they have already been successfully applied in many industries. Since neural networks are best at identifying patterns or trends in data, they are well suited for prediction or forecasting needs including: sales forecasting, industrial process control, customer research, data validation, risk management, target marketing.

ANN are also used in the following specific paradigms: recognition of speakers in communications; diagnosis of hepatitis; recovery of telecommunications from faulty software; interpretation of multi-meaning Chinese words; undersea mine detection; texture analysis; three-dimensional object recognition; hand-written word recognition; and facial recognition.

Hopfield-type (additive) networks have been studied intensively during the last two decades and have been applied to optimization problems [61, 62, 65, 70], and [96]. The original model used two-state threshold "neurons" that followed a stochastic algorithm: each model neuron i had two states, characterized by the values V_i^0 or V_i^1 (which may often be taken as 0 and 1, respectively). The input of each neuron came from two sources, external inputs I_i and inputs from other neurons. The total input to neuron i is then

$$\text{Input to } i = H_i = \sum_{i \neq j} T_{ij} V_j + I_i,$$

where T_{ij} can be biologically viewed as a description of the synaptic interconnection strength from neuron j to neuron i. The motion of the state of a system of N neurons in state space describes the computation that the set of neurons is performing. A model therefore must describe how the state evolves in time, and the original model describes this in terms of a stochastic evolution. Each neuron samples its input at random times. It changes the value of its output or leaves it fixed according to a threshold rule with thresholds U_i [71, 72]:

$$V_i \to V_i^0 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i < U_i,$$
$$V_i \to V_i^1 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i > U_i.$$

In order to solve problems in the fields of optimization, neural control and signal processing, neural networks have to be designed such that there is only one equilibrium point and this equilibrium point is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima. In the case of global stability, there is no need to be specific about the initial conditions for the neural circuits since all trajectories starting from anywhere settle down at the same unique equilibrium. If the equilibrium is exponentially asymptotically stable, the convergence is fast for real-time computations. The unique equilibrium depends on the external stimulus. The nonlinear neural activation functions $f_i(\cdot)$, $i = \overline{1, m}$, are usually chosen to be continuous and differentiable nonlinear sigmoid functions satisfying the following conditions:

(a) $f_i(x) \to \mp 1$ as $x \to \mp\infty$;

(b) $f_i(x)$ is bounded above by 1 and below by -1;

(c) $f_i(x) = 0$ at a unique point $x = 0$;

(d) $f_i'(x) > 0$ and $f_i'(x) \to 0$ as $x \to \mp\infty$;

(e) $f_i'(x)$ has a global maximum value of 1 at the unique point $x = 0$.

Some examples of activation functions $f_i(\cdot)$ are

$$f_i(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad f_i(x) = \frac{1 - e^{-x}}{1 + e^{-x}} = \tanh(x/2),$$
$$f_i(x) = \frac{2}{\pi} \arctan\left(\frac{\pi}{2} x\right), \qquad f_i(x) = \frac{x^2}{1 + x^2} \operatorname{sgn}(x),$$

where $\operatorname{sgn}(\cdot)$ is a signum function and all the above nonlinear functions are bounded, monotonic and nondecreasing functions. It has been shown that the absolute capacity of an associative memory network can be improved by replacing the usual sigmoid activation functions. There, it seems appropriate that nonmonotonic functions might be better candidates for neuron activation in designing and implementing an artificial neural network. In many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input-output functions are frequently adapted.

In [95] the global stability characteristic of a system of equations modelling the dynamics of additive Hopfield-type neural networks both in the continuous and discrete-time cases is investigated. In particular, a novel method of obtaining a discrete-time dynamical system whose dynamics is inherited from the continuous-time dynamical system is studied. This aspect is important since numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behaviour if either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems [88].

Cohen-Grossberg neural network [43] and its various generalizations with or without transmission delays and impulsive state displacements have been the subject of intense investigation recently [19, 40, 41, 107, 113, 116]. In a Cohen-Grossberg neural network model, the feedback terms consist of amplification and stabilizing functions which are generally nonlinear. These terms provide the model with a special kind of generalization wherein many neural network models that are capable for content addressable memory such as additive neural networks, cellular neural networks and bidirectional associative memory networks and also biological models such as Lotka-Volterra models of population dynamics are included as special cases.

Most widely studied and used neural networks can be classified as either continuous or discrete. Recently, there has been a somewhat new category of neural networks which are neither purely continuous-time nor purely dicrete-time. This third category of neural networks called impulsive neural networks displays a combination of characteristics of both the continuous and discrete systems [64].

In the present chapter we consider various generalizations of Hopfield and Cohen-Grossberg neural networks with delays and impulses and their discrete-time counterparts.

3.1 Global Exponential Stability of Equilibrium Points of Continuous-Time Neural Networks

We investigate the global stability characteristics of a system of equations modelling the dynamics of additive Hopfield-type neural networks with impulses. We find sufficient conditions for the existence of a unique equilibrium point and its global exponential stability. Next we study impulsive Cohen-Grossberg neural networks with S-type distributed delays. This type of delays in the presence of impulses is more general than the usual types of delays studied in the literature. Using analysis techniques we prove the existence of a unique equilibrium point. By means of simple and efficient Lyapunov functions we present some sufficient conditions for the exponential stability of the equilibrium. Further on, an impulsive Cohen-Grossberg neural network with time-varying and S-type distributed delays and reaction-diffusion terms is considered. By using Hardy-Poincaré inequality instead of Hardy-Sobolev inequality or just the nonpositivity of the reaction-diffusion operators, under suitable conditions in terms of $M-$matrices which involve the reaction-diffusion coefficients and the dimension and size of the spatial domain, improved stability estimates for the system with zero Dirichlet boundary conditions are obtained. Examples are given.

Finally, we obtain sufficient conditions in terms of minimal Lipschitz constants and nonlinear measures for the existence of a unique equilibrium point and its exponential stability for impulsive neural networks which are generalizations of Cohen-Grossberg neural networks, with time-varying delays.

3.1.1 Additive Hopfield-type neural networks

The impulsive continuous-time neural network consists of m elementary processing units (or neurons) whose state variables x_i $(i = \overline{1,m})$ are governed by the system

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(t-\tau_{ij})) \qquad (3.1.1.1)\\ &+ \sum_{j=1}^{m} d_{ij} h_j\left(\int_0^\infty K_{ij}(s) x_j(t-s)\,ds\right) + I_i, \quad t>0, \quad t\neq t_k,\end{aligned}$$

$$\Delta x_i(t_k) = -B_{ik}x_i(t_k) + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)x_i(s)\,ds + \gamma_{ik}, \quad i = \overline{1,m}, \quad k \in \mathbb{N}, \tag{3.1.1.2}$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, 0]$. In (3.1.1.1), the coefficient $a_i > 0$ is the rate with which the $i-$th unit self-regulates or resets its potential when isolated from other units and inputs; $f_j(\cdot)$, $g_j(\cdot)$, $h_j(\cdot)$ denote activation functions; the parameters b_{ij}, c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the $j-$th unit and the $i-$th unit; the real constant I_i represents an input signal introduced from outside the network to the $i-$th unit; τ_{ij} are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t - \tau_{ij}$ from the $j-$th unit to the unit i; and the delay kernels K_{ij} incorporate the fading past effects (or fading memories) of the $j-$th unit on the $i-$th unit. In (3.1.1.2), $\Delta x_i(t_k) = x_i(t_k+0) - x_i(t_k-0)$ denote impulsive state displacements at fixed instants of time t_k ($k \in \mathbb{N}$) involving integral terms whose kernels $\psi_{ik} : [t_{k-1}, t_k] \to \mathbb{R}$ are measurable functions, essentially bounded on the respective interval. Here it is assumed that the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$; B_{ik} and γ_{ik} are some real constants.

The assumptions that accompany the impulsive network (3.1.1.1), (3.1.1.2) are given as follows:

A3.1.1.1. For the activation functions $f_j, g_j, h_j : \mathbb{R} \to \mathbb{R}$ there exist positive constants F_j, G_j, H_j such that

$$\begin{aligned} F_j &= \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|, \qquad G_j = \sup_{x \neq y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right|, \\ H_j &= \sup_{x \neq y} \left| \frac{h_j(x) - h_j(y)}{x - y} \right| \quad \text{for} \quad x, y \in \mathbb{R}, \quad j = \overline{1,m}. \end{aligned}$$

A3.1.1.2. $a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0, \quad i = \overline{1,m}.$

A3.1.1.3. $K_{ij} : [0, \infty) \to [0, \infty)$ are bounded and piecewise continuous ($i, j = \overline{1,m}$).

A3.1.1.4. $\int_0^\infty K_{ij}(s)\,ds = 1$ ($i, j = \overline{1,m}$).

A3.1.1.5. There exists a positive number μ such that $\int_0^\infty K_{ij}(s)e^{\mu s}\,ds < \infty$ $(i,j = \overline{1,m})$.

An equilibrium point of the impulsive network (3.1.1.1), (3.1.1.2) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*) + \sum_{j=1}^{m} d_{ij} h_j(x_j^*) + I_i, \quad i = \overline{1,m}, \quad (3.1.1.3)$$

and satisfy the linear equations

$$\left(-B_{ik} + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)\,ds\right) x_i^* + \gamma_{ik} = 0, \quad k \in \mathbb{N},\ i = \overline{1,m}. \qquad (3.1.1.4)$$

Lemma 3.1.1.1. *Let conditions* **A3.1.1.1**, **A3.1.1.2** *be satisfied. Then system* (3.1.1.3) *has a unique solution* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

In other words, if conditions **A3.1.1.1**–**A3.1.1.4** are satisfied, the system without impulses (3.1.1.1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Proof. In system (3.1.1.3) we perform the substitution $y_i = a_i x_i^*$, $i = \overline{1,m}$. Thus we obtain the system

$$y_i = \Phi_i(y) \equiv \sum_{j=1}^{m} \left[b_{ij} f_j\left(\frac{y_j}{a_j}\right) + c_{ij} g_j\left(\frac{y_j}{a_j}\right) + d_{ij} h_j\left(\frac{y_j}{a_j}\right) \right] + I_i, \quad i = \overline{1,m}.$$

We shall show that the mapping $y \mapsto \Phi(y) = (\Phi_1(y), \Phi_2(y), \ldots, \Phi_m(y))^T$ acts as a contraction in the space $\mathbb{R}^m$ equipped with the norm $\|y\| = \sum_{i=1}^{m} |y_i|$. In fact, for any $y, z \in \mathbb{R}^m$ by virtue of **A3.1.1.1** we have

$$|\Phi_i(y) - \Phi_i(z)| \le \sum_{j=1}^{m} \left(|b_{ij}|F_j + |c_{ij}|G_j + |d_{ij}|H_j\right) \frac{|y_j - z_j|}{a_j}, \qquad i = \overline{1,m}.$$

A summation with respect to i and changing the order of summation yield

$$\|\Phi(y) - \Phi(z)\| = \sum_{i=1}^{m} |\Phi_i(y) - \Phi_i(z)|$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} (|b_{ij}|F_j + |c_{ij}|G_j + |d_{ij}|H_j) \frac{|y_j - z_j|}{a_j}$$

$$= \sum_{i=1}^{m} \frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}| \right) |y_i - z_i|.$$

From condition **A3.1.1.2**

$$\frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}| \right) < 1, \qquad i = \overline{1, m},$$

thus

$$\alpha \equiv \max_{i=\overline{1,m}} \frac{1}{a_i} \left(F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}| \right) < 1$$

and

$$\|\Phi(y) - \Phi(z)\| \leq \sum_{i=1}^{m} \alpha |y_i - z_i| = \alpha \|y - z\|.$$

This proves the contraction property of the mapping Φ. □

Our main result in the present subsection is the following

Theorem 3.1.1.1. *Let system* (3.1.1.1), (3.1.1.2) *have an equilibrium point* $x^* = (x_1^*, \dots, x_m^*)^T$ *and satisfy the conditions* **A3.1.1.1**–**A3.1.1.5**. *Then there exist constants* $M > 1$ *and* $\lambda > 0$ *such that all solutions* $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ *of system* (3.1.1.1), (3.1.1.2) *satisfy the estimate*

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \leq M e^{-\lambda t} \prod_{k=1}^{i(0,t)} \left\{ \max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)} \, ds \right\} \times \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*| \quad \text{for all } \; t > 0, \tag{3.1.1.5}$$

where $i(0,t) = \max\{k \in \{0\} \cup \mathbb{N} : t_k < t\}$ *is the number of instants of impulse effect* t_k *in the interval* $(0, t)$.

Proof. First we notice that the equilibrium point x^* is unique by virtue of condition **A3.1.1.2** and Lemma 3.1.1.1.

Let us consider the functions $\Phi_i : [0, \mu] \to \mathbb{R}$ defined by

$$\Phi_i(\lambda) = a_i - \lambda - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| e^{\lambda \tau_{ji}} - H_i \sum_{j=1}^{m} |d_{ji}| \int_0^\infty K_{ji}(s) e^{\lambda s}\, ds,$$

$i = \overline{1, m}$. We have

$$\Phi_i(0) = a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0$$

by virtue of condition **A3.1.1.2**. Now, because of the assumptions **A3.1.1.3**–**A3.1.1.5** each $\Phi_i(\cdot)$ is well defined, continuous and decreasing on $[0, \mu]$. Thus there exists $\lambda_i^* \in (0, \mu]$ such that $\Phi_i(\lambda) > 0$ for $\lambda \in (0, \lambda_i^*)$, $i = \overline{1, m}$. Choosing $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\}$, then we have

$$\Phi_i(\lambda) > 0, \qquad \lambda \in (0, \lambda^*), \quad i = \overline{1, m}. \tag{3.1.1.6}$$

We have from (3.1.1.1) and (3.1.1.3) that

$$D^+ |x_i(t) - x_i^*| \le -a_i |x_i(t) - x_i^*| + \sum_{j=1}^{m} |b_{ij}|\, F_j\, |x_j(t) - x_j^*| \tag{3.1.1.7}$$

$$+ \quad \sum_{j=1}^{m} |c_{ij}|\, G_j\, |x_j(t - \tau_{ij}) - x_j^*| + \sum_{j=1}^{m} |d_{ij}| H_j \int_0^\infty K_{ij}(s) |x_j(t - s) - x_j^*|\, ds$$

for $i = \overline{1, m}$, $t > 0$, $t \neq t_k$, where $D^+ f(t)$ denotes the upper right Dini derivative of a continuous function $f(t)$ defined by

$$D^+ f(t) = \lim_{h \to 0^+} \sup_{0 < \Delta t \le h} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Let us note that if the continuous function $f(t)$ is differentiable at t_0, then

$$D^+ |f(t_0)| = \begin{cases} \dot{f}(t_0) & \text{when } f(t_0) > 0, \\ -\dot{f}(t_0) & \text{when } f(t_0) < 0, \\ |\dot{f}(t_0)| & \text{when } f(t_0) = 0. \end{cases}$$

Next we define

$$y_i(t) = |x_i(t) - x_i^*|e^{\lambda t}, \qquad (3.1.1.8)$$

where $i = \overline{1,m}$, $t \in \mathbb{R}$, and from (3.1.1.7) we derive

$$D^+ y_i(t) \le -(a_i - \lambda)y_i(t) + \sum_{j=1}^{m} |b_{ij}|F_j y_j(t)$$
$$+ \sum_{j=1}^{m} |c_{ij}|G_j e^{\lambda\tau_{ij}} y_j(t-\tau_{ij}) + \sum_{j=1}^{m} |d_{ij}|H_j \int_0^\infty K_{ij}(s)e^{\lambda s} y_j(t-s)\, ds$$

for $t > 0$, $t \ne t_k$. We define a Lyapunov functional $V(\cdot)$ by

$$V(t) = \sum_{i=1}^{m} \left\{ y_i(t) + \sum_{j=1}^{m} |c_{ij}|G_j e^{\lambda\tau_{ij}} \int_{t-\tau_{ij}}^{t} y_j(s)\, ds \right. \qquad (3.1.1.9)$$
$$\left. + \sum_{j=1}^{m} |d_{ij}|H_j \int_0^\infty K_{ij}(s)e^{\lambda s} \left(\int_{t-s}^{t} y_j(\sigma)\, d\sigma \right) ds \right\}, \quad t \ge 0.$$

It is easily seen that $V(t) \ge 0$ for $t > 0$ and that

$$V(0) \le \sum_{i=1}^{m} \left\{ y_i(0) + \sum_{j=1}^{m} |c_{ij}|G_j e^{\lambda\tau_{ij}} \cdot \sup_{s\in[-\tau,0]} y_j(s) \right.$$
$$\left. + \sum_{j=1}^{m} |d_{ij}|H_j \int_0^\infty K_{ij}(s)e^{\lambda s} s\, ds \cdot \sup_{s\in(-\infty,0]} y_j(s) \right\},$$

where $\tau = \max\{\tau_{ij} : i,j = \overline{1,m}\}$, that is,

$$V(0) \le M \sum_{i=1}^{m} \sup_{s\in(-\infty,0]} |x_i(s) - x_i^*| \qquad (3.1.1.10)$$

with

$$M = \max_{i=\overline{1,m}} \left\{ 1 + G_i \sum_{j=1}^{m} |c_{ji}|e^{\lambda\tau_{ji}} + H_i \sum_{j=1}^{m} |d_{ji}| \int_0^\infty K_{ji}(s)e^{\lambda s} s\, ds \right\}.$$

This implies that $V(0) < \infty$ since $\int_0^\infty K_{ij}(s)e^{\lambda s} s\, ds < \infty$ for $\lambda < \mu$.

We can now calculate the rate of change of $V(t)$ along the solutions of (3.1.1.1):

$$\begin{aligned} D^+V(t) \le \sum_{i=1}^m \Bigg\{ -(a_i-\lambda)y_i(t) + \sum_{j=1}^m |c_{ij}|G_j e^{\lambda\tau_{ij}} y_j(t) \\ + \sum_{j=1}^m |d_{ij}|H_j \left(\int_0^\infty K_{ij}(s)e^{\lambda s}\,ds\right) y_j(t)\Bigg\} \end{aligned}$$

$$\begin{aligned} &= -\sum_{i=1}^m \left\{ a_i - \lambda - G_i \sum_{j=1}^m |c_{ji}|e^{\lambda\tau_{ji}} - H_i \sum_{j=1}^m |d_{ji}| \int_0^\infty K_{ji}(s)e^{\lambda s}\,ds \right\} y_i(t) \\ &\equiv -\sum_{i=1}^m \Phi_i(\lambda)y_i(t) \le 0 \quad \text{for} \;\; t>0,\; t\ne t_k, \end{aligned}$$

by virtue of (3.1.1.6). This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$V(t) \le V(t_{k-1}+0) \quad \text{for} \quad t \in (t_{k-1}, t_k],\; k\in\mathbb{N}. \tag{3.1.1.11}$$

In particular,

$$V(t_k) \le V(t_{k-1}+0), \qquad k\in\mathbb{N}. \tag{3.1.1.12}$$

Further on, making use of the equalities (3.1.1.2) and (3.1.1.4), for an arbitrary moment of impulse effect t_k, $k\in\mathbb{N}$, we successively find

$$\begin{aligned} \Delta x_i(t_k) &= -B_{ik}(x_i(t_k)-x_i^*) + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)(x_i(s)-x_i^*)\,ds, \\ |x_i(t_k+0)-x_i^*| &\le |1-B_{ik}|\,|x_i(t_k)-x_i^*| + \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)|\,|x_i(s)-x_i^*|\,ds, \\ y_i(t_k+0) &\le |1-B_{ik}|y_i(t_k) + \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)}|\psi_{ik}(s)|y_i(s)\,ds,\; i=\overline{1,m}. \end{aligned}$$

Making use of (3.1.1.11) and (3.1.1.12), we obtain

$$\begin{aligned} V(t_k+0) &\le \max_{i=\overline{1,m}} |1-B_{ik}|V(t_k) + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)}|\psi_{ik}(s)|\,ds\; V(t_{k-1}+0) \\ &\le \left(\max_{i=\overline{1,m}} |1-B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)}|\psi_{ik}(s)|\,ds \right) V(t_{k-1}+0) \end{aligned}$$

and

$$V(t) \le \prod_{k=1}^{i(0,t)} \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} |\psi_{ik}(s)|\, ds \right) V(0) \tag{3.1.1.13}$$

for all $t > 0$. Finally, from (3.1.1.8) and (3.1.1.9) we have

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| = e^{-\lambda t} \sum_{i=1}^{m} y_i(t) \le e^{-\lambda t} V(t).$$

The last inequality combined with (3.1.1.13) and (3.1.1.10) yields (3.1.1.5). □

Definition 3.1.1.1. The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.1.1.1), (3.1.1.2) is said to be *globally exponentially stable* (with Lyapunov exponent λ) if there exist constants $\lambda > 0$ and $M \ge 1$ and any solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of system (3.1.1.1), (3.1.1.2) is defined for all $t > 0$ and we have

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le M e^{-\lambda t} \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*| \quad \text{for all} \quad t \ge 0. \tag{3.1.1.14}$$

For three sets of additional assumptions on the impulse effects we will show that inequality (3.1.1.5) implies global exponential stability of the equilibrium point x^* of the impulsive system (3.1.1.1), (3.1.1.2).

Corollary 3.1.1.1. *Let all conditions of Theorem 3.1.1.1 hold. Let there exist $\lambda \in (0, \lambda^*)$ such that*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)}\, ds \le 1$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the impulsive system (3.1.1.1), (3.1.1.2) *is globally exponentially stable with Lyapunov exponent λ.*

The proof of this corollary is obvious. The global exponential stability is provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that these do not occur too often.

Corollary 3.1.1.2. *Let all conditions of Theorem 3.1.1.1 hold and*

$$\limsup_{t\to\infty} \frac{i(0,t)}{t} = p < +\infty. \tag{3.1.1.15}$$

Let there exist positive constants $\lambda \in (0, \lambda^*)$ *and* B *satisfying the inequalities*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)}\, ds \le B \tag{3.1.1.16}$$

for all sufficiently large values of $k \in \mathbb{N}$*, and* $p \ln B < \lambda$*. Then for any* $\tilde{\lambda} \in (0, \lambda - p\ln B)$ *the equilibrium point* x^* *of the impulsive system* (3.1.1.1), (3.1.1.2) *is globally exponentially stable with Lyapunov exponent* $\tilde{\lambda}$*.*

Proof. Inequalities (3.1.1.5) and (3.1.1.16) yield

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le M e^{-\lambda t} B^{i(0,t)} \sum_{i=1}^{m} \sup_{s\in(-\infty,0]} |x_i(s) - x_i^*| \quad \text{for all} \;\; t > 0.$$

Condition (3.1.1.15) means that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that the inequality

$$\frac{i(0,t)}{t} \le p + \varepsilon$$

is satisfied for all $t \ge T$. For such t we have $i(0,t) \le (p+\varepsilon)t$ and

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le M e^{-(\lambda - (p+\varepsilon)\ln B)t} \sum_{i=1}^{m} \sup_{s\in(-\infty,0]} |x_i(s) - x_i^*|.$$

It suffices to choose $\varepsilon > 0$ such that $(p+\varepsilon)\ln B < \lambda$ and $\tilde{\lambda} = \lambda - (p+\varepsilon)\ln B$. Then inequality (3.1.1.14) will be satisfied with $\tilde{\lambda}$ instead of λ and a possibly bigger constant M. □

Corollary 3.1.1.3. *Let all conditions of Theorem 3.1.1.1 hold and there exist constants* $\lambda \in (0, \lambda^*)$ *and* $\kappa \in (0, \lambda)$ *such that*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}}^{t_k} |\psi_{ik}(s)| e^{\lambda(t_k - s)}\, ds \le e^{\kappa(t_k - t_{k-1})} \tag{3.1.1.17}$$

for all sufficiently large values of $k \in \mathbb{N}$*. Then the equilibrium point* x^* *of the impulsive system* (3.1.1.1), (3.1.1.2) *is globally exponentially stable with Lyapunov exponent* $\lambda - \kappa$*.*

Proof. By virtue of condition (2.1.1.17) for $t \in (t_k, t_{k+1}]$ inequality (3.1.1.5) implies

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \leq M e^{-\lambda t} e^{\kappa t_k} \sum_{i=1}^{m} \sup_{s \in (-\infty, 0]} |x_i(s) - x_i^*|$$

with a possibly larger constant M. Since $t_k < t$, we have $e^{-\lambda t} e^{\kappa t_k} < e^{-(\lambda-\kappa)t}$ and inequality (3.1.1.14) will be satisfied with $\lambda - \kappa$ instead of λ. □

A similar condition was later introduced in the paper [97].

The results of the present subsection were essentially given in our paper [4] where impulse conditions were provided for the continuous-time neural networks considered in [95]. The exposition here follows the pattern of some of our more recent papers.

3.1.2 Cohen-Grossberg neural networks with S-type distributed delays

In the present subsection we study impulsive Cohen-Grossberg neural networks with finite S-type distributed delays. This type of delays in the presence of impulses is more general than the usual types of delays studied in the literature. In fact, concentrated delays correspond to the points of discontinuity of the bounded variation functions. Neural networks with S-type delays without impulses were considered, for instance, in [26, 66, 77, 111].

We consider the impulsive Cohen-Grossberg neural network with S-type delays consisting of m elementary processing units (or neurons) whose state variables x_i ($i = \overline{1, m}$) are governed by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= a_i(x_i(t)) \Bigg[-b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) \qquad (3.1.2.1) \\ &+ \sum_{j=1}^{m} d_{ij} \int_{-\tau}^{0} g_j(x_j(t+\theta))\, d\eta_{ij}(\theta) + I_i \Bigg], \quad t > t_0 = 0,\ t \neq t_k, \\ \Delta x_i(t_k) &= -B_{ik} x_i(t_k) + \int_{-\tau}^{0} x_i(t_k + \theta)\, d\zeta_k(\theta) + \gamma_{ik}, \qquad (3.1.2.2) \\ & i = \overline{1, m}, \quad k \in \mathbb{N}, \end{aligned}$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ with discontinuities of the first kind for $s \in [-\tau, 0]$. In (3.1.2.1), $a_i(x_i)$

denotes an amplification function; $b_i(x_i)$ denotes an appropriate function which supports the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i; $f_j(x_j)$, $g_j(x_j)$ denote activation functions; the parameters c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the $j-$th unit and the $i-$th unit; the real constant I_i represents an input signal introduced from outside the network to the $i-$th unit; the past effect of the $j-$th unit on the $i-$th unit is given by the Lebesgue-Stieltjes integral $\int_{-\tau}^{0} g_j(x_j(t+\theta))\, d\eta_{ij}(\theta)$; $\Delta x_i(t_k) = x_i(t_k+0) - x_i(t_k-0)$ denote impulsive state displacements at fixed moments of time t_k, $k \in \mathbb{N}$, involving Lebesgue-Stieltjes integrals. Here it is assumed that the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$.

The assumptions that accompany the impulsive network (3.1.2.1), (3.1.2.2) are given as follows:

A3.1.2.1. The amplification functions $a_i : \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded in the sense that

$$0 < \underline{a}_i \le a_i(x) \le \overline{a}_i \quad \text{for } x \in \mathbb{R},\ i = \overline{1,m}.$$

A3.1.2.2. The stabilizing functions $b_i : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing, namely,

$$0 < \underline{b}_i \le \frac{b_i(x) - b_i(y)}{x - y} \quad \text{for } x \ne y,\ x, y \in \mathbb{R},\ i = \overline{1,m}.$$

A3.1.2.3. The activation functions $f_j, g_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in the sense of

$$\sup_{x \ne y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right| = F_j, \qquad \sup_{x \ne y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right| = G_j$$

for $x, y \in \mathbb{R}$, $j = \overline{1,m}$, where F_j, G_j denote positive constants.

A3.1.2.4. $\eta_{ij}(\theta)$ $(i, j = \overline{1,m})$, $\zeta_k(\theta)$ $(k \in \mathbb{N})$ are nondecreasing bounded variation functions on $[-\tau, 0]$, $t_{k+1} - t_k \ge \tau$ for $k \in \{0\} \cup \mathbb{N}$ and $\int_{-\tau}^{0} d\eta_{ij}(\theta) = 1$ (without loss of generality), $\int_{-\tau}^{0} d\zeta_k(\theta) = \beta_k$.

Under these assumptions and the given initial conditions, there is a unique solution of the impulsive network (3.1.2.1), (3.1.2.2). The solution is a vector

$x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ in which $x_i(t)$ are piecewise continuous for $t \in (0, \beta)$, where β is some positive number, possibly ∞, such that the limits $x_i(t_k + 0)$ and $x_i(t_k - 0)$ exist and $x_i(t)$ are differentiable for $t \in (t_{k-1}, t_k) \subset (0, \beta)$. An equilibrium point of the impulsive network (3.1.2.1), (3.1.2.2) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^{m} c_{ij} f_j(x_j^*) + \sum_{j=1}^{m} d_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1, m}, \tag{3.1.2.3}$$

and satisfy the linear equations

$$(-B_{ik} + \beta_k)x_i^* + \gamma_{ik} = 0, \quad k \in \mathbb{N},\ i = \overline{1, m}. \tag{3.1.2.4}$$

Definition 3.1.2.1. The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of the impulsive network (3.1.2.1), (3.1.2.2) is said to be globally exponentially stable with a Lyapunov exponent λ if there exist constants $M \geq 1$ and $\lambda > 0$ and any other solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of (3.1.2.1), (3.1.2.2) is defined for all $t > 0$ and satisfies the estimate

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \leq M e^{-\lambda t} \sum_{i=1}^{m} \sup_{s \in [-\tau, 0]} |x_i(s) - x_i^*|, \qquad t \geq 0. \tag{3.1.2.5}$$

Our first task is to prove the existence and uniqueness of the solution x^* of the algebraic system (3.1.2.3). To this end we will need the following lemma.

Lemma 3.1.2.1. [56] *A locally invertible C^0 map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is a homeomorphism of $\mathbb{R}^m$ onto itself if and only if it is proper.*

In fact, this assertion is due to Hadamard [68]. A mapping is proper if the pre-image of every compact is compact. In the finite-dimensional case it suffices to show that $\|\Phi(x)\| \to \infty$ as $\|x\| \to \infty$.

Theorem 3.1.2.1. *Let the assumptions* **A3.1.2.1**–**A3.1.2.4** *hold. Suppose, further, that the following inequalities are valid:*

$$\underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}| - G_i \sum_{j=1}^{m} |d_{ji}| > 0, \quad i = \overline{1, m}. \tag{3.1.2.6}$$

Then the system without impulses (3.1.2.1) *has a unique equilibrium point* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Proof. Let us define a mapping $\Phi :\ \mathbb{R}^m \to \mathbb{R}^m$ by $\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x))^T$ for $x \in \mathbb{R}^m$, where

$$\Phi_i(x) = -b_i(x_i) + \sum_{j=1}^{m} c_{ij} f_j(x_j) + \sum_{j=1}^{m} d_{ij} g_j(x_j) + I_i, \quad i = \overline{1,m}.$$

The space $\mathbb{R}^m$ is endowed with the norm $\|x\| = \sum_{i=1}^{m} |x_i|$. Under the assumptions **A3.1.2.2**, **A3.1.2.3**, $\Phi(x) \in C^0$. It is known that if $\Phi(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$, then there is a unique point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \in \mathbb{R}^m$ such that $\Phi(x^*) = 0$, that is, $\Phi_i(x^*) = 0,\ i = \overline{1,m}$. The last equalities are, in fact, (3.1.2.3), so $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ is the equilibrium point we are looking for.

To demonstrate the one-to-one property of $\Phi(x)$, we take arbitrary vectors $x, y \in \mathbb{R}^m$ and assume that $\Phi(x) = \Phi(y)$. From

$$b_i(x_i) - b_i(y_i) = \sum_{j=1}^{m} c_{ij} \left(f_j(x_j) - f_j(y_j)\right) + \sum_{j=1}^{m} d_{ij} \left(g_j(x_j) - g_j(y_j)\right), \ \ i = \overline{1,m},$$

one obtains

$$\underline{b}_i |x_i - y_i| \le \sum_{j=1}^{m} |c_{ij}| F_j |x_j - y_j| + \sum_{j=1}^{m} |d_{ij}| G_j |x_j - y_j|, \qquad i = \overline{1,m},$$

under the given assumptions. Adding together the above inequalities, we derive

$$\begin{aligned} \sum_{i=1}^{m} \underline{b}_i |x_i - y_i| &\le \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ |c_{ij}| F_j + |d_{ij}| G_j \right\} |x_j - y_j| \\ &= \sum_{i=1}^{m} \left\{ F_i \sum_{j=1}^{m} |c_{ji}| + G_i \sum_{j=1}^{m} |d_{ji}| \right\} |x_i - y_i|, \end{aligned}$$

that is,

$$\sum_{i=1}^{m} \left\{ \underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}| - G_i \sum_{j=1}^{m} |d_{ji}| \right\} |x_i - y_i| \le 0.$$

Now the assertion $x_i = y_i$, $i = \overline{1,m}$, follows by virtue of inequalities (3.1.2.6). Thus, $\Phi(x) = \Phi(y)$ implies $x = y$.

Next we show that $\|\Phi(x)\| \to \infty$ as $\|x\| \to \infty$. It suffices to show that $\|\tilde{\Phi}(x)\| \to \infty$, where $\tilde{\Phi}(x) = \Phi(x) - \Phi(0)$. We have $\tilde{\Phi}(x) = (\tilde{\Phi}_1(x), \tilde{\Phi}_2(x), \ldots, \tilde{\Phi}_m(x))^T$, where

$$\tilde{\Phi}_i(x) = -(b_i(x_i) - b_i(0)) + \sum_{j=1}^{m} c_{ij}(f_j(x_j) - f_j(0)) + \sum_{j=1}^{m} d_{ij}(g_j(x_j) - g_j(0)).$$

These equalities imply

$$|\tilde{\Phi}_i(x)| \geq \underline{b}_i |x_i| - \sum_{j=1}^{m} |c_{ij}|\, F_j\, |x_j| - \sum_{j=1}^{m} |d_{ij}|\, G_j\, |x_j|.$$

As above we deduce

$$\|\tilde{\Phi}(x)\| \geq \sum_{i=1}^{m} \left\{ \underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}| - G_i \sum_{j=1}^{m} |d_{ji}| \right\} |x_i|.$$

By virtue of inequalities (3.1.2.6) there exists a number $\mu > 0$ such that

$$\underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}| - G_i \sum_{j=1}^{m} |d_{ji}| \geq \mu, \qquad i = \overline{1, m}.$$

Then $\|\tilde{\Phi}(x)\| \geq \mu \|x\|$ and $\|\tilde{\Phi}(x)\| \to \infty$ as $\|x\| \to \infty$.

According to Lemma 3.1.2.1, $\Phi(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$. Thus, there is a unique point $x^* \in \mathbb{R}^m$ such that $\Phi(x^*) = 0$. The point represents a unique solution of the algebraic system (3.1.2.3). □

Theorem 3.1.2.2. *Let the assumptions* **A3.1.2.1**–**A3.1.2.4** *hold. Suppose, further, that the inequalities*

$$\underline{a}_i \underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}| \overline{a}_j - G_i \sum_{j=1}^{m} |d_{ji}| \overline{a}_j > 0, \quad i = \overline{1, m}, \tag{3.1.2.7}$$

are valid and the system (3.1.2.1) *has a unique equilibrium point* x^* *whose components* x_i^*, $i = \overline{1, m}$, *satisfy the linear equations* (3.1.2.4). *Then there exist constants* $M \geq 1$ *and* $\lambda > 0$ *and any other solution* $x(t) = (x_1(t), x_2(t),$

$\ldots, x_m(t))^T$ *of* (3.1.2.1), (3.1.2.2) *is defined for all* $t > 0$ *and satisfies the estimate*

$$\begin{aligned}\sum_{i=1}^{m} |x_i(t) - x_i^*| \;&\leq\; M e^{-\lambda t} \prod_{k=1}^{i(0,t)} \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \right) \\ &\times \sum_{i=1}^{m} \sup_{s\in[-\tau,0]} |x_i(s) - x_i^*|, \qquad t \geq 0. \end{aligned} \tag{3.1.2.8}$$

Proof. Upon introducing the translations

$$u_i(t) = x_i(t) - x_i^*, \qquad \varphi_i(s) = \phi_i(s) - x_i^*$$

we derive the system

$$\frac{du_i(t)}{dt} = \tilde{a}_i(u_i(t)) \Bigg[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij} \tilde{f}_j(u_j(t)) \tag{3.1.2.9}$$

$$+ \sum_{j=1}^{m} d_{ij} \int_{-\tau}^{0} \tilde{g}_j(u_j(t+\theta))\, d\eta_{ij}(\theta) \Bigg], \qquad t > t_0 = 0, \quad t \neq t_k,$$

$$\Delta u_i(t_k) = -B_{ik} u_i(t_k) + \int_{-\tau}^{0} u_i(t_k + \theta)\, d\zeta_k(\theta),\ i = \overline{1,m},\ k \in \mathbb{N}, \tag{3.1.2.10}$$

$$u_i(s) = \varphi_i(s), \quad s \in [-\tau, 0],$$

where

$$\tilde{a}_i(u_i) = a_i(u_i + x_i^*), \qquad \tilde{b}_i(u_i) = b_i(u_i + x_i^*) - b_i(x_i^*),$$
$$\tilde{f}_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), \qquad \tilde{g}_j(u_j) = g_j(u_j + x_j^*) - g_j(x_j^*).$$

This system inherits the assumptions **A3.1.2.1**–**A3.1.2.4** given before. It suffices to examine the exponential stability characteristics of the trivial equilibrium point $u^* = 0$ of system (3.1.2.9), (3.1.2.10).

From equation (3.1.2.9) we derive an estimate for the upper right Dini derivative

$$\begin{aligned} D^+|u_i(t)| \;&\leq\; -\underline{a}_i \underline{b}_i |u_i(t)| + \overline{a}_i \sum_{j=1}^{m} |c_{ij}|\, F_j\, |u_j(t)| \\ &+\; \overline{a}_i \sum_{j=1}^{m} |d_{ij}|\, G_j \int_{-\tau}^{0} |u_j(t+\theta)|\, d\eta_{ij}(\theta), \quad i = \overline{1,m}. \end{aligned} \tag{3.1.2.11}$$

Next we define the following functions of $\lambda \geq 0$:

$$H_i(\lambda) = \underline{a}_i\underline{b}_i - \lambda - F_i \sum_{j=1}^{m} |c_{ji}|\, \overline{a}_j - G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta}\, d\eta_{ji}(\theta), \quad i = \overline{1,m}.$$

By virtue of the inequalities (3.1.2.7) we find

$$H_i(0) = \underline{a}_i\underline{b}_i - F_i \sum_{j=1}^{m} |c_{ji}|\, \overline{a}_j - G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j > 0, \quad i = \overline{1,m}.$$

By a lemma proved in [26] the integrals in $H_i(\lambda)$ depend continuously on λ. Since $H_i(\lambda)$ are a finite number of continuous functions, there is $\lambda^* > 0$ such that $H_i(\lambda) > 0$ for $\lambda \in [0, \lambda^*]$, that is,

$$\underline{a}_i\underline{b}_i - \lambda - F_i \sum_{j=1}^{m} |c_{ji}|\, \overline{a}_j - G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta}\, d\eta_{ji}(\theta) > 0, \quad i = \overline{1,m}. \tag{3.1.2.12}$$

For any $\lambda \in (0, \lambda^*]$ define $y_i(t) = e^{\lambda t}|u_i(t)|$. Then by virtue of (3.1.2.11) we find

$$\begin{aligned} D^+ y_i(t) &\leq -(\underline{a}_i\underline{b}_i - \lambda) y_i(t) + \overline{a}_i \sum_{j=1}^{m} |c_{ij}|\, F_j y_j(t) \qquad (3.1.2.13) \\ &+ \overline{a}_i \sum_{j=1}^{m} |d_{ij}|\, G_j \int_{-\tau}^{0} e^{-\lambda\theta} y_j(t+\theta)\, d\eta_{ij}(\theta) \quad i = \overline{1,m}. \end{aligned}$$

We consider a Lyapunov functional

$$V(t) = \sum_{i=1}^{m} \left\{ y_i(t) + \overline{a}_i \sum_{j=1}^{m} |d_{ij}|\, G_j \int_{-\tau}^{0} e^{-\lambda\theta} \left(\int_{t+\theta}^{t} y_j(s)\, ds \right) d\eta_{ij}(\theta) \right\}.$$

We note that $V(t) > 0$ for $t \geq 0$ and

$$\begin{aligned} V(0) &= \sum_{i=1}^{m} \left\{ y_i(0) + \overline{a}_i \sum_{j=1}^{m} |d_{ij}|\, G_j \int_{-\tau}^{0} e^{-\lambda\theta} \left(\int_{\theta}^{0} y_j(s)\, ds \right) d\eta_{ij}(\theta) \right\} \\ &= \sum_{i=1}^{m} \left\{ y_i(0) + G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta} \left(\int_{\theta}^{0} y_i(s)\, ds \right) d\eta_{ji}(\theta) \right\} \\ &\leq \sum_{i=1}^{m} \left\{ 1 + G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta}(-\theta)\, d\eta_{ji}(\theta) \right\} \sup_{s\in[-\tau,0]} y_i(s), \end{aligned}$$

thus

$$V(0) \leq M \sum_{i=1}^{m} \sup_{s\in[-\tau,0]} y_i(s) \tag{3.1.2.14}$$

with

$$M = \max_{i=\overline{1,m}} \left\{ 1 + G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta}(-\theta)\, d\eta_{ji}(\theta) \right\}.$$

Calculating the rate of change of $V(t)$ along the solutions of (3.1.2.9), by virtue of (3.1.2.13) and (3.1.2.7) we obtain

$$D^+V(t) \leq \sum_{i=1}^{m} \left\{ D^+y_i(t) + \overline{a}_i \sum_{j=1}^{m} |d_{ij}| G_j \int_{-\tau}^{0} e^{-\lambda\theta}\big(y_j(t) - y_j(t+\theta)\big)\, d\eta_{ij}(\theta) \right\}$$

$$\leq -\sum_{i=1}^{m}(\underline{a}_i\underline{b}_i - \lambda) y_i(t) + \sum_{i=1}^{m} \overline{a}_i \left\{ \sum_{j=1}^{m} |c_{ij}|\, F_j y_j(t) + \sum_{j=1}^{m} |d_{ij}|\, G_j \int_{-\tau}^{0} e^{-\lambda\theta}\, d\eta_{ij}(\theta)\, y_j(t) \right\}$$

$$= -\sum_{i=1}^{m} \left\{ \underline{a}_i\underline{b}_i - \lambda - F_i \sum_{j=1}^{m} |c_{ji}|\, \overline{a}_j - G_i \sum_{j=1}^{m} |d_{ji}|\, \overline{a}_j \int_{-\tau}^{0} e^{-\lambda\theta}\, d\eta_{ji}(\theta) \right\} y_i(t) \leq 0.$$

This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$V(t) \leq V(t_{k-1}+0) \quad \text{for} \quad t_{k-1} < t \leq t_k. \tag{3.1.2.15}$$

In particular,

$$V(t_k) \leq V(t_{k-1}+0), \qquad k \in \mathbb{N}. \tag{3.1.2.16}$$

Further on, we have

$$u_i(t_k+0) = (1 - B_{ik})u_i(t_k) + \int_{-\tau}^{0} u_i(t_k+\theta)\, d\zeta_k(\theta)$$

and

$$y_i(t_k+0) \leq |1 - B_{ik}|\, y_i(t_k) + \int_{-\tau}^{0} e^{-\lambda\theta} y_i(t_k+\theta)\, d\zeta_k(\theta).$$

Making use of (3.1.2.15) and (3.1.2.16), we obtain

$$\begin{aligned} V(t_k+0) &\leq \max_{i=\overline{1,m}} |1 - B_{ik}| V(t_k) + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta)\, V(t_{k-1}+0) \\ &\leq \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \right) V(t_{k-1}+0). \end{aligned}$$

Combining the last estimate with (3.1.2.15), (3.1.2.16) and (3.1.2.14), we derive (3.1.2.8). □

For three sets of additional assumptions we show that inequality (3.1.2.8) implies global exponential stability of the equilibrium point x^* of the impulsive system (3.1.2.1), (3.1.2.2).

Corollary 3.1.2.1. *Let all conditions of Theorem 3.1.2.2 hold. Let there exist $\lambda > 0$ such that inequalities* (3.1.2.12) *are valid and*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \le 1 \tag{3.1.2.17}$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the impulsive system (3.1.2.1), (3.1.2.2) *is globally exponentially stable with Lyapunov exponent λ.*

Corollary 3.1.2.2. *Let all conditions of Theorem 3.1.2.2 hold and*

$$\limsup_{t\to\infty} \frac{i(0,t)}{t} = p < +\infty.$$

Let there exist positive constants λ and B satisfying the inequalities (3.1.2.12),

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \le B$$

for all sufficiently large values of $k \in \mathbb{N}$ and $p \ln B < \lambda$. Then for any $\tilde{\lambda} \in (0, \lambda - p \ln B)$ the equilibrium point x^* of the impulsive system (3.1.2.1), (3.1.2.2) *is globally exponentially stable with Lyapunov exponent $\tilde{\lambda}$.*

Corollary 3.1.2.3. *Let all conditions of Theorem 3.1.2.2 hold and there exist constants $\lambda > \kappa > 0$ satisfying the inequalities* (3.1.2.12) *and*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) \le e^{\kappa(t_k - t_{k-1})} \tag{3.1.2.18}$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the impulsive system (3.1.2.1), (3.1.2.2) *is globally exponentially stable with Lyapunov exponent $\lambda - \kappa$.*

The formulations of the above corollaries and their proofs are similar to those of Corollaries 3.1.1.1–3.1.1.3.

Example. Consider the system

$$\begin{aligned}\dot{x}_1(t) &= (2+\sin x_1(t))\Big(-2x_1(t)+0.1\arctan x_1(t)+0.15\arctan x_2(t)\\ &\quad +0.1\int_{-1}^{0} x_1(t+\theta)\,de^{\theta}+0.15\int_{-1}^{0} x_2(t+\theta)\,de^{\theta}\Big),\\ \dot{x}_2(t) &= (3+\sin x_2(t))\Big(-3x_2(t)+0.15\arctan x_1(t)-0.2\arctan x_2(t)\\ &\quad +0.1\int_{-1}^{0} x_1(t+\theta)\,de^{\theta}-0.2\int_{-1}^{0} x_2(t+\theta)\,de^{\theta}\Big).\end{aligned} \tag{3.1.2.19}$$

For this system assumptions **A3.1.2.1**–**A3.1.2.3** hold with $\underline{a}_1=1$, $\overline{a}_1=3$, $\underline{a}_2=2$, $\overline{a}_2=4$, $\underline{b}_1=2$, $\underline{b}_2=3$, $F_1=F_2=G_1=G_2=1$. Moreover, $c_{11}=0.1$, $c_{12}=c_{21}=0.15$, $c_{22}=-0.2$, $d_{11}=d_{21}=0.1(1-e^{-1})$, $d_{12}=0.15(1-e^{-1})$, $d_{22}=-0.2(1-e^{-1})$. Inequalities (3.1.2.7) reduce to

$$0.4+0.7e^{-1}>0 \qquad \text{and} \qquad 3.5+1.25e^{-1}>0.$$

Further on,

$$H_1(\lambda)=1.1-\lambda-0.7\frac{1-e^{\lambda-1}}{1-\lambda} \quad \text{and} \quad H_2(\lambda)=4.75-\lambda-1.25\frac{1-e^{\lambda-1}}{1-\lambda}.$$

Since $H_1(0.5)\approx 0.05>0$ and $H_2(0.5)\approx 3.27>0$, we can take $\lambda^*=0.5$. Theorem 3.1.2.2 is valid for system (3.1.2.19) with any impulse conditions of the form (3.1.2.2) such that $\gamma_{ik}=0$, $i=\overline{1,m}$, $k\in\mathbb{N}$.

Let us consider the impulse conditions

$$\begin{aligned}\Delta x_1(t_k) &= -\frac{1}{2}x_1(t_k)+\frac{1}{4}\int_{-1}^{0} x_1(t_k+\theta)\,de^{\theta},\\ \Delta x_2(t_k) &= -\frac{1}{4}x_2(t_k)+\frac{1}{4}\int_{-1}^{0} x_2(t_k+\theta)\,de^{\theta}.\end{aligned} \tag{3.1.2.20}$$

Now

$$\max_{i=\overline{1,2}}|1-B_{ik}|+\int_{-\tau}^{0} e^{-\lambda\theta}\,d\zeta_k(\theta)=\frac{3}{4}+\frac{1}{4}\int_{-1}^{0} e^{-\lambda\theta}\,de^{\theta}=\begin{cases}\frac{3}{4}+\frac{1-e^{\lambda-1}}{4(1-\lambda)}, & \lambda\neq 1,\\ 1, & \lambda=1.\end{cases}$$

Obviously, inequalities (3.1.2.17) are valid for all $k \in \mathbb{N}$ and all $\lambda \in (0, 1]$, in particular, for $\lambda = 0.5$. According to Corollary 3.1.3.1, the equilibrium point $(0, 0)^T$ of the impulsive system (3.1.2.19), (3.1.2.20) is globally exponentially stable with Lyapunov exponent 0.5.

Next consider the impulse conditions

$$\begin{aligned} \Delta x_1(t_k) &= -100x_1(t_k) + \int_{-1}^{0} x_1(t_k + \theta)\, de^{\theta}, \\ \Delta x_2(t_k) &= -50x_2(t_k) + \int_{-1}^{0} x_2(t_k + \theta)\, de^{\theta}, \\ & t_k = 10k, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.1.2.21}$$

Now

$$\max_{i=\overline{1,2}} |1 - B_{ik}| + \int_{-\tau}^{0} e^{-\lambda\theta}\, d\zeta_k(\theta) = 99 + \int_{-1}^{0} e^{-\lambda\theta}\, de^{\theta} = \begin{cases} 99 + \frac{1-e^{\lambda-1}}{1-\lambda}, & \lambda \neq 1, \\ 100, & \lambda = 1, \end{cases}$$

and we can take $B = 100$. Further on, $p = 0.1$, for $\lambda = 0.5$ we have $\lambda - p \ln B \approx 0.5 - 0.1 \times 4.605 = 0.0395$. According to Corollary 3.1.3.2, the equilibrium point $(0, 0)^T$ of the impulsive system (3.1.2.19), (3.1.2.21) is globally exponentially stable with Lyapunov exponent 0.039.

Finally, let us consider the impulse conditions

$$\begin{aligned} \Delta x_1(t_k) &= -(k + 1)x_1(t_k) + k \int_{-1}^{0} x_1(t_k + \theta)\, de^{\theta}, \\ \Delta x_2(t_k) &= -(k^2 + 1)x_2(t_k) + k^2 \int_{-1}^{0} x_2(t_k + \theta)\, de^{\theta}, \\ & t_k = k^2, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.1.2.22}$$

Now for $\lambda = 0.5$ inequality (3.1.2.18) becomes $k^2(3 - e^{0.5}) \leq e^{\kappa(2k-1)}$. Obviously, for any $\kappa > 0$ this inequality is valid for all natural k large enough. For instance, for $\kappa = 0.4$ inequality (3.1.2.18) holds for $k \geq 6$, while for $\kappa = 0.1$ it holds for $k \geq 41$. Thus, according to Corollary 3.1.3.3, the equilibrium point $(0, 0)^T$ of the impulsive system (3.1.2.19), (3.1.2.22) is globally exponentially stable with Lyapunov exponent any $\lambda \in (0, 0.5)$.

The results of the present subsection were reported at the Conference on Differential and Difference Equations and Applications, Rajecké Teplice, Slovakia, 2010, and published in [8].

3.1.3 Cohen-Grossberg neural networks with S-type distributed delays and reaction-diffusion terms

It is well known that diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields [86], so the activations must be considered to vary in space as well as in time. The papers [83, 84] are devoted to the exponential stability of impulsive Cohen-Grossberg neural networks with, respectively, time-varying and distributed delays and reaction-diffusion terms. In the above cited papers and many others as well as in our recent paper [16] the stability conditions were independent of the diffusion. On the other hand, in [98, 118, 119] the estimate of the exponential convergence rate depends on the reaction-diffusion.

In the present subsection we consider an impulsive Cohen-Grossberg neural network with both time-varying and S-type distributed delays [26, 66, 77, 111] and reaction-diffusion terms as in [105, 118, 119] which are of a form more general than in [83, 84], and zero Dirichlet boundary conditions. By using Hardy-Poincaré inequality as in [119], under suitable conditions in terms of $M-$matrices which involve the reaction-diffusion coefficients and the dimension and size of the spatial domain, it is proved that for the system with zero Dirichlet boundary conditions the equilibrium point is globally exponentially stable. The estimate of the Lyapunov exponent is more precise than those obtained by using Hardy-Sobolev inequality or just the nonpositivity of the reaction-diffusion operators. Examples are given.

We consider the impulsive Cohen-Grossberg neural network with time-varying and S-type distributed delays and reaction-diffusion terms, and zero Dirichlet boundary conditions:

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} &= \sum_{\nu=1}^{n}\frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right)+\alpha_i(u_i(t,x))\Bigg[-\beta_i(u_i(t,x)) \\ &+ \sum_{j=1}^{m}a_{ij}f_j(u_j(t,x))+\sum_{j=1}^{m}b_{ij}g_j(u_j(t-\tau_{ij}(t),x)) \qquad (3.1.3.1)\\ &+ \sum_{j=1}^{m}c_{ij}\int_{-\infty}^{0}h_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)+J_i\Bigg], \quad t>0, \quad t\neq t_k,\\ \Delta u_i(t_k,x) &= -B_{ik}u_i(t_k,x)+\int_{t_{k-1}-t_k}^{0}u_i(t_k+\theta)\,d\zeta_{ik}(\theta), \quad k\in\mathbb{N}, \qquad (3.1.3.2)\end{aligned}$$

$$u_i|_{\partial\Omega} = 0, \quad u_i(s,x) = \phi_i(s,x),\ s \le 0,\ x \in \Omega,\ i = \overline{1,m},$$

where $m \ge 2$ is the number of neurons in the network; $\Omega \subset \mathbb{R}^n$ is a bounded open set containing the origin, with smooth boundary $\partial\Omega$ and $\operatorname{mes}\Omega > 0$; $D_{i\nu}(t,x,u) > 0$ are smooth functions corresponding to the transmission diffusion operator along the $i-$th neuron; $\alpha_i(u_i)$ represent amplification functions; $\beta_i(u_i)$ are appropriately behaving functions which support the stabilizing feedback term $-\alpha_i(u_i)\beta_i(u_i)$ of the $i-$th neuron; a_{ij}, b_{ij}, c_{ij} denote the connection weights (or strengths) of the synaptic connections between the $j-$th neuron and the $i-$th neuron; $f_j(u_j)$, $g_j(u_j)$, $h_j(u_j)$ denote the activation functions of the $j-$th neuron; J_i denotes external input to the $i-$th neuron; $\tau_{ij}(t)$ correspond to the transmission delays; the past effect of the $j-$th neuron on the $i-$th neuron is given by the Lebesgue-Stieltjes integral $\int_{-\infty}^0 h_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)$; $\Delta u_i(t_k,x) = u_i(t_k+0,x) - u_i(t_k-0,x)$ denote impulsive state displacements at fixed moments (instants) of time t_k, $k \in \mathbb{N}$, involving Lebesgue-Stieltjes integrals; B_{ik} are real numbers. Here it is assumed that $u_i(t_k-0,x)$ and $u_i(t_k+0,x)$ denote respectively the left-hand and right-hand limit at t_k, and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k \to +\infty$ as $k \to +\infty$. The initial data $\phi(s,x) = (\phi_1(s,x),\ldots,\phi_m(s,x))^T$ is such that

$$\sup_{s\le 0}\sum_{i=1}^{m}\int_{\Omega}\phi_i^2(s,x)\,dx < \infty.$$

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto u(t,x)$ we assume that $u_i(t_k,x) \equiv u_i(t_k-0,x)$ (while in [83, 84] continuity from the right is assumed). It is clear that, in general, the derivatives $\frac{\partial u_i}{\partial t}(t_k,x)$ do not exist. On the other hand, according to (3.1.3.1), there do exist the limits $\frac{\partial u_i}{\partial t}(t_k \mp 0,x)$. According to the above convention, we assume $\frac{\partial u_i}{\partial t}(t_k,x) \equiv \frac{\partial u_i}{\partial t}(t_k-0,x)$.

Throughout the subsection we assume that:

A3.1.3.1. $n \ge 3$ and the positive constants ω and R_Ω are such that for $x = (x_1,\ldots,x_n)^T \in \Omega \subset \mathbb{R}^n$ we have $|x|^2 = \sum_{\nu=1}^n x_\nu^2 < \omega^2$ and $\operatorname{mes}\{x \in \mathbb{R}^n : |x| < R_\Omega\} = \operatorname{mes}\Omega$.

A3.1.3.2. There exist constants $\underline{D}_i > 0$ ($i = \overline{1,m}$) such that $D_{i\nu}(t,x,u) \ge \underline{D}_i$ for $\nu = \overline{1,n}$, $t \ge 0$, $x \in \Omega$ and $u \in \mathbb{R}^m$.

A3.1.3.3. The amplification functions $\alpha_i : \mathbb{R} \to (0, +\infty)$ are continuous and bounded in the sense that $0 < \underline{\alpha}_i \le \alpha_i(u) \le \overline{\alpha}_i$ for $u \in \mathbb{R}$, $i = \overline{1, m}$.

A3.1.3.4. The stabilizing functions $\beta_i : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing, namely, $0 < \underline{\beta}_i \le \dfrac{\beta_i(u) - \beta_i(v)}{u - v}$ for $u, v \in \mathbb{R}$, $u \ne v$, $i = \overline{1, m}$.

A3.1.3.5. For the activation functions $f_i(u), g_i(u), h_i(u)$ there exist positive constants F_i, G_i, H_i such that

$$\begin{aligned} F_i &= \sup_{u \ne v} \left| \frac{f_i(u) - f_i(v)}{u - v} \right|, \qquad G_i = \sup_{u \ne v} \left| \frac{g_i(u) - g_i(v)}{u - v} \right|, \\ H_i &= \sup_{u \ne v} \left| \frac{h_i(u) - h_i(v)}{u - v} \right| \qquad \text{for all } \ u, v \in \mathbb{R}, \ u \ne v, \ i = \overline{1, m}. \end{aligned}$$

A3.1.3.6. $\tau_{ij}(t)$ satisfy $0 \le \tau_{ij}(t) \le \tau_{ij}$, $0 \le \dot{\tau}_{ij}(t) \le \mu_{ij} < 1$ $(i, j = \overline{1, m})$.

A3.1.3.7. $\eta_{ij}(\theta)$ $(i, j = \overline{1, m})$, $\zeta_{ik}(\theta)$ $(i = \overline{1, m},\ k \in \mathbb{N})$ are nondecreasing bounded variation functions on $(-\infty, 0]$ and $[t_{k-1} - t_k, 0]$, respectively, and

$$\int_{-\infty}^{0} e^{-\lambda\theta}\, d\eta_{ij}(\theta) = K_{ij}(\lambda)$$

are continuous functions on $[0, \lambda_0)$ for some $\lambda_0 > 0$ and $K_{ij}(0) = 1$ (without loss of generality).

Due to the zero Dirichlet boundary conditions system (3.1.3.1) can have just one equilibrium point $\mathbf{0} = (0, 0, \ldots, 0)^T$. It is really an equilibrium point of system (3.1.3.1) if and only if

$$-\beta_i(0) + \sum_{j=1}^{m} \left(a_{ij} f_j(0) + b_{ij} g_j(0) + c_{ij} h_j(0)\right) + J_i = 0, \quad i = \overline{1, m}. \quad (3.1.3.3)$$

From equations (3.1.3.1) and (3.1.3.3) we deduce

$$
\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= \sum_{\nu=1}^{n} \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right) + \alpha_i(u_i(t,x))\Bigg[-\tilde{\beta}_i(u_i(t,x)) \\
&+ \sum_{j=1}^{m} a_{ij}\tilde{f}_j(u_j(t,x)) + \sum_{j=1}^{m} b_{ij}\tilde{g}_j(u_j(t-\tau_{ij}(t),x)) \\
&+ \sum_{j=1}^{m} c_{ij}\int_{-\infty}^{0}\tilde{h}_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)\Bigg], \quad t>0, \quad t\neq t_k, \quad (3.1.3.4)
\end{aligned}
$$

where $\tilde{\beta}_i(u) = \beta_i(u) - \beta_i(0)$, $\tilde{f}_i(u) = f_i(u) - f_i(0)$, $\tilde{g}_i(u) = g_i(u) - g_i(0)$, $\tilde{h}_i(u) = h_i(u) - h_i(0)$, $i = \overline{1,m}$. Now conditions **A3.1.3.4**, **A3.1.3.5** imply

$$
\tilde{\beta}_i(u)u \geq \underline{\beta}_i u^2, \quad |\tilde{f}_i(u)| \leq F_i|u|, \quad |\tilde{g}_i(u)| \leq G_i|u|, \quad |\tilde{h}_i(u)| \leq H_i|u|
$$

for all $u \in \mathbb{R}$ and $i = \overline{1,m}$.

Denote

$$
\|u_i(t,\cdot)\| = \left(\int_\Omega u_i^2(t,x)\,dx\right)^{1/2}.
$$

Definition 3.1.3.1. The equilibrium point $u = \mathbf{0}$ of system (3.1.3.1), (3.1.3.2) is said to be *globally exponentially stable* (with Lyapunov exponent λ) if there exist constants $\lambda > 0$ and $M \geq 1$ such that for any solution $u(t,x) = (u_1(t,x),\ldots,u_m(t,x))^T$ of system (3.1.3.1), (3.1.3.2) we have

$$
\sum_{i=1}^{m}\|u_i(t,\cdot)\| \leq M\sup_{s\leq 0}\sum_{i=1}^{m}\|\phi_i(s,\cdot)\|e^{-\lambda t} \quad \text{for all } t \geq 0, \quad x \in \Omega.
$$

Definition 3.1.3.2. [27] A real matrix $A = (a_{ij})_{m\times m}$ is said to be an $M-$matrix if $a_{ij} \leq 0$ for $i,j = \overline{1,m}$, $i \neq j$, and all successive principal minors of A are positive.

Lemma 3.1.3.1. [27] *Let $A = (a_{ij})_{m\times m}$ be a real matrix with non-positive off-diagonal elements. Then A is an $M-$matrix if and only if one of the following conditions holds:*

(1) *There exists a vector $\xi = (\xi_1,\xi_2,\ldots,\xi_m)^T$ with $\xi_i > 0$ such that every component of $\xi^T A$ is positive — that is, $\sum\limits_{i=1}^{m}\xi_i a_{ij} > 0$, $j = \overline{1,m}$.*

(2) *There exists a vector $\xi = (\xi_1,\xi_2,\ldots,\xi_m)^T$ with $\xi_i > 0$ such that every component of $A\xi$ is positive — that is, $\sum\limits_{j=1}^{m} a_{ij}\xi_j > 0$, $i = \overline{1,m}$.*

For more details about $M-$matrices the reader is referred to [55, 73].

Further on we will need the following lemma.

Lemma 3.1.3.2. (Hardy-Poincaré inequality [39]) *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ *be a bounded open set containing the origin and* $u \in H_0^1(\Omega)$*. Then*

$$\int_\Omega |\nabla u|^2\, dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\, dx + \frac{\Lambda_2}{R_\Omega^2}\int_\Omega u^2\, dx,$$

where $\Lambda_2 = 5.783\ldots$ *is the first eigenvalue of the Dirichlet Laplacian of the unit disc in* $\mathbb{R}^2$ *and* R_Ω *is the radius of a ball in* $\mathbb{R}^n$ *having the same measure as* Ω.

Hardy-Poincaré inequality implies Hardy-Sobolev inequality [1]

$$\int_\Omega |\nabla u|^2\, dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\, dx.$$

Now let us introduce the following matrices: $\underline{D} = \operatorname{diag}(\underline{D}_1, \ldots, \underline{D}_m)$,

$$\begin{array}{lll} \underline{\alpha} = \operatorname{diag}(\underline{\alpha}_1, \ldots, \underline{\alpha}_m), & \overline{\alpha} = \operatorname{diag}(\overline{\alpha}_1, \ldots, \overline{\alpha}_m), & \underline{\beta} = \operatorname{diag}(\underline{\beta}_1, \ldots, \underline{\beta}_m), \\ F = \operatorname{diag}(F_1, \ldots, F_m), & G = \operatorname{diag}(G_1, \ldots, G_m), & H = \operatorname{diag}(H_1, \ldots, H_m), \\ |A| = (|a_{ij}|)_{m\times m}, & |B(\mu)| = \left(\frac{|b_{ij}|}{1-\mu_{ij}}\right)_{m\times m}, & |C| = (|c_{ij}|)_{m\times m}. \end{array}$$

Theorem 3.1.3.1. *Suppose that system* (3.1.3.1), (3.1.3.2) *satisfies assumptions* **A3.1.3.1**–**A3.1.3.7** *and equalities* (3.1.3.3) *hold. If there exists a vector* $\xi = (\xi_1, \ldots, \xi_m)^T$ *with* $\xi_i > 0$ *and a number* $\lambda \in (0, \lambda_0)$ *such that*

$$\begin{aligned} &\sum_{i=1}^m \left\{ \left[\lambda - \left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}_i - \underline{\alpha}_i \underline{\beta}_i\right]\delta_{ij} \right. && (3.1.3.5) \\ &+ \left. \overline{\alpha}_i \left[|a_{ij}|F_j + |b_{ij}|G_j \frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}} + |c_{ij}|H_j K_{ij}(\lambda)\right]\right\}\xi_i < 0,\ j = \overline{1,m}, \end{aligned}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$ *for* $j \neq i$*, then there exists a constant* $M \geq 1$ *such that for any solution* $u(t,x) = (u_1(t,x), \ldots, u_m(t,x))^T$ *of system* (3.1.3.1), (3.1.3.2) *we have*

$$\begin{aligned} \sum_{i=1}^m \|u_i(t,\cdot)\| \;\leq\; & M e^{-\lambda t} \prod_{k=1}^{i(0,t)} \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^0 e^{-\lambda\theta}\, d\zeta_{ik}(\theta)\right) \\ & \times \sup_{s\leq 0} \sum_{i=1}^m \|u_i(s,\cdot)\|, \qquad t \geq 0. \qquad (3.1.3.6) \end{aligned}$$

Remark 3.1.3.1. If in the subsequent proof we choose to use Hardy-Sobolev inequality rather than Hardy-Poincaré inequality, the inequalities (3.1.3.5) should be replaced by

$$\sum_{i=1}^{m}\left\{\left[\lambda-\left(\frac{n-2}{2\omega}\right)^{2}\underline{D}_i-\underline{\alpha}_i\underline{\beta}_i\right]\delta_{ij}\right. \tag{3.1.3.7}$$
$$+\quad \left.\overline{\alpha}_i\left[|a_{ij}|F_j+|b_{ij}|G_j\frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}}+|c_{ij}|H_jK_{ij}(\lambda)\right]\right\}\xi_i<0,\ j=\overline{1,m}.$$

Further on, if we use just the nonpositivity of the reaction-diffusion operators, then inequalities (3.1.3.5) should be replaced by

$$\sum_{i=1}^{m}\left\{\left(\lambda-\underline{\alpha}_i\underline{\beta}_i\right)\delta_{ij}+\overline{\alpha}_i\left[|a_{ij}|F_j+|b_{ij}|G_j\frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}}+|c_{ij}|H_jK_{ij}(\lambda)\right]\right\}\xi_i<0 \tag{3.1.3.8}$$

for $j=\overline{1,m}$. It is clear that if ξ and λ satisfy inequalities (3.1.3.7) or (3.1.3.8), they satisfy (3.1.3.5). On the other hand, we can find ξ and λ satisfying inequalities (3.1.3.5) but not satisfying any of the sets of inequalities (3.1.3.7) and (3.1.3.8). Thus by using Hardy-Poincaré inequality, we can prove global exponential stability with a larger Lyapunov exponent than by using Hardy-Sobolev inequality or the nonpositivity of the reaction-diffusion operators, and in some cases when the last two methods do not work we can still prove global exponential stability.

Proof. Let us note that there exists a vector $\xi=(\xi_1,\ldots,\xi_m)^T$ with $\xi_i>0$ and a number $\lambda\in(0,\lambda_0)$ such that inequalities (3.1.3.5) hold if and only if

$$\mathcal{A}=\left(\frac{(n-2)^2}{4\omega^2}+\frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}+\underline{\alpha}\underline{\beta}-\overline{\alpha}\left(|A|F+|B(\mu)|G+|C|H\right)$$

is an $M-$matrix. In fact, if $\mathcal{A}$ is an $M-$matrix, from Lemma 3.1.3.1 there exists a vector $\xi>0$ such that every component of $-\xi^T\mathcal{A}$ is negative. By continuity, there exists a $\lambda\in(0,\lambda_0)$ such that (3.1.3.5) hold. Conversely, if (3.1.3.5) hold for some $\lambda^*\in(0,\lambda_0)$, then they still hold for all $\lambda\in[0,\lambda^*]$. For $\lambda=0$, from Lemma 3.1.3.1 we deduce that $\mathcal{A}$ is an $M-$matrix.

First we shall derive the estimate

$$D^+\|u_i(t,\cdot)\| \leq -\left[\left(\frac{(n-2)^2}{4\omega^2}+\frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}_i+\underline{\alpha}_i\underline{\beta}_i\right]\|u_i(t,\cdot)\| \qquad (3.1.3.9)$$
$$+ \overline{\alpha}_i\sum_{j=1}^m\Big\{|a_{ij}|\,F_j\,\|u_j(t,\cdot)\|+|b_{ij}|\,G_j\,\|u_j(t-\tau_{ij}(t),\cdot)\|$$
$$+ |c_{ij}|\,H_j\int_{-\infty}^0\|u_j(t+\theta,\cdot)\|\,d\eta_{ij}(\theta)\Big\},\quad t>0,\quad t\neq t_k,$$

where D^+ denotes the upper right Dini derivative.

Let t be such that $\|u_i(t,\cdot)\|\neq 0$. We multiply the $i-$th differential equation in (3.1.3.4) by $u_i(t,x)$ and integrate over the domain Ω:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_i^2(t,x)\,dx=\int_\Omega\sum_{\nu=1}^n\frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right)u_i(t,x)\,dx$$
$$- \int_\Omega\alpha_i(u_i(t,x))\tilde{\beta}_i(u_i(t,x))u_i(t,x)\,dx$$
$$+ \int_\Omega\alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^m a_{ij}\tilde{f}_j(u_j(t,x))\,dx$$
$$+ \int_\Omega\alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^m b_{ij}\tilde{g}_j(u_j(t-\tau_{ij}(t),x))\,dx$$
$$+ \int_\Omega\alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^m c_{ij}\int_{-\infty}^0\tilde{h}_j(u_j(t+\theta,x))\,d\eta_{ij}(\theta)\;dx.$$

By using Green's formula, the zero Dirichlet boundary conditions, Lemma 3.1.3.2 and assumptions **A3.1.4.1**, **A3.1.4.2** we have

$$\int_\Omega\sum_{\nu=1}^n\frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right)u_i(t,x)\,dx$$

$$=-\int_\Omega\sum_{\nu=1}^n D_{i\nu}(t,x,u)\left(\frac{\partial u_i(t,x)}{\partial x_\nu}\right)^2dx\leq-\underline{D}_i\int_\Omega\sum_{\nu=1}^n\left(\frac{\partial u_i(t,x)}{\partial x_\nu}\right)^2dx$$

$$=-\underline{D}_i\int_\Omega|\nabla u_i(t,x)|^2\,dx\leq-\underline{D}_i\left[\left(\frac{n-2}{2}\right)^2\int_\Omega\frac{u_i^2(t,x)}{|x|^2}\,dx+\frac{\Lambda_2}{R_\Omega^2}\int_\Omega u_i^2(t,x)\,dx\right]$$

$$\leq -\left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right) \underline{D}_i \int_\Omega u_i^2(t,x)\,dx = -\left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right) \underline{D}_i \|u_i(t,\cdot)\|^2.$$

If we prefer to use Hardy-Sobolev inequality, we obtain

$$\int_\Omega \sum_{\nu=1}^n \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right) u_i(t,x)\,dx \leq -\left(\frac{n-2}{2\omega}\right)^2 \underline{D}_i \|u_i(t,\cdot)\|^2$$

and we can complete the proof by using inequalities (3.1.3.7).

Finally, in the case of zero Dirichlet or Neumann boundary conditions and no restrictions on the dimension of the spatial domain Ω, we have

$$\begin{aligned} &\int_\Omega \sum_{\nu=1}^n \frac{\partial}{\partial x_\nu}\left(D_{i\nu}(t,x,u)\frac{\partial u_i(t,x)}{\partial x_\nu}\right) u_i(t,x)\,dx \\ = \;& -\int_\Omega \sum_{\nu=1}^n D_{i\nu}(t,x,u)\left(\frac{\partial u_i(t,x)}{\partial x_\nu}\right)^2 dx \leq 0 \end{aligned}$$

and we can complete the proof by using inequalities (3.1.3.8).

Next, we have

$$\int_\Omega \alpha_i(u_i(t,x))\tilde{\beta}_i(u_i(t,x))u_i(t,x)\,dx \geq \underline{\alpha}_i\underline{\beta}_i \int_\Omega u_i^2(t,x)\,dx = \underline{\alpha}_i\underline{\beta_i}\|u_i(t,\cdot)\|^2;$$

$$\begin{aligned} &\int_\Omega \alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^m a_{ij}\tilde{f}_j(u_j(t,x))\,dx \\ &\leq \overline{\alpha}_i \sum_{j=1}^m |a_{ij}| \int_\Omega |u_i(t,x)|\,F_j\,|u_j(t,x)|\,dx \\ \leq \;& \overline{\alpha}_i \sum_{j=1}^m |a_{ij}|F_j \left(\int_\Omega u_i^2(t,x)\,dx\right)^{1/2}\left(\int_\Omega u_j^2(t,x)\,dx\right)^{1/2} \\ &= \overline{\alpha}_i \sum_{j=1}^m |a_{ij}|F_j \|u_i(t,\cdot)\|\;\|u_j(t,\cdot)\|. \end{aligned}$$

Similarly,

$$\int_\Omega \alpha_i(u_i(t,x))u_i(t,x)\sum_{j=1}^m b_{ij}\tilde{g}_j(u_j(t-\tau_{ij}(t),x))\,dx$$

$$\leq \overline{\alpha}_i \sum_{j=1}^{m} |b_{ij}| G_j \|u_i(t,\cdot)\| \, \|u_j(t-\tau_{ij}(t),\cdot)\|$$

and

$$\int_\Omega \alpha_i(u_i(t,x)) u_i(t,x) \sum_{j=1}^{m} c_{ij} \int_{-\infty}^{0} \tilde{h}_j(u_j(t+\theta,x))\, d\eta_{ij}(\theta)\, dx$$

$$\leq \overline{\alpha}_i \sum_{j=1}^{m} |c_{ij}| H_j \|u_i(t,\cdot)\| \int_{-\infty}^{0} \|u_j(t+\theta,\cdot)\|\, d\eta_{ij}(\theta).$$

Combining the above inequalities, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_i(t,\cdot)\|^2 \;\; &\leq \;\; -\left[\left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}_i + \underline{\alpha}_i \underline{\beta}_i\right] \|u_i(t,\cdot)\|^2 \\
&+ \;\; \overline{\alpha}_i \sum_{j=1}^{m} \Big\{ |a_{ij}|\, F_j \, \|u_j(t,\cdot)\| + |b_{ij}|\, G_j \, \|u_j(t-\tau_{ij}(t),\cdot)\| \\
&+ \;\; |c_{ij}|\, H_j \int_{-\infty}^{0} \|u_j(t+\theta,\cdot)\|\, d\eta_{ij}(\theta) \Big\} \|u_i(t,\cdot)\|,
\end{aligned}$$

which implies (3.1.3.9) in view of $\|u_i(t,\cdot)\| \neq 0$.

Next, let us suppose that t is such that $\|u_i(t,\cdot)\| = 0$. If this equality holds for all t in some open interval, then $\frac{d}{dt}\|u_i(t,\cdot)\| = 0$ in this interval, and inequality (3.1.3.9) reduces to

$$\begin{aligned}
0 \leq \overline{\alpha}_i \sum_{j=1}^{m} \Big\{ |a_{ij}|\, F_j \, \|u_j(t,\cdot)\| \;\; &+ \;\; |b_{ij}|\, G_j \, \|u_j(t-\tau_{ij}(t),\cdot)\| \\
&+ \;\; |c_{ij}|\, H_j \int_{-\infty}^{0} \|u_j(t+\theta,\cdot)\|\, d\eta_{ij}(\theta) \Big\},
\end{aligned}$$

which is trivially satisfied. If this is not the case, we can find a sequence of times $\{t_l'\}$ such that $t_l' \to t$ and $\|u_i(t_l',\cdot)\| \neq 0$. Then the validity of inequality (3.1.3.9) for t_l', $l \in \mathbb{N}$ implies its validity for t.

If we introduce the notation $y_i(t) = e^{\lambda t}\|u_i(t,\cdot)\|$, then from inequality

(3.1.3.9) which we just proved we find

$$
\begin{aligned}
D^+ y_i(t) &\le \left[\lambda - \left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}_i - \underline{\alpha}_i\underline{\beta}_i\right] y_i(t) \\
&+ \overline{\alpha}_i \sum_{j=1}^m \Bigg\{ |a_{ij}| F_j y_j(t) + |b_{ij}| G_j y_j(t-\tau_{ij}(t)) e^{\lambda \tau_{ij}(t)} \\
&+ |c_{ij}| H_j \int_{-\infty}^0 e^{-\lambda\theta} y_j(t+\theta)\, d\eta_{ij}(\theta) \Bigg\}. \qquad (3.1.3.10)
\end{aligned}
$$

We consider a Lyapunov functional

$$
\begin{aligned}
V(t) &= \sum_{i=1}^m \Bigg\{ y_i(t) + \overline{\alpha}_i \sum_{j=1}^m |b_{ij}| G_j \frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}} \int_{t-\tau_{ij}(t)}^t y_j(s)\, ds \\
&+ \overline{\alpha}_i \sum_{j=1}^m |c_{ij}| H_j \int_{-\infty}^0 e^{-\lambda\theta} \left(\int_{t+\theta}^t y_j(s)\, ds \right) d\eta_{ij}(\theta) \Bigg\} \xi_i,
\end{aligned}
$$

where λ and ξ_i, $i = \overline{1,m}$, are as in inequalities (3.1.3.5).

We note that $V(t) \ge 0$ for $t \ge 0$ and

$$
V(0) \le M \sum_{i=1}^m \sup_{s \le 0} y_i(s) \qquad (3.1.3.11)
$$

with

$$
M = \max_{i=\overline{1,m}} \left\{ \xi_i + G_i \sum_{j=1}^m |b_{ji}| \frac{\overline{\alpha}_j e^{\lambda\tau_{ji}} \tau_{ji}}{1-\mu_{ji}}\, \xi_j + H_i \sum_{j=1}^m |c_{ji}| \overline{\alpha}_j \int_{-\infty}^0 e^{-\lambda\theta}(-\theta)\, d\eta_{ji}(\theta)\, \xi_j \right\}.
$$

The above integral is convergent because of $\lambda < \lambda_0$.

Calculating the rate of change of $V(t)$ along the solutions of system (3.1.3.1), by using successively inequalities (3.1.3.10), (3.1.3.5) and condition **A3.1.3.6** we obtain

$$
\begin{aligned}
D^+ V(t) &\le \sum_{j=1}^m y_j(t) \sum_{i=1}^m \Bigg\{ \left[\lambda - \left(\frac{(n-2)^2}{4\omega^2} + \frac{\Lambda_2}{R_\Omega^2}\right)\underline{D}_i - \underline{\alpha}_i\underline{\beta}_i\right] \delta_{ij} \\
&\qquad + \overline{\alpha}_i \left[|a_{ij}| F_j + |b_{ij}| G_j \frac{e^{\lambda\tau_{ij}}}{1-\mu_{ij}} + |c_{ij}| H_j K_{ij}(\lambda) \right] \Bigg\} \xi_i \\
&+ \sum_{i=1}^m \overline{\alpha}_i \xi_i \sum_{j=1}^m |b_{ij}| G_j y_j(t-\tau_{ij}(t)) \left(e^{\lambda\tau_{ij}(t)} - e^{\lambda\tau_{ij}} \frac{1-\dot{\tau}_{ij}(t)}{1-\mu_{ij}} \right) \le 0.
\end{aligned}
$$

This implies that $V(t)$ is nonincreasing on every interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$V(t) \le V(t_{k-1} + 0) \quad \text{for} \quad t_{k-1} < t \le t_k. \tag{3.1.3.12}$$

In particular,

$$V(t_k) \le V(t_{k-1} + 0), \qquad k \in \mathbb{N}. \tag{3.1.3.13}$$

Further on, for $k \in \mathbb{N}$ we find successively

$$u_i(t_k + 0, x) = (1 - B_{ik}) u_i(t_k, x) + \int_{t_{k-1}-t_k}^{0} u_i(t_k + \theta, x)\, d\zeta_{ik}(\theta),$$

$$\|u_i(t_k + 0, \cdot)\| \le |1 - B_{ik}|\, \|u_i(t_k, \cdot)\| + \int_{t_{k-1}-t_k}^{0} \|u_i(t_k + \theta, \cdot)\|\, d\zeta_{ik}(\theta)$$

and

$$y_i(t_k + 0) \le |1 - B_{ik}| y_i(t_k) + \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta} y_i(t_k + \theta)\, d\zeta_{ik}(\theta).$$

Making use of (3.1.3.12) and (3.1.3.13), we obtain

$$\begin{aligned} V(t_k + 0) &\le \max_{i=\overline{1,m}} |1 - B_{ik}| V(t_k) + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta)\, V(t_{k-1} + 0) \\ &\le \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \right) V(t_{k-1} + 0). \end{aligned}$$

Combining the last estimate with (3.1.3.12) and (3.1.3.13), we derive

$$V(t) \le \prod_{k=1}^{i(0,t)} \left(\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \right) V(0), \quad t \ge 0.$$

Finally, by using inequality (3.1.3.11) and the definitions of $V(t)$ and $y_i(t)$, we obtain estimate (3.1.3.6). □

It is clear that inequality (3.1.3.6) guarantees global exponential stability of the equilibrium point $\mathbf{0}$ of the system without impulses (3.1.3.1). Further on, for three sets of additional assumptions on the impulse effects we can show that inequality (3.1.3.6) implies global exponential stability of the equilibrium point $\mathbf{0}$ of the impulsive system (3.1.3.1), (3.1.3.2).

Corollary 3.1.3.1. *Let all conditions of Theorem 3.1.3.1 hold and*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \le 1 \qquad (3.1.3.14)$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point $\mathbf{0}$ of the impulsive system (3.1.3.1), (3.1.3.2) *is globally exponentially stable with Lyapunov exponent λ.*

Corollary 3.1.3.2. *Let all conditions of Theorem 3.1.3.1 hold and*

$$\limsup_{t\to\infty} \frac{i(0,t)}{t} = p < +\infty.$$

Let there exist a positive constant B satisfying the inequalities

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \le B$$

for all sufficiently large values of $k \in \mathbb{N}$, and $p \ln B < \lambda$. Then for any $\tilde{\lambda} \in (0, \lambda - p\ln B)$ the equilibrium point $\mathbf{0}$ of the impulsive system (3.1.3.1), (3.1.3.2) *is globally exponentially stable with Lyapunov exponent $\tilde{\lambda}$.*

Corollary 3.1.3.3. *Let all conditions of Theorem 3.1.3.1 hold and there exists a constant $\kappa \in (0, \lambda)$ satisfying the inequality*

$$\max_{i=\overline{1,m}} |1 - B_{ik}| + \max_{i=\overline{1,m}} \int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) \le e^{\kappa(t_k - t_{k-1})} \qquad (3.1.3.15)$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point $\mathbf{0}$ of the impulsive system (3.1.3.1), (3.1.3.2) *is globally exponentially stable with Lyapunov exponent $\lambda - \kappa$.*

The above corollaries are similar to those in §3.1.1 and §3.1.2.

Example. Denote $\varphi(t) = (|t+1| - |t-1|)/2$. Let Ω be the unit ball in $\mathbb{R}^3$: $\Omega = \{x \in \mathbb{R}^3 \,|\, |x| < 1\}$ and let ∇^2 denote the Laplacian in $\mathbb{R}^3$: $\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}$, $i = 1, 2$. Consider the system

$$
\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= 16\nabla^2 u_1(t,x) + \big(2+\sin u_1(t,x)\big)\Big\{-2u_1(t,x) \\
&\quad +0.5\arctan u_1(t,x) + 0.3\varphi(u_2(t,x)) \\
&\quad +0.1u_1(t-\frac{1}{2}\arctan t, x) + 0.12\arctan u_2(t-\frac{2}{3}\varphi(t),x) \\
&\quad +0.1\int_{-\infty}^{0} u_1(t+\theta,x)\,de^{\theta} + 0.15\int_{-\infty}^{0} u_2(t+\theta,x)\,de^{\theta}\Big\}, \\
\frac{\partial u_2(t,x)}{\partial t} &= 20\nabla^2 u_2(t,x) + \big(3+\sin u_2(t,x)\big)\Big\{-3u_2(t,x) \\
&\quad -0.6\varphi(u_1(t,x)) + 0.5\arctan u_2(t,x) \\
&\quad +0.16u_1(t-1-\frac{1}{3}\varphi(t),x) - 0.3\arctan u_2(t-2-\frac{3}{4}\varphi(t),x) \\
&\quad +0.1\int_{-\infty}^{0} u_1(t+\theta,x)\,de^{\theta} - 0.2\int_{-\infty}^{0} u_2(t+\theta,x)\,de^{\theta}\Big\}, \\
\Delta u_i(t_k,x) &= -B_{ik}u_i(t_k,x) + \int_{t_{k-1}-t_k}^{0} u_i(t_k+\theta,x)\,d\zeta_{ik}(\theta), \quad k\in\mathbb{N}, \\
u_i|_{\partial\Omega} = 0, &\quad u_i(s,x) = \phi_i(s,x), \quad s\le 0, \quad x\in\Omega, \quad i=1,2, \qquad (3.1.3.16)
\end{aligned}
$$

where the initial data $\phi_1(s,x)$, $\phi_2(s,x)$ satisfies

$$
\sup_{s\le 0}\int_{\Omega}\left(\phi_1^2(s,x)+\phi_2^2(s,x)\right)\,dx < \infty.
$$

This system has a unique equilibrium point $(0,0)^T$ and assumptions **A3.1.3.1**–**A3.1.3.7** hold with $n=3$, $\omega=1$, $R_\Omega=1$,

$$
\underline{D} = \begin{pmatrix} 16 & 0 \\ 0 & 20 \end{pmatrix}, \quad \underline{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \overline{\alpha} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \quad \underline{\beta} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},
$$

$\tau_{11}=\pi/4,\ \tau_{12}=2/3,\ \tau_{21}=4/3,\ \tau_{22}=11/4,\ \mu_{11}=1/2,\ \mu_{12}=2/3,\ \mu_{21}=1/3,$ $\mu_{22}=3/4,\ K_{ij}(\lambda)=1/(1-\lambda),\ i,j=1,2,\ \lambda_0=1,\ F=G=H=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$

$$
|A| = \begin{pmatrix} 0.5 & 0.3 \\ 0.6 & 0.5 \end{pmatrix}, \quad |B(\mu)| = \begin{pmatrix} 0.2 & 0.36 \\ 0.24 & 1.2 \end{pmatrix}, \quad |C| = \begin{pmatrix} 0.1 & 0.15 \\ 0.1 & 0.2 \end{pmatrix},
$$

the matrix

$$
\mathcal{A} = \begin{pmatrix} 96.528 & -2.43 \\ -3.76 & 119.06 \end{pmatrix}
$$

is an $M-$matrix. Further on, the vector $\xi = (1,\, 1)^T$ is such that $\xi^T \mathcal{A} = (92.768,\, 116.63)$ has positive components. Let us denote by $\Phi_j(\lambda)$, $j = 1, 2$, the left-hand sides of inequalities (3.1.3.5) for the given vector ξ. Then

$$\begin{aligned} \Phi_1(\lambda) &= \lambda + 0.6e^{\pi\lambda/4} + 0.96e^{4\lambda/3} + \frac{0.7}{1-\lambda} - 94.628, \\ \Phi_2(\lambda) &= \lambda + 1.08e^{2\lambda/3} + 4.8e^{11\lambda/4} + \frac{1.25}{1-\lambda} - 123.76. \end{aligned}$$

Since $\Phi_1(0.975279) = -60.52221507 < 0$ and $\Phi_2(0.975279) = -0.00143964 < 0$, we can take $\lambda = 0.975279$. Theorem 3.1.3.1 is valid for system (3.1.3.16).

If we use Hardy-Sobolev inequality, then inequalities (3.1.3.7) are satisfied with $\xi = (6,\, 5)^T$ and $\lambda = 0.02849$. Using just the nonpositivity of the reaction-diffusion operators cannot provide an estimate of the form (3.1.3.6).

Let us consider the impulsive conditions

$$\begin{aligned} \Delta u_1(t_k, x) &= -\frac{1}{2}u_1(t_k, x) + \frac{1}{4}\int_{-1}^{0} u_1(t_k+\theta, x)\, de^{\theta}, \qquad (3.1.3.17) \\ \Delta u_2(t_k, x) &= -\frac{1}{4}u_2(t_k, x) + \frac{1}{4}\int_{-1}^{0} u_2(t_k+\theta, x)\, de^{\theta}, \quad t_k = k,\ k \in \mathbb{N}. \end{aligned}$$

Now

$$\max_{i=1,2}|1-B_{ik}| + \max_{i=1,2}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) = \frac{3}{4} + \frac{1}{4}\int_{-1}^{0} e^{-\lambda\theta}\, de^{\theta} = \frac{3}{4} + \frac{1-e^{\lambda-1}}{4(1-\lambda)},$$

$\lambda < 1$. Obviously, inequalities (3.1.3.14) are valid for all $k \in \mathbb{N}$ and all $\lambda \in (0, 1)$, in particular, for $\lambda = 0.975279$. According to Corollary 3.1.3.1, the equilibrium point $(0, 0)^T$ of system (3.1.3.16) with impulsive conditions (3.1.3.17) is globally exponentially stable with Lyapunov exponent 0.975279.

Next consider the impulsive conditions

$$\begin{aligned} \Delta u_1(t_k, x) &= -100u_1(t_k, x) + \int_{-10}^{0} u_1(t_k+\theta, x)\, de^{\theta}, \qquad (3.1.3.18) \\ \Delta u_2(t_k, x) &= -50u_2(t_k, x) + \int_{-10}^{0} u_2(t_k+\theta, x)\, de^{\theta},\ t_k = 10k,\ k \in \mathbb{N}. \end{aligned}$$

Now

$$\max_{i=1,2}|1-B_{ik}| + \max_{i=1,2}\int_{t_{k-1}-t_k}^{0} e^{-\lambda\theta}\, d\zeta_{ik}(\theta) = 99 + \int_{-10}^{0} e^{-\lambda\theta}\, de^{\theta} = 99 + \frac{1-e^{10(\lambda-1)}}{1-\lambda}$$

for $\lambda \in (0,1)$ and we can take $B = 107.8598086$ which is the value of the above expression for $\lambda = 0.975279$. Further on, $p = 0.1$, for $\lambda = 0.975279$ we have $\lambda - p \ln B \approx 0.975279 - 0.1 \times 4.680832315 \approx 0.507196$. According to Corollary 3.1.3.2, the equilibrium point $(0,0)^T$ of system (3.1.3.16) with impulsive conditions (3.1.3.18) is globally exponentially stable with Lyapunov exponent any $\lambda \in (0, 0.507196)$, we can take $\lambda = 0.5$.

Finally, let us consider the impulsive conditions

$$\begin{aligned} \Delta u_1(t_k,x) &= -(k\ +1)u_1(t_k,x) + k \int_{-2k+1}^{0} u_1(t_k+\theta,x)\, de^{\theta}, \\ \Delta u_2(t_k,x) &= -(k^2+1)u_2(t_k,x) + k^2 \int_{-2k+1}^{0} u_2(t_k+\theta,x)\, de^{\theta}, \\ & \qquad t_k = k^2, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.1.3.19}$$

Now for $\lambda = 0.975279$ inequality (3.1.3.15) becomes $42.45143805k^2 \leq e^{\kappa(2k-1)}$. Obviously, for any $\kappa > 0$ this inequality is valid for all natural k large enough. For instance, for $\kappa = 0.1$ inequality (3.1.3.15) holds for $k \geq 61$. Thus, according to Corollary 3.1.3.3, the equilibrium point $(0,0)^T$ of system (3.1.3.16) with impulsive conditions (3.1.3.19) is globally exponentially stable with Lyapunov exponent any $\lambda \in (0, 0.975279)$.

We first considered a similar problem with one-dimensional spatial domain and zero Neumann conditions. The reaction-diffusion terms were approximated by a difference operator. The results were reported at the 5th International Conference: 2009 — Dynamical Systems and Applications, Constanţa, Romania, 2009, and were published in [11]. Next, the problem was studied for a spatial domain of arbitrary dimension using the non-positivity of the reaction-diffusion operator. The results were reported at the 6th International Conference: 2010 — Dynamical Systems and Applications, Antalya, Turkey, 2010, and were published in [16]. The problem of the present subsection (with zero Dirichlet conditions) was first treated using Hardy-Sobolev inequality. The results were reported at the 7th European Conference on Elliptic and Parabolic Problems, Gaeta, Italy, 2012, and published in [6]. The results of the present subsection using Hardy-Poincaré inequality were reported at the Conference on Differential and Difference Equations and Applications, Terchová, Slovakia, 2012, and were published in [15].

3.1.4 Stability of neural networks with time-varying delays via minimal Lipschitz constants and nonlinear measures

In the present subsection we consider a neural networks model described by the following system of differential equations with time-varying delays and impulses:

$$
\begin{aligned}
\frac{d}{dt}u_i(t) &= -a_i(u_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} v_{ij} g_j(u_j(t-\tau_{ij}(t))) + I_i, \\
& \quad t > t_0, \quad t \neq t_k, \qquad i = \overline{1,n}, \qquad\qquad (3.1.4.1)\\
& \quad u_i(t_k + 0) = J_{ik}(u_i(t_k)), \qquad i = \overline{1,n}, \quad k = 1,2,3,\ldots, \\
& \quad t_0 < t_1 < t_2 < \cdots < t_k \to +\infty \quad \text{as} \quad k \to +\infty,
\end{aligned}
$$

where $n \geq 2$ is the number of neurons in the network; $u_i(t)$ is the state potential of the i-th neuron at time t; $f_j(\cdot)$ and $g_j(\cdot)$ are the normal and the delayed activation functions; w_{ij} and v_{ij} are the normal and the delayed synaptic connection weights from the j-th neuron on the i-th neuron; I_i are constant external inputs from outside the network to the i-th neuron; the functions $a_i(\cdot)$ show how the neuron self-regulates or resets its potential when isolated from other neurons and inputs; $\tau_{ij}(t) \geq 0$ are the time-varying transmission delays; t_k $(k = 1,2,3,\ldots)$ are the instants of impulse effect.

The model (3.1.4.1) is a generalization of Cohen-Grossberg neural networks first proposed by Cohen and Grossberg [43]. Special cases of this model are Hopfield-type neural networks with time-varying delays [101], cellular neural networks with time-varying delays and bi-directional associative memory neural networks with discrete delays.

Throughout the present subsection, we will use the following assumptions.

A3.1.4.1. The functions $a_i : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $\lambda_i > 0$ such that

$$(x_1 - x_2)[a_i(x_1) - a_i(x_2)] \geq \lambda_i (x_1 - x_2)^2$$

for all $x_1, x_2 \in \mathbb{R}$ and $i = \overline{1,n}$.

A3.1.4.2. The transfer functions f_i and g_i are continuous and monotonically increasing.

A3.1.4.3. The delays $\tau_{ij}(t)$ are bounded, that is, there exists a constant b such that $0 \le \tau_{ij} \le b$ for all $t \ne t_k$ and $i, j = \overline{1, n}$.

A3.1.4.4. The number $i(t_0, t)$ of instants of impulse effect between t_0 and t satisfies

$$\limsup_{t \to +\infty} \frac{i(t_0, t)}{t} = p < +\infty,$$

and the impulse functions J_{ik} are monotonically increasing.

The results of the present subsection were reported at the Sixth International Symposium on Neural Networks, Wuhan, China, 2009, and published in its proceedings [12]. They generalize the results of our previous paper [2], where $a_i(u_i) = a_i u_i$ with positive constants a_i $(i = \overline{1, n})$, $f_i \equiv 0$, g_i are Lipschitz continuous and monotonically increasing. Here the Lipschitz continuity of f_i and g_i will be replaced by a weaker property. However, in [2] we were able to obtain conditions for the existence of an equilibrium point of (3.1.4.1). Here we have to assume the existence and uniqueness of the solution of the initial value problem for (3.1.4.1), and the existence of an equilibrium point for (3.1.4.1).

Here we follow [101, 108] adapting the approach expounded therein to impulsive systems. In order to study the stability analysis of the general time delay system we rewrite system (3.1.4.1) as

$$\frac{du(t)}{dt} = F(u(t)) + G(u_\tau(t)), \quad t > t_0, \ t \ne t_k, \tag{3.1.4.2}$$

$$u(t_k + 0) = J_k(u(t_k)), \quad k = 1, 2, 3, \ldots, \tag{3.1.4.3}$$

where $J_k(u) = (J_{1k}(u_1), \ldots, J_{nk}(u_n))^T$ and the operators J_{ik} satisfy the condition **A3.1.4.4**, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ is the state vector of the neural network, both F and G are continuous mappings from an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, and $G(u_\tau(t))$ is defined as $G(u) = (G_1(u), G_2(u), \ldots, G_n(u))^T$ and

$$G_i(u_\tau(t)) = G_i(u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t)), \ldots, u_n(t - \tau_{in}(t))),$$

τ_{ij} $(i, j = \overline{1, n})$ are the delays which satisfy the condition **A3.1.4.3**. Also, we suppose that the initial value problem for (3.1.4.2), (3.1.4.3) has a unique solution.

Let $\mathbb{R}^n$ be the n-dimensional real vector space with vector norm $\| \cdot \|$ defined as follows: If $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, then $\|x\| = \sum_{i=1}^{n} |x_i|$.

Definition 3.1.4.1. Let $\Omega \subset \mathbb{R}^n$, $f: \ \Omega \to \mathbb{R}^n$ and $x^0 \in \Omega$. The *minimal Lipschitz constant of* f *with respect to* x^0 is defined by

$$L_\Omega(f, x^0) = \inf \{\alpha > 0: \ \|f(x) - f(x^0)\| \leq \alpha \|x - x^0\|, \ x \in \Omega\}.$$

Definition 3.1.4.2. Let $\Omega \subset \mathbb{R}^n$, $f: \ \Omega \to \mathbb{R}^n$ and $x^0 \in \Omega$. Then the constant

$$m_\Omega(f, x^0) = \sup_{x \in \Omega \setminus \{x^0\}} \frac{\langle f(x) - f(x^0), \operatorname{sgn}(x - x^0) \rangle}{\|x - x^0\|}$$

is called the *relative nonlinear measure of* f *with respect to* x^0 *on* Ω. Here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$ and $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \dots, \operatorname{sgn}(x_n))^T$.

Remark 3.1.4.1. If $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^T$ and f_i $(i = \overline{1, n})$ are monotonically increasing, then

$$m_\Omega(f, x^0) = \sup_{x \in \Omega \setminus \{x^0\}} \frac{\|f(x) - f(x^0)\|}{\|x - x^0\|}.$$

We will also need the following simple assertion from calculus.

Lemma 3.1.4.1. [101] *If* $a > c \geq 0$*, then, for each nonnegative real number* b*, the equation*

$$\lambda - a + ce^{\lambda b} = 0$$

has a unique positive solution.

In fact, the left-hand side of this equation is a strictly monotonic function for $\lambda \geq 0$, which is nonnegative at $\lambda = a$ and negative at $\lambda = 0$.

If u^* is an equilibrium point of system (3.1.4.2), then it satisfies the equation

$$F(u^*) + G(u^*) = 0. \tag{3.1.4.4}$$

If u^* is an equilibrium point of the impulsive system (3.1.4.2), (3.1.4.3), then it satisfies (3.1.4.4) and

$$u^* = J_k(u^*), \qquad k = 1, 2, 3, \dots,$$

that is, u^* is a fixed point of the impulse operators J_k.

Definition 3.1.4.3. The time-delay impulsive system (3.1.4.2), (3.1.4.3) is said to be *exponentially stable* on a neighbourhood Ω of an equilibrium point u^* if there are two positive constants α and M such that

$$\|u(t) - u^*\| \le M e^{-\alpha(t-t_0)} \sup_{t_0-b\le s\le t_0} \|u_0(s) - u^*\|, \quad t \ge t_0,$$

where $u(t)$ is the unique trajectory of the system initiated from $u_0(s) \in \Omega$ with $s \in [t_0 - b, t_0]$.

Proposition 3.1.4.1. *Let $u^* \in \Omega$ be an equilibrium point of system* (3.1.4.2)*. If $m_\Omega(F + G, u^*) < 0$, then u^* is the unique equilibrium point in Ω.*

Proof. Suppose that x^* is another equilibrium point of system (3.1.4.2) in Ω. Then both u^* and x^* satisfy equation (3.1.4.4) and

$$\begin{aligned} 0 > m_\Omega(F+G,u^*) &= \sup_{u\in\Omega\setminus\{u^*\}} \frac{\langle F(u)+G(u)-F(u^*)-G(u^*), \operatorname{sgn}(u-u^*)\rangle}{\|u-u^*\|} \\ &\ge \frac{\langle F(x^*)+G(x^*), \operatorname{sgn}(x^*-u^*)\rangle}{\|x^*-u^*\|} = 0, \end{aligned}$$

which is a contradiction. □

Let us introduce some notation:

$$\begin{aligned} d &= \sup_{k\in\mathbb{N}} m_\Omega(J_k, u^*), \\ D &= \max\left\{\sup_{k\in\mathbb{N}} m_\Omega(J_k^{-1}, u^*),\ 1\right\}, \\ \nu &= \sup_{t>t_0} \left(i(t_0,t) - i(t_0, t-b)\right). \end{aligned}$$

From condition **A3.1.4.4** it follows that $\nu < +\infty$.

Proposition 3.1.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of an equilibrium point u^* of the system* (3.1.4.2), (3.1.4.3)*. If $D < +\infty$ and for some matrix $A = \operatorname{diag}(a_1, a_2, \dots, a_n)$ with $a_i > 0$ we have*

$$m_{A^{-1}(\Omega)}(FA, A^{-1}u^*) + D^\nu L_{A^{-1}(\Omega)}(GA, A^{-1}u^*) < 0, \tag{3.1.4.5}$$

then by Lemma 3.1.4.1 the equation

$$\lambda \min_{i=\overline{1,n}} a_i + m_{A^{-1}(\Omega)}(FA, A^{-1}u^*) + D^\nu L_{A^{-1}(\Omega)}(GA, A^{-1}u^*) e^{b\lambda} = 0 \tag{3.1.4.6}$$

has a unique positive solution λ. *If*

$$d < e^{\lambda/p}, \tag{3.1.4.7}$$

that is, $p \ln d < \lambda$*, then system* (3.1.4.2), (3.1.4.3) *is exponentially stable on* Ω*. More precisely, for any* $\tilde{\lambda} \in (0, \lambda - p \ln d)$ *there exists a constant* M *such that*

$$\|u(t) - u^*\| \le M e^{-\tilde{\lambda}(t-t_0)} \sup_{t_0 - b \le s \le t_0} \|u_0(s) - u^*\| \quad \text{for all } t \ge t_0. \tag{3.1.4.8}$$

Proof. For any vector $w \in \mathbb{R}^n$ we have $\|w\| = \langle w, \operatorname{sgn}(w)\rangle$ and $\|w\| \ge \langle w, \operatorname{sgn}(z)\rangle$ for all $z \in \mathbb{R}^n$. Therefore for any $s \in \mathbb{R},\ s > 0$ we have

$$\frac{\|u(t) - u^*\| - \|u(t-s) - u^*\|}{s} \le \frac{1}{s}\langle u(t) - u(t-s), \operatorname{sgn}(u(t) - u^*)\rangle.$$

So, from system (3.1.4.2) we have

$$\begin{aligned}
& \frac{d\|u(t) - u^*\|}{dt} \le \langle \frac{du(t)}{dt}, \operatorname{sgn}(u(t) - u^*)\rangle \\
= \ & \langle F(u(t)) + G(u_\tau(t)), \operatorname{sgn}(u(t) - u^*)\rangle \\
= \ & \langle F(u(t)) - F(u^*), \operatorname{sgn}(u(t) - u^*)\rangle \\
+ \ & \langle G(u_\tau(t)) - G(u^*), \operatorname{sgn}(u(t) - u^*)\rangle \\
= \ & \langle FA(A^{-1}u(t)) - FA(A^{-1}u^*), \operatorname{sgn}(A^{-1}u(t) - A^{-1}u^*)\rangle \\
+ \ & \langle GA(A^{-1}u_\tau(t)) - GA(A^{-1}u^*), \operatorname{sgn}(A^{-1}u(t) - A^{-1}u^*)\rangle \\
\le \ & m_{A^{-1}(\Omega)}(FA, A^{-1}u^*)\|A^{-1}(u(t) - u^*)\| \\
+ \ & L_{A^{-1}(\Omega)}(GA, A^{-1}u^*) \sup_{t-b \le s \le t} \|A^{-1}(u(s) - u^*)\| \\
\le \ & \Big\{ m_{A^{-1}(\Omega)}(FA, A^{-1}u^*)\|u(t) - u^*\| \\
+ \ & L_{A^{-1}(\Omega)}(GA, A^{-1}u^*) \sup_{t-b \le s \le t} \|u(s) - u^*\| \Big\} \Big(\min_{i=\overline{1,n}} a_i \Big)^{-1}.
\end{aligned}$$

By virtue of (3.1.4.5), by Halanay's inequality [53] and taking into account the presence of impulses, we have

$$\|u(t) - u^*\| \le e^{-\lambda(t-t_0)} d^{i(t_0,t)} \sup_{t_0 - b \le s \le t_0} \|u(s) - u^*\|, \tag{3.1.4.9}$$

where λ is the unique positive solution of equation (3.1.4.6).

In fact, for $t \in [t_0, t_1]$ by Halanay's inequality we derive

$$\|u(t) - u^*\| \le e^{-\lambda(t-t_0)} \sup_{t_0 - b \le s \le t_0} \|u(s) - u^*\|.$$

Now let $t \in (t_k, t_{k+1}]$ for some $k \in \mathbb{N}$. In order to apply Halanay's inequality, we extend $u(t)$ as a continuous function from $(t_k, t_{k+1}]$ back to $t_0 - b$ as follows:

$$v(t) = \begin{cases} u(t), & t \in (t_k, t_{k+1}], \\ J_k J_{k-1} \cdots J_\ell u(t), & t \in (t_{\ell-1}, t_\ell], \quad \ell = \overline{2, k}, \\ J_k J_{k-1} \cdots J_1 u(t), & t \in [t_0 - b, t_1]. \end{cases}$$

If $t_{k-\mu-1} < t - b \le t_{k-\mu} < \cdots < t_k < t$, then

$$\sup_{t-b \le s \le t} \|u(s) - u^*\| \le D^\mu \sup_{t-b \le s \le t} \|v(s) - u^*\|$$

and $\mu \le \nu$. Thus we have

$$\frac{d\|v(t) - u^*\|}{dt} \le \Big\{ m_{A^{-1}(\Omega)}(FA, A^{-1}u^*) \|v(t) - u^*\|$$

$$+ \quad D^\nu L_{A^{-1}(\Omega)}(GA, A^{-1}u^*) \sup_{t-b \le s \le t} \|v(s) - u^*\| \Big\} \Big(\min_{i=\overline{1,n}} a_i \Big)^{-1}.$$

Now from Halanay's inequality we have

$$\|v(t) - u^*\| \le e^{-\lambda(t-t_0)} \sup_{t_0 - b \le s \le t_0} \|v(s) - u^*\|.$$

To derive (3.1.4.9) it suffices to notice that

$$\|v(s) - u^*\| \le d^k \|u(s) - u^*\|$$

for $s \in [t_0 - b, t_0]$.

Let $\varepsilon > 0$ be such that $\lambda - (p+\varepsilon) \ln d > 0$. Then $i(t_0, t) \le (p+\varepsilon)(t - t_0)$ for all t large enough and there exists a constant $M \ge 1$ such that $i(t, t_0) \le (p+\varepsilon)(t-t_0) + \ln M / \ln d$ for all $t \ge t_0$. Then

$$d^{i(t_0,t)} \le M \exp\{[(p+\varepsilon)\ln d]\,(t - t_0)\}$$

and the desired estimate (3.1.4.8) follows with $\tilde{\lambda} = \lambda - (p+\varepsilon) \ln d$. □

Now, extending some results of [2, 101, 108], we present some sufficient conditions for uniqueness and exponential stability of the equilibrium of the impulsive network (3.1.4.1). In order to apply Propositions 3.1.4.1 and 3.1.4.2, we define F, G : $\mathbb{R}^n \to \mathbb{R}^n$ by $F_i(u) = -a_i(u_i) + \sum\limits_{j=1}^{n} w_{ij} f_j(u_j)$ and $G_i(u) = \sum\limits_{j=1}^{n} v_{ij} g_j(u_j) + I_i$. For $\Omega \subset \mathbb{R}^n$ we denote by Ω_i the projection of Ω on the i-th axis of $\mathbb{R}^n$.

Theorem 3.1.4.1. *Let Ω be a neighbourhood of an equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of system* (3.1.4.1), *$m_i = m_{\Omega_i}(f_i, u_i^*)$ and $M_i = m_{\Omega_i}(g_i, u_i^*)$. If r_i $(i = \overline{1,n})$ are positive real numbers such that*

$$\max_{i=\overline{1,n}} \frac{1}{\lambda_i} \left\{ m_i \sum_{j=1}^{n} \frac{r_j}{r_i} |w_{ji}| + D^\nu M_i \sum_{j=1}^{n} \frac{r_j}{r_i} |v_{ji}| \right\} < 1, \tag{3.1.4.10}$$

then the equilibrium point u^* of system (3.1.4.1) *is unique in Ω.*

Proof. For each $i = \overline{1,n}$ the transfer functions f_i and g_i are increasing, or equivalently

$$\begin{aligned} &\big(f_i(t) - f_i(s)\big)\operatorname{sgn}(t-s) = |f_i(t) - f_i(s)|, \\ &\big(g_i(t) - g_i(s)\big)\operatorname{sgn}(t-s) = |g_i(t) - g_i(s)| \qquad \text{for all } \ t, s \in \mathbb{R}. \end{aligned}$$

Moreover, from condition **A3.1.4.1** we have

$$\big(a_i(t) - a_i(s)\big)\operatorname{sgn}(t-s) = \big|a_i(t) - a_i(s)\big| \geq \lambda_i |t-s|.$$

An equilibrium point u^* of system (3.1.4.1) corresponds to a solution of the equation (3.1.4.4). Let us suppose that u and u^* are two distinct solutions of (3.1.4.4) in Ω. Then for $R = \operatorname{diag}(r_1, r_2, \ldots, r_n)$ we have

$$\begin{aligned} 0 &= \langle R\big(F(u) + G(u) - F(u^*) - G(u^*)\big), \operatorname{sgn}(u - u^*) \rangle \\ &= \sum_{i=1}^{n} r_i \left\{ -\big(a_i(u_i) - a_i(u_i^*)\big) + \sum_{j=1}^{n} w_{ij} \big(f_j(u_j) - f_j(u_j^*)\big) \right. \\ &\qquad \left. + \sum_{j=1}^{n} v_{ij} \big(g_j(u_j) - g_j(u_j^*)\big) \right\} \operatorname{sgn}(u_i - u_i^*) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} r_i \Bigg\{ -\big|a_i(u_i) - a_i(u_i^*)\big| + \sum_{j=1}^{n} \big[|w_{ij}|\big|f_j(u_j) - f_j(u_j^*)\big| \\
&\qquad\qquad + |v_{ij}|\big|g_j(u_j) - g_j(u_j^*)\big|\big] \Bigg\} \\
&\leq -\sum_{i=1}^{n} r_i\lambda_i|u_i - u_i^*| + \sum_{j=1}^{n}\sum_{i=1}^{n} r_i\big[|w_{ij}|m_j|u_j - u_j^*| + |v_{ij}|M_j|u_j - u_j^*|\big] \\
&= -\sum_{i=1}^{n} r_i\lambda_i|u_i - u_i^*| + \sum_{j=1}^{n}\left[m_j\sum_{i=1}^{n} r_i|w_{ij}| + M_j\sum_{i=1}^{n} r_i|v_{ij}|\right]|u_j - u_j^*| \\
&= -\sum_{i=1}^{n}\left\{r_i\lambda_i - m_i\sum_{j=1}^{n} r_j|w_{ji}| - M_j\sum_{j=1}^{n} r_j|v_{ji}|\right\}|u_i - u_i^*| \\
&= -\sum_{i=1}^{n} r_i\lambda_i\left\{1 - \frac{1}{\lambda_i}\left[m_i\sum_{j=1}^{n}\frac{r_j}{r_i}|w_{ji}| + M_j\sum_{j=1}^{n}\frac{r_j}{r_i}|v_{ji}|\right]\right\}|u_i - u_i^*| < 0.
\end{aligned}
$$

As in Proposition 3.1.4.1, the contradiction obtained proves the uniqueness of the equilibrium point u^* of system (3.1.4.1) in Ω. □

Theorem 3.1.4.2. *Let all assumptions of Theorem 3.1.4.1 hold. Suppose further that the unique positive solution* λ *of the equation*

$$
\lambda \min_{i=\overline{1,n}} \frac{1}{p_i} - 1 + D^{\nu} q e^{\lambda b} = 0
$$

with

$$
p_i = \lambda_i - m_i \sum_{j=1}^{n} \frac{r_j}{r_i}|w_{ji}|
$$

and

$$
q = \max_{i=\overline{1,n}} \left\{ \frac{M_i}{p_i} \sum_{j=1}^{n} \frac{r_j}{r_i}|v_{ji}| \right\}
$$

satisfies (3.1.4.7). *If* $u(t)$ *is the trajectory of system* (3.1.4.1) *initiated from* $u_0(s) \in \Omega$ *with* $s \in [t_0 - b, t_0]$, *then*

$$
\|u(t) - u^*\| \leq M e^{-\tilde{\lambda}(t-t_0)} \frac{\max_{i=\overline{1,n}} r_i}{\min_{i=\overline{1,n}} r_i} \sup_{t_0-b\leq s\leq t_0} \|u_0(s) - u^*\|, \qquad (3.1.4.11)
$$

where $\tilde{\lambda} \in (0, \lambda - p \ln d)$.

Proof. We can first note that by virtue of Theorem 3.1.4.1 the equilibrium of system (3.1.4.1) is unique.

By the change $x = Ru$ system (3.1.4.2), (3.1.4.3) takes the form

$$\begin{cases} \frac{d}{dt}x(t) = RF(R^{-1}x(t)) + RG(R^{-1}x_\tau(t)), & t \neq t_k, \\ x(t_k+) = RJ_k(R^{-1}x(t_k)), & k = 1, 2, 3, \dots \end{cases} \tag{3.1.4.12}$$

It is easy to see that $x^* = Ru^*$ is an equilibrium point of system (3.1.4.12) and $m_{R(\Omega)}(RJ_kR^{-1}, Ru^*) = m_\Omega(J_k, u^*) \leq \Gamma$.

Denote $P = \operatorname{diag}(p_1, p_2, \dots, p_n)$. From inequality (3.1.4.10) it follows that $p_i > 0$, $i = \overline{1, n}$.

As in Theorem 3.1.4.1, for all $x \in PR(\Omega)$ we have

$$\begin{aligned} & \langle RF(R^{-1}P^{-1}x) - RF(u^*), \operatorname{sgn}(x - PRu^*)\rangle \\ \leq\ & \sum_{i=1}^n r_i \left\{ -\left|a_i(r_i^{-1}p_i^{-1}x_i) - a_i(u_i^*)\right| + \sum_{j=1}^n |w_{ij}|\left|f_j(r_j^{-1}p_j^{-1}x_j) - f_j(u_j^*)\right| \right\} \\ \leq\ & \sum_{i=1}^n r_i \left\{ -\frac{\lambda_i}{r_ip_i}\left|x_i - p_ir_iu_i^*\right| + \sum_{j=1}^n |w_{ij}|\frac{m_j}{r_jp_j}\left|x_j - p_jr_ju_j^*\right| \right\} \\ =\ & -\sum_{i=1}^n p_i^{-1}\left(\lambda_i - m_i\sum_{j=1}^n \frac{r_j}{r_i}|w_{ji}|\right)\left|x_i - p_ir_iu_i^*\right| \\ =\ & -\sum_{i=1}^n p_i^{-1}p_i\left|x_i - p_ir_iu_i^*\right| = -\|x - PRu^*\| \end{aligned}$$

and thus

$$m_{PR(\Omega)}(RFR^{-1}P^{-1}, PRu^*) \leq -1. \tag{3.1.4.13}$$

Further on, for all $x \in PR(\Omega)$

$$\begin{aligned} & \|RG(R^{-1}P^{-1}x) - RG(u^*)\| = \sum_{i=1}^n \left| r_i \sum_{j=1}^n v_{ij}\big(g_j(r_j^{-1}p_j^{-1}x_j) - g_j(u_j^*)\big)\right| \\ \leq\ & \sum_{i=1}^n r_i \sum_{j=1}^n |v_{ij}|M_jr_j^{-1}p_j^{-1}\left|x_j - p_jr_ju_j^*\right| = \sum_{i=1}^n \frac{M_i}{p_i}\sum_{j=1}^n \frac{r_j}{r_i}|v_{ji}|\left|x_i - p_ir_iu_i^*\right| \\ \leq\ & \max_{i=\overline{1,n}}\left\{\frac{M_i}{p_i}\sum_{j=1}^n \frac{r_j}{r_i}|v_{ji}|\right\}\sum_{i=1}^n\left|x_i - p_ir_iu_i^*\right| = q\|x - PRu^*\|, \end{aligned}$$

which implies that

$$L_{PR(\Omega)}(RGR^{-1}P^{-1}, PRu^*) \leq q. \tag{3.1.4.14}$$

From inequalities (3.1.4.13) and (3.1.4.14) we deduce

$$\begin{aligned}
&m_{PR(\Omega)}(RFR^{-1}P^{-1}, PRu^*) + D^\nu L_{PR(\Omega)}(RGR^{-1}P^{-1}, PRu^*) \leq -1 + D^\nu q \\
&= -1 + D^\nu \max_{i=\overline{1,n}} \left\{ \frac{M_i}{p_i} \sum_{j=1}^{n} \frac{r_j}{r_i} |v_{ji}| \right\} = \max_{i=\overline{1,n}} \left\{ p_i^{-1} \left[-p_i + D^\nu M_i \sum_{j=1}^{n} \frac{r_j}{r_i} |v_{ji}| \right] \right\} \\
&= \max_{i=\overline{1,n}} \left\{ p_i^{-1} \left[m_i \sum_{j=1}^{n} \frac{r_j}{r_i} |w_{ji}| + D^\nu M_i \sum_{j=1}^{n} \frac{r_j}{r_i} |v_{ji}| - \lambda_i \right] \right\} < 0
\end{aligned}$$

in view of inequality (3.1.4.10).

Thus we can apply Proposition 3.1.4.2 to system (3.1.4.12) with $A = P^{-1}$ and deduce the estimate

$$\|x(t) - Ru^*\| \leq M e^{-\tilde{\lambda}(t-t_0)} \sup_{t_0 - b \leq s \leq t} \|x_0(s) - Ru^*\|$$

for all $t \geq t_0$. Since $x = Ru$, this yields estimate (3.1.4.11). □

3.2 Global Exponential Stability of Equilibrium Points and Periodic Solutions of Discrete-Time Counterparts of Continuous Neural Networks

We formulate a discrete-time analogue of the additive Hopfield-type neural network with impulses considered in §3.1.1 by the semi-discretization method and investigate its global stability characteristics. An extension of the method is used to formulate a discrete-time analogue of an impulsive Cohen-Grossberg neural network with transmission delay. The convergence estimates are obtained in the respective ℓ_p-norm.

Further on, for two different classes (respectively with constant and periodic rates at which the neurons reset their states when isolated from the system) of Hopfield neural networks with periodic impulses and finite distributed delays discrete-time counterparts are introduced. Using different methods, we find sufficient conditions for the existence and global exponential stability of a unique periodic solution of the discrete systems considered. An example is given.

It is known from the literature on population dynamics [59] that time delays in the stabilizing negative feedback terms have a tendency to destabilize the system. Due to some theoretical and technical difficulties [58], so far there have been very few existing works with time delay in leakage (or "forgetting") terms [79, 59, 58, 102, 85].

We introduce the discrete-time counterpart of a class of Hopfield neural networks with impulses and concentrated and infinite distributed delays as well as a small delay in the leakage terms. We obtain sufficient conditions for the existence and global exponential stability of a unique equilibrium point of the discrete-time system considered. Note that conditions of smallness of the leakage delays have been introduced in [58, 85].

3.2.1 Equilibrium points of additive Hopfield-type neural networks

Let us consider again the impulsive continuous-time neural network (3.1.1.1), (3.1.1.2) satisfying the assumptions **A3.1.1.1–A3.1.1.5**. Recall that the components x_i^* of an equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of (3.1.1.1), (3.1.1.2) are governed by the algebraic system (3.1.1.3) and satisfy the linear equations (3.1.1.4). According to Lemma 3.1.1.1, if conditions **A3.1.1.1**–**A3.1.1.4** are satisfied, the system without impulses (3.1.1.1) has a unique equilibrium point x^*.

Following [95], we will obtain a discrete-time counterpart of system (3.1.1.1). Let $h > 0$ denote a uniform discretization step size and $[t/h]$ denote the greatest integer in t/h. For convenience, we denote $[t/h] = n$, $n \in \{0\} \cup \mathbb{N}$, and, by an abuse of notation, write $x_i(n)$ instead of $x_i(nh)$. Further on, we denote $\kappa_{ij} = [\tau_{ij}/h]$, $i, j = \overline{1,m}$. Finally, we replace the integral terms $\int_0^\infty K_{ij}(s)x_j(t-s)\,ds$, $i, j = \overline{1,m}$, by sums of the form $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p)x_j(n-p)$, where $p = [s/h]$, $\mathcal{K}_{ij}(p)$ stands for $\mathcal{K}_{ij}(ph)$ and $x_j(n-p)$ for $x_j((n-p)h)$, and the discrete kernels $\mathcal{K}_{ij}(\cdot)$, $i, j = \overline{1,m}$, satisfy the following conditions:

A3.2.1.1. $\mathcal{K}_{ij}(p) \geq 0$ and is bounded for $p \in \mathbb{N}$.

A3.2.1.2. $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) = 1$.

A3.2.1.3. There exists a number $\nu > 1$ such that $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p)\nu^p < \infty$.

Now, on the interval $[nh, (n+1)h)$ ($n \in \{0\} \cup \mathbb{N}$) we approximate system (3.1.1.1) by

$$\begin{aligned} \frac{dx_i(s)}{ds} &= -a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(n-\kappa_{ij})) \\ &+ \sum_{j=1}^{m} d_{ij} h_j \left(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p) \right) + I_i, \quad i = \overline{1,m}. \end{aligned} \tag{3.2.1.1}$$

We rewrite equation (3.2.1.1) in the form

$$\begin{aligned} \frac{d}{ds}\left(x_i(s)e^{a_i s}\right) &= e^{a_i s} \left(\sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(n-\kappa_{ij})) \right. \\ &+ \left. \sum_{j=1}^{m} d_{ij} h_j \left(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p) \right) + I_i \right), \quad i = \overline{1,m}, \end{aligned}$$

and integrate it over the interval $[nh,(n+1)h]$ to obtain

$$x_i(n+1)e^{a_i(n+1)h} - x_i(n)e^{a_inh} = \frac{e^{a_i(n+1)h} - e^{a_inh}}{a_i}\left(\sum_{j=1}^{m} b_{ij}f_j(x_j(n))\right.$$
$$+ \left.\sum_{j=1}^{m} c_{ij}g_j(x_j(n-\kappa_{ij})) + \sum_{j=1}^{m} d_{ij}h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right) + I_i\right)$$

or

$$x_i(n+1) = e^{-a_ih}x_i(n) + \frac{1-e^{-a_ih}}{a_i}\left(\sum_{j=1}^{m} b_{ij}f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij}g_j(x_j(n-\kappa_{ij}))\right.$$
$$+\left.\sum_{j=1}^{m} d_{ij}h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right) + I_i\right), \quad n\in\{0\}\cup\mathbb{N}, \quad i=\overline{1,m}. \tag{3.2.1.2}$$

This system is the discrete-time analogue of the system without impulses (3.1.1.1). It is provided with initial values of the form $x_i(-\ell) = \varphi(-\ell)$ ($\ell \in \{0\}\cup\mathbb{N}$), where the sequences $\{\varphi(-\ell)\}_{\ell=0}^{\infty}$ are bounded for all $i=\overline{1,m}$. The method used here is called *semi-discretization* [95].

An equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of (3.2.1.2) satisfies the system

$$\frac{1-e^{-a_ih}}{a_i}\left\{a_ix_i^* - \left(\sum_{j=1}^{m} b_{ij}f_j(x_j^*) + \sum_{j=1}^{m} c_{ij}g_j(x_j^*) + \sum_{j=1}^{m} d_{ij}h_j(x_j^*) + I_i\right)\right\} = 0,$$
$$i=\overline{1,m}. \tag{3.2.1.3}$$

Obviously the quantities

$$\phi_i(h) = \frac{1-e^{-a_ih}}{a_i}, \qquad i=\overline{1,m},$$

satisfy $\phi_i(h) > 0$, thus (3.2.1.3) implies (3.1.1.3), *i.e.*, the equilibria of the systems (3.1.1.1) and (3.2.1.2) coincide. We write the system (3.2.1.2) in the form

$$x_i(n+1) = e^{-a_ih}x_i(n) + \phi_i(h)\left(\sum_{j=1}^{m} b_{ij}f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij}g_j(x_j(n-\kappa_{ij}))\right.$$
$$+\left.\sum_{j=1}^{m} d_{ij}h_j\left(\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)x_j(n-p)\right) + I_i\right), \quad n\in\{0\}\cup\mathbb{N}, \quad i=\overline{1,m}. \tag{3.2.1.4}$$

In [95] no restrictions were imposed on the step size h. Neither are such restrictions required to obtain the stability result for system (3.2.1.4).

However, we are investigating the impulsive system (3.1.1.1), (3.1.1.2). We find it appropriate to assume there is not more than one instant of impulse effect in a step. To this end we suppose that

$$\theta = \inf_{k \in \mathbb{N}} (t_{k+1} - t_k) > 0 \tag{3.2.1.5}$$

and $h > 0$ satisfies

$$h < \theta. \tag{3.2.1.6}$$

We denote $[t_k/h] = n_k$, $k \in \mathbb{N}$, and obtain a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ satisfying $n_1 < n_2 < \cdots < n_k \to \infty$. With each such integer n_k we associate two values of the solution $x(n)$, namely, $x(n_k)$ which can be regarded as the value of the solution before the impulse effect and whose components are evaluated by equations (3.2.1.4), and $x(n_k^+)$ which can be regarded as the value of the solution after the impulse effect and whose components are evaluated by the equations

$$x_i(n_k^+) - x_i(n_k) = \sum_{\ell = n_{k-1}+1}^{n_k} B_{ik\ell} x_i(\ell) + \gamma_{ik}, \qquad i = \overline{1, m}, \quad k \in \mathbb{N}, \tag{3.2.1.7}$$

where, for convenience, $n_0 = -1$ and $B_{ik\ell}$ are suitably chosen constants.

Further on we will call system (3.2.1.4), (3.2.1.7) the discrete-time analogue of the system with impulses (3.1.1.1), (3.1.1.2). Whenever a value $x_i(n_k)$ appears in the right-hand side of (3.2.1.4), we mean $x_i(n_k^+)$.

The components of an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.2.1.4), (3.2.1.7) must satisfy the equations (3.1.1.3) and

$$\sum_{\ell = n_{k-1}+1}^{n_k} B_{ik\ell} x_i^* + \gamma_{ik} = 0. \tag{3.2.1.8}$$

To ensure that systems (3.1.1.1), (3.1.1.2) and (3.2.1.4), (3.2.1.7) have the same equilibrium points if any, we choose the constants $B_{ik\ell}$ so that

$$\sum_{\ell = n_{k-1}+1}^{n_k} B_{ik\ell} = -B_{ik} + \int_{t_{k-1}}^{t_k} \psi_{ik}(s)\, ds, \qquad i = \overline{1, m}, \quad k \in \mathbb{N}.$$

Our main result in the present subsection is the following

Theorem 3.2.1.1. *Let system* (3.2.1.4), (3.2.1.7) *satisfy conditions* **A3.1.1.1**, **A3.1.1.2**, **A3.2.1.1**–**A3.2.1.3** *and the components of the unique equilibrium point* $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *of system* (3.2.1.4) *satisfy* (3.2.1.8). *Then there exist constants* $M > 1$ *and* $\lambda \in (1, \nu)$ *and any other solution* $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ *of system* (3.2.1.4), (3.2.1.7) *is defined for all* $n \in \mathbb{N}$ *and satisfies the estimate*

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \leq M\lambda^{-n} \prod_{k=1}^{i(1,n)} B_k(\lambda) \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}, \qquad (3.2.1.9)$$

where $i(1,n) = \begin{cases} 0, & n \leq n_1, \\ \max\{k \in \mathbb{N} : n_k < n\}, & n > n_1, \end{cases}$ *and*

$$B_k(\lambda) = \max_{i=\overline{1,m}} |1 + B_{ikn_k}| + \sum_{\ell=n_{k-1}+1}^{n_k-1} \max_{i=\overline{1,m}} |B_{ik\ell}|\lambda^{n_k-\ell}, \quad k \in \mathbb{N}.$$

Proof. From the conditions of the theorem it follows that system (3.2.1.4), (3.2.1.7) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

Let us consider the functions $\Phi_i : [1, \nu] \to \mathbb{R}$ defined by

$$\Phi_i(\lambda) = 1 - \lambda e^{-a_i h}$$
$$- \quad \phi_i(h) \left[F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}|\lambda^{\kappa_{ji}+1} + H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p)\lambda^{p+1} \right], \ i = \overline{1,m}.$$

By virtue of conditions **A3.2.1.2** and **A3.1.1.2** we have

$$\begin{aligned} \Phi(1) &= 1 - \lambda e^{-a_i h} - \phi_i(h) \left[F_i \sum_{j=1}^{m} |b_{ji}| + G_i \sum_{j=1}^{m} |c_{ji}| + H_i \sum_{j=1}^{m} |d_{ji}| \right] \\ &= \phi_i(h) \left[a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| \right] > 0. \end{aligned}$$

Now, because of the assumptions **A3.2.1.1** and **A3.2.1.3** each $\Phi_i(\cdot)$ is well defined, continuous and decreasing on $[1, \nu]$. Thus there exists $\lambda_i^* \in (1, \nu]$ such that $\Phi_i(\lambda) > 0$ for $\lambda \in [1, \lambda_i^*)$, $i = \overline{1,m}$. Choosing $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*\}$, we have

$$\Phi_i(\lambda) > 0, \qquad \lambda \in (1, \lambda^*), \quad i = \overline{1,m}. \qquad (3.2.1.10)$$

From (3.2.1.4) and (3.2.1.3) it follows that

$$|x_i(n+1) - x_i^*| \le e^{-a_i h}|x_i(n) - x_i^*| + \phi_i(h) \sum_{j=1}^{m} \Bigg[|b_{ij}|\, F_j\, |x_j(n) - x_j^*| \tag{3.2.1.11}$$

$$+ |c_{ij}|\, G_j\, |x_j(n-\kappa_{ij}) - x_j^*| + |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)|x_j(n-p) - x_j^*| \Bigg],$$

$i = \overline{1,m}$. If for any $\lambda \in (0, \lambda^*)$ we denote

$$y_i(n) = \lambda^n \frac{|x_i(n) - x_i^*|}{\phi_i(h)}, \qquad i = \overline{1,m}, \quad n \in \mathbb{Z}, \tag{3.2.1.12}$$

then from (3.2.1.11) we have

$$y_i(n+1) \le \lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m} \phi_j(h) \Bigg[|b_{ij}| F_j y_j(n) \tag{3.2.1.13}$$

$$+ \quad |c_{ij}| G_j \lambda^{\kappa_{ij}+1} y_j(n-\kappa_{ij}) + |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) \lambda^{p+1} y_j(n-p) \Bigg]$$

for $n \in \{0\} \cup \mathbb{N}$ and $i = \overline{1,m}$. Next consider a Lyapunov functional $V(n) = V(y_1, y_2, \ldots, y_m)(n)$ defined by

$$V(n) = \sum_{i=1}^{m} \Bigg\{ y_i(n) + \sum_{j=1}^{m} \phi_j(h) \Bigg[|c_{ij}| G_j \lambda^{\kappa_{ij}+1} \sum_{\ell=n-\kappa_{ij}}^{n-1} y_j(\ell)$$

$$+ \ |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) \lambda^{p+1} \sum_{r=n-p}^{n-1} y_j(r) \Bigg] \Bigg\}, \quad n \in \{0\} \cup \mathbb{N}. \tag{3.2.1.14}$$

It is easy to see that $V(n) \ge 0$ and $V(0) < \infty$ by **A3.2.1.3**. More precisely,

$$V(0) \le M \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}, \tag{3.2.1.15}$$

where

$$M = \max_{i=\overline{1,m}} \Bigg\{ 1 + \phi_i(h) \Bigg[G_i \sum_{j=1}^{m} |c_{ji}| \sum_{\ell=1}^{\kappa_{ji}} \lambda^{\ell} + H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \sum_{r=1}^{p} \lambda^r \Bigg] \Bigg\}.$$

Further on, by virtue of (3.2.1.13) we obtain

$$\begin{aligned} V(n+1) \leq & \sum_{i=1}^{m}\Bigg\{\lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m}\phi_j(h)\Bigg[|b_{ij}|F_j y_j(n) \\ + & |c_{ij}|G_j\lambda^{\kappa_{ij}+1} y_j(n-\kappa_{ij}) + |d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1} y_j(n-p)\Bigg] \end{aligned}$$

$$+\sum_{j=1}^{m}\phi_j(h)\left[|c_{ij}|G_j\lambda^{\kappa_{ij}+1}\sum_{\ell=n+1-\kappa_{ij}}^{n} y_j(\ell) + |d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1}\sum_{r=n+1-p}^{n} y_j(r)\right]\Bigg\}$$

$$= \sum_{i=1}^{m}\Bigg\{\lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m}\phi_j(h)\Bigg[|b_{ij}|F_j y_j(n)$$

$$+ |c_{ij}|G_j\lambda^{\kappa_{ij}+1}\sum_{\ell=n-\kappa_{ij}}^{n} y_j(\ell) + |d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1}\sum_{r=n-p}^{n} y_j(r)\Bigg]\Bigg\}$$

and

$$\begin{aligned} V(n+1)-V(n) \leq & \sum_{i=1}^{m}\Bigg\{\left(\lambda e^{-a_i h}-1\right) y_i(n) + \sum_{j=1}^{m}\phi_j(h)\Bigg[|b_{ij}|F_j y_j(n) \\ + & |c_{ij}|G_j\lambda^{\kappa_{ij}+1} y_j(n) + |d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1} y_j(n)\Bigg]\Bigg\} \\ = & -\sum_{i=1}^{m}\Bigg\{1-\lambda e^{-a_i h} - \phi_i(h)\Bigg[F_i\sum_{j=1}^{m}|b_{ji}| \\ + & G_i\sum_{j=1}^{m}|c_{ji}|\lambda^{\kappa_{ji}+1} + H_i\sum_{j=1}^{m}|d_{ji}|\sum_{p=1}^{\infty}\mathcal{K}_{ji}(p)\lambda^{p+1}\Bigg]\Bigg\} y_i(n) \\ = & -\sum_{i=1}^{m}\Phi_i(\lambda) y_i(n) \leq 0 \end{aligned}$$

by virtue of (3.2.1.10). This implies $V(n+1) \leq V(n)$ for $n \neq n_k$ and $V(n_k+1) \leq V(n_k^+)$, where $V(n_k^+)$ contains $|x_i(n_k^+)-x_i^*|$ instead of $|x_i(n_k)-x_i^*|$. The above inequalities yield

$$V(n) \le \begin{cases} V(n_k^+) & \text{for } \ n_k < n \le n_{k+1}, \\ V(0) & \text{for } \ 0 < n \le n_1. \end{cases} \tag{3.2.1.16}$$

Further on, making use of equalities (3.2.1.7) and (3.2.1.8), for any $k \in \mathbb{N}$ we find successively

$$\begin{aligned} |x_i(n_k^+) - x_i^*| &\le |1 + B_{ikn_k}|\,|x_i(n_k) - x_i^*| + \sum_{\ell=n_{k-1}+1}^{n_k-1} |B_{ik\ell}|\,|x_i(\ell) - x_i^*|, \\ y_i(n_k^+) &\le |1 + B_{ikn_k}| y_i(n_k) + \sum_{\ell=n_{k-1}+1}^{n_k-1} |B_{ik\ell}|\lambda^{n_k-\ell} y_i(\ell), \\ V(n_k^+) &\le \max_{i=\overline{1,m}} |1 + B_{ikn_k}| V(n_k) + \sum_{\ell=n_{k-1}+1}^{n_k-1} \max_{i=\overline{1,m}} |B_{ik\ell}|\lambda^{n_k-\ell} V(\ell), \end{aligned}$$

thus $V(n_k^+) \le B_k(\lambda)V(n_{k-1}^+)$ for $k \ge 2$ and $V(n_1^+) \le B_1(\lambda)V(0)$, where the quantities $B_k(\lambda)$ were introduced in the statement of Theorem 3.2.1.1.

Combining the last inequalites and (3.2.1.16), we derive the estimate

$$V(n) \le \prod_{k=1}^{i(1,n)} B_k(\lambda)V(0). \tag{3.2.1.17}$$

Finally, from the inequalities

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le \lambda^{-n} V(n),$$

(3.2.1.17) and (3.2.1.15) we deduce (3.2.1.9) for any $\lambda \in (1, \lambda^*)$. □

Definition 3.2.1.4. The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.2.1.4), (3.2.1.7) is said to be *globally exponentially stable with multiplier* ρ if there exist constants $M \ge 1$ and $\rho \in (0, 1)$ and any other solution $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ of system (3.2.1.4), (3.2.1.7) is defined for all $n \in \mathbb{N}$ and satisfies the estimate

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le M\rho^n \sum_{i=1}^{m} \sup_{\ell \in \{0\}\cup\mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)} \quad \text{for all } \ n \in \{0\} \cup \mathbb{N}. \tag{3.2.1.18}$$

For three sets of additional assumptions on the impulse effects we will show that inequality (3.2.1.9) implies global exponential stability of the equilibrium point x^* of the system (3.2.1.4), (3.2.1.7).

Corollary 3.2.1.1. *Let all conditions of Theorem 3.2.1.1 hold. Let there exist* $\lambda \in (1, \lambda^*)$ *such that* $B_k(\lambda) \le 1$ *for all sufficiently large values of* $k \in \mathbb{N}$*. Then the equilibrium point* x^* *of the discrete-time system* (3.2.1.4), (3.2.1.7) *is globally exponentially stable with multiplier* $1/\lambda$*.*

The proof of this corollary is obvious. The global exponential stability is provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large and even unbounded magnitudes of the impulse effects provided that these do not occur too often.

Corollary 3.2.1.2. *Let all conditions of Theorem 3.2.1.1 hold and*

$$\limsup_{n\to\infty} \frac{i(1,n)}{n} = p < \infty. \tag{3.2.1.19}$$

Let there exist positive constants $\lambda \in (1, \lambda^*)$ *and* B *satisfying the inequalities*

$$B_k(\lambda) \le B \tag{3.2.1.20}$$

for all sufficiently large values of $k \in \mathbb{N}$*, and* $B^p < \lambda$*. Then for any* $\rho \in \left(\frac{B^p}{\lambda}, 1\right)$ *the equilibrium point* x^* *of the discrete-time system* (3.2.1.4), (3.2.1.7) *is globally exponentially stable with multiplier* ρ*.*

Proof. Inequalities (3.2.1.9) and (3.2.1.20) yield

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le M\lambda^{-n} B^{i(1,n)} \sum_{i=1}^{m} \sup_{\ell\in\{0\}\cup\mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)} \quad \text{for all } n \in \mathbb{N}.$$

Condition (3.2.1.19) means that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that the inequality

$$\frac{i(1,n)}{n} \le p + \varepsilon$$

is satisfied for all $n \ge N$. For such n we have $i(1,n) \le (p+\varepsilon)n$ and

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le M \left(\frac{B^{p+\varepsilon}}{\lambda}\right)^n \sum_{i=1}^{m} \sup_{\ell\in\{0\}\cup\mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}.$$

It suffices to choose $\varepsilon > 0$ such that $B^{p+\varepsilon} < \lambda$ and $\rho = \frac{B^{p+\varepsilon}}{\lambda}$. Then inequality (3.2.1.18) will be satisfied with a possibly bigger constant M. □

Corollary 3.2.1.3. *Let all conditions of Theorem 3.2.1.1 hold and there exist constants $\lambda \in (1, \lambda^*)$ and $\mu \in (1, \lambda)$ such that*

$$B_k(\lambda) \le \mu^{n_k - n_{k-1}} \tag{3.2.1.21}$$

for all sufficiently large values of $k \in \mathbb{N}$. Then the equilibrium point x^* of the discrete-time system (3.2.1.4), (3.2.1.7) *is globally exponentially stable with multiplier μ/λ.*

Proof. By virtue of condition (2.2.1.21) for $n_k < n \le n_{k+1}$ inequality (3.2.1.9) implies

$$\sum_{i=1}^{m} \frac{|x_i(t) - x_i^*|}{\phi_i(h)} \le M\lambda^{-n}\mu^{n_k} \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} \frac{|x_i(-\ell) - x_i^*|}{\phi_i(h)}$$

with a possibly larger constant M. Since $n_k < n$, we have $\lambda^{-n}\mu^{n_k} < \left(\frac{\mu}{\lambda}\right)^n$ and inequality (3.2.1.18) will be satisfied with $\rho = \mu/\lambda$. □

These results were first reported in a very concise form at the Conference on Differential Equations and Applications, Žilina, Slovakia, 2003, and published in [47]. The results of the present subsection were essentially given in our paper [3] where impulse conditions were provided for the continuous-time neural networks considered in [95]. The exposition here follows the pattern of some of our more recent papers. In particular, the discretization of the impulse conditions used here is different from the one used in [3]. For more details, see the next subsection.

3.2.2 Equilibrium points of Cohen-Grossberg neural networks with transmission delays

In the present subsection we consider an impulsive continuous-time neural network consisting of m elementary processing units (or neurons) whose state variables x_i $(i = \overline{1, m})$ are governed by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= a_i(x_i(t)) \Bigg[-b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t - \tau_{ij})) \qquad (3.2.2.1) \\ &+ \sum_{j=1}^{m} d_{ij} \int_0^\infty K_{ij}(s) g_j(x_j(t-s))\, ds + I_i \Bigg], \qquad t > t_0, \quad t \ne t_k, \\ \Delta x_i(t_k) &= r_{ik}(x_i(t_k)), \qquad i = \overline{1, m}, \quad k \in \mathbb{N}, \end{aligned}$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, t_0]$. In (3.2.2.1), $a_i(x_i)$, $b_i(x_i)$, c_{ij}, d_{ij} and I_i are as in §3.1.2, τ_{ij} are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t - \tau_{ij}$ from the $j-$th unit to the unit i; and the delay kernels K_{ij} incorporate the fading past effects (or fading memories) of the $j-$th unit on the $i-$th unit; $\Delta x_i(t_k) = x_i(t_k+0) - x_i(t_k-0)$ denote impulsive state displacements characterized by the nonlinear functions $r_{ik}(x_i(t_k))$ at fixed moments of time t_k, $k \in \mathbb{N}$. Here it is assumed that the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$ and $t_k - t_{k-1} \geq \theta$, where $\theta > 0$ denotes the minimum time interval between successive impulses. In other words, the value $\theta > 0$ means that the impulses do not occur too often, but $\theta = \infty$ means that the network (3.2.2.1) is free of impulses.

The assumptions that accompany the impulsive network (3.2.2.1) are **A3.1.2.1**–**A3.1.2.3** as well as

A3.2.2.1. For the impulse functions $r_{ik} : \mathbb{R} \to \mathbb{R}$ there exist positive numbers γ_{ik} such that

$$\gamma_{ik} = \sup_{u \neq v} \left| \frac{r_{ik}(u) - r_{ik}(v)}{u - v} \right| \text{ for } u, v \in \mathbb{R},\ k \in \mathbb{N},\ i = \overline{1, m}.$$

A3.2.2.2. The delay kernels $K_{ij} : [0, \infty) \to [0, \infty)$ are piecewise-continuous functions that satisfy

$$\int_0^\infty K_{ij}(s)\, ds = \kappa_{ij} \quad \text{and} \quad \int_0^\infty K_{ij}(s) e^{\nu_0 s}\, ds < \infty,$$

where κ_{ij} denote nonnegative constants and ν_0 is some positive number.

Under these assumptions and the given initial conditions, there is a unique solution of the impulsive network (3.2.2.1). The solution is a vector $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ in which $x_i(t)$ are piecewise continuous for $t \in (t_0, \beta)$, where $\beta > t_0$ is some positive number, possibly ∞, such that the limits $x_i(t_k + 0)$ and $x_i(t_k - 0)$ exist and $x_i(t)$ are differentiable for $t \in (t_{k-1}, t_k) \subset (t_0, \beta)$. An equilibrium point of the impulsive network (3.2.2.1) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^m c_{ij} f_j(x_j^*) + \sum_{j=1}^m d_{ij} \kappa_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1, m}, \tag{3.2.2.2}$$

and satisfy the equalities

$$r_{ik}(x_i^*) = 0, \quad k \in \mathbb{N},\ i = \overline{1,m}. \tag{3.2.2.3}$$

Till 2008, the semi-discretization method [95] had not been exploited for obtaining a discrete-time analogue of Cohen-Grossberg neural network mainly due to the nonlinearity of the feedback terms $-a_i(x_i)b_i(x_i)$. An appropriate extension of the method is presented here. We begin by rewriting the differential system in (3.2.2.1) as

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -\beta_i x_i(t) + \Bigg\{\beta_i x_i(t) + a_i(x_i(t))\Bigg[-b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t-\tau_{ij}))\\ &+ \sum_{j=1}^{m} d_{ij}\int_0^\infty K_{ij}(s) g_j(x_j(t-s))\,ds + I_i\Bigg]\Bigg\},\ i=\overline{1,m},\ t>t_0,\ t\neq t_k,\end{aligned} \tag{3.2.2.4}$$

where $\beta_i = \underline{a}_i\underline{b}_i > 0$. Let the value $h \in (0,\theta)$ be fixed, and $n_0 = [t_0/h]$, $n = [t/h]$, $\sigma_{ij} = [\tau_{ij}/h]$ and $\ell = [s/h]$, where $[r]$ denotes the greatest integer contained in the real number r. On any interval $[nh,(n+1)h)$ not containing a moment of impulse effect t_k the equation (3.2.2.1) can be approximated by equations with constant arguments of the form

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -\beta_i x_i(t) + \Bigg\{\beta_i x_i([t/h]h)\\ &+ a_i(x_i([t/h]h))\Bigg[-b_i(x_i([t/h]h)) + \sum_{j=1}^{m} c_{ij} f_j(x_j([t/h]h - [\tau_{ij}/h]h))\\ &+ \sum_{j=1}^{m} d_{ij}\sum_{[s/h]=1}^{\infty}\mathcal{K}_{ij}([s/h]h) g_j(x_j([t/h]h-[s/h]h)) + I_i\Bigg],\ i=\overline{1,m},\end{aligned} \tag{3.2.2.5}$$

with

$$\begin{aligned}&[t/h]h = nh \to t, \quad [\tau_{ij}/h]h = \sigma_{ij}h \to \tau_{ij}, \quad [s/h]h = \ell h \to s,\\ &\sum_{[s/h]=1}^{\infty}\mathcal{K}_{ij}([s/h]h) g_j(x_j([t/h]h-[s/h]h)) \to \int_0^\infty K_{ij}(s) g_j(x_j(t-s))\,ds\end{aligned}$$

for a fixed time t as $h \to 0$. Moreover,

A3.2.2.3. The delay kernels $\mathcal{K}_{ij} : \mathbb{N} \to [0, \infty)$ $(i, j = \overline{1, m})$ satisfy

$$\sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) = \kappa_{ij} \quad \text{and} \quad \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\nu \ell h} < \infty,$$

where the positive real number $\nu = \nu(h)$ for a given $h \in (0, \theta)$ is related to the positive number ν_0 (cf. assumption **A3.2.2.2**) by $\nu \to \nu_0$ (from below) as $h \to 0$.

For simplicity, we write system (3.2.2.5) as

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -\beta_i x_i(t) + \Bigg\{ \beta_i x_i(n) + a_i(x_i(n)) \Bigg[-b_i(x_i(n)) \qquad (3.2.2.6) \\ &+ \sum_{j=1}^{m} c_{ij} f_j(x_j(n - \sigma_{ij})) + \sum_{j=1}^{m} d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j(x_j(n - \ell)) + I_i \Bigg] \Bigg\} \end{aligned}$$

for $i = \overline{1, m}$, $t \in [nh, (n+1)h)$, $t \neq t_k$, $n = n_0, n_0 + 1, \ldots$ wherein the notation $w(n) \equiv w(nh)$ has been adopted. Upon integrating (3.2.2.6) over the interval $[nh, (n+1)h)$, one obtains a discrete analogue of the differential system in (3.2.2.1) given by

$$\begin{aligned} x_i(n+1) &= e^{-\beta_i h} x_i(n) + \psi_i(h) \Bigg\{ \beta_i x_i(n) + a_i(x_i(n)) \Bigg[-b_i(x_i(n)) \qquad (3.2.2.7) \\ &+ \sum_{j=1}^{m} c_{ij} f_j(x_j(n - \sigma_{ij})) + \sum_{j=1}^{m} d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j(x_j(n - \ell)) + I_i \Bigg] \Bigg\} \end{aligned}$$

for $i = \overline{1, m}$, $n \geq n_0$, $n \neq n_k$, where $\psi_i(h) = \frac{1 - e^{-\beta_i h}}{\beta_i}$ denotes the associated denominator function. Observe that $0 < \psi_i(h) < \frac{1}{\beta_i}$ for $0 < h < \theta$ and $\psi_i(h) \approx h + O(h^2)$ for small $h > 0$.

The analogue (3.2.2.7) is supplemented with an initial vector sequence $\phi(\ell) = (\phi_1(\ell), \phi_2(\ell), \ldots, \phi_m(\ell))^T$ for $\ell = n_0, n_0 - 1, n_0 - 2, \ldots$ and is subject to impulsive state displacements characterized by the map

$$x_i(n_k^+) = x_i(n_k) + r_{ik}(x_i(n_k)), \qquad i = \overline{1, m}, \quad k \in \mathbb{N}. \qquad (3.2.2.8)$$

The iterations involved in (3.2.2.7) and (3.2.2.8) are described as follows: The values $x_i(n_k)$ generated by system (3.2.2.7) at time $n = n_k - 1$ are mapped impulsively by (3.2.2.8) to give the values $x_i(n_k^+)$. The mapped values $x_i(n_k^+)$ together with the past values $x_i(\ell)$ for $\ell = n_k - 1, n_k - 2, \ldots$ are then supplied back to system (3.2.2.7) as initial values required for the next successive iterations of $x_i(n+1)$ for $n = n_k, n_k+1, \ldots, n_{k+1}-1$. The existence of a unique solution $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ of the impulsive analogue (3.2.2.7), (3.2.2.8) for $n > n_0$ is therefore justified.

The impulsive map (3.2.2.8) was introduced in [91] wherein the impulse functions r_{ik} were defined linearly by $r_{ik}(x_i(n_k)) = -\delta_{ik}(x_i(n_k) - x_i^*)$ for $i = \overline{1,m},\ k \in \mathbb{N}$ with δ_{ik} denoting real numbers. An iteration scheme similar to (3.2.2.7), (3.2.2.8) was introduced in [49]. This map, for a given $h \in (0, \theta)$, differs from the impulsive state displacements described by difference equations of the form

$$x_i(n_k+1) = x_i(n_k) + r_{ik}(x_i(n_k)), \qquad i = \overline{1,m}, \quad k \in \mathbb{N},$$

considered in [3, 47, 60]. However, in the limit $h \to 0$, both characterizations operate in a similar manner in accordance with the continuous-time characterization (3.2.2.1) at the impulse moment $t = t_k$.

One can verify that an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of the impulsive analogue (3.2.2.7), (3.2.2.8) satisfies the same algebraic system (3.2.2.2) and (3.2.2.3) under the assumptions **A3.1.2.1**–**A3.1.2.3**, **A3.2.2.1**, **A3.2.2.3** for any given value $h \in (0, \theta)$. To prove the global exponential stability of the point x^* we will use Definition 3.1.3.2 and Lemmas 3.1.3.1 and 3.1.2.1.

Our first task is to prove the existence and uniqueness of the solution x^* of the algebraic system (3.2.2.2).

Theorem 3.2.2.1. *Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions* **A3.1.2.1**–**A3.1.2.3**, **A3.2.2.1**, **A3.2.2.3** *hold. Suppose the matrix*

$$\Xi_0 = B_0 - \frac{p-1}{p}(C_0^* + D_0^*) - \frac{1}{p}(C_0 F + D_0 G) \tag{3.2.2.9}$$

is an $M-$matrix, where

$$\begin{aligned} B_0 &= \operatorname{diag}(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m), \quad C_0 = (|c_{ij}|)_{m\times m}, \quad D_0 = (|d_{ij}|\kappa_{ij})_{m\times m}, \\ C_0^* &= \operatorname{diag}\left(\sum_{j=1}^m |c_{1j}|F_j, \sum_{j=1}^m |c_{2j}|F_j, \ldots, \sum_{j=1}^m |c_{mj}|F_j\right), \end{aligned}$$

$$D_0^* = \operatorname{diag}\left(\sum_{j=1}^m |d_{1j}|\kappa_{1j}G_j, \sum_{j=1}^m |d_{2j}|\kappa_{2j}G_j, \ldots, \sum_{j=1}^m |d_{mj}|\kappa_{mj}G_j\right),$$
$$F = \operatorname{diag}(F_1, F_2, \ldots, F_m), \qquad G = \operatorname{diag}(G_1, G_2, \ldots, G_m).$$

Then the algebraic system (3.2.2.2) *has a unique solution* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Sketch of the proof. Define a mapping $\mathcal{F}: \mathbb{R}^m \to \mathbb{R}^m$ by $\mathcal{F}(x) = (\mathcal{F}_1(x), \mathcal{F}_2(x), \ldots, \mathcal{F}_m(x))^T$ for $x \in \mathbb{R}^m$, where

$$\mathcal{F}_i(x) = -b_i(x_i) + \sum_{j=1}^m c_{ij}f_j(x_j) + \sum_{j=1}^m d_{ij}\kappa_{ij}g_j(x_j) + I_i, \quad i = \overline{1,m}.$$

The space $\mathbb{R}^m$ is endowed with the norm $\|x\|_p = \left(\sum_{i=1}^m |x_i|^p\right)^{1/p}$. Under the assumptions **A3.1.2.2**, **A3.1.2.3**, $\mathcal{F}(x) \in C^0$. It is known that if $\mathcal{F}(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$, then there is a unique point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \in \mathbb{R}^m$ such that $\mathcal{F}(x^*) = 0$, that is, $\mathcal{F}_i(x^*) = 0,\ i = \overline{1,m}$.

To demonstrate the one-to-one property of $\mathcal{F}$, we take two arbitrary vectors $x, y \in \mathbb{R}^m$ and assume that $\mathcal{F}(x) = \mathcal{F}(y)$. From

$$b_i(x_i) - b_i(y_i) = \sum_{j=1}^m c_{ij}\left(f_j(x_j) - f_j(y_j)\right) + \sum_{j=1}^m d_{ij}\kappa_{ij}\left(g_j(x_j) - g_j(y_j)\right),\ i = \overline{1,m},$$

one obtains

$$\underline{b}_i|x_i - y_i| \le \sum_{j=1}^m |c_{ij}|F_j|x_j - y_j| + \sum_{j=1}^m |d_{ij}|\kappa_{ij}G_j|x_j - y_j|, \qquad i = \overline{1,m},$$

under the given assumptions. Multiplying both sides of the last inequality by $|x_i - y_i|^{p-1}$ and applying the inequality

$$\eta_1^{p-1}\eta_2 \le \frac{p-1}{p}\eta_1^p + \frac{1}{p}\eta_2^p, \qquad \eta_1, \eta_2 \ge 0, \quad p \ge 1,$$

we derive

$$\underline{b}_i|x_i - y_i|^p \le \sum_{j=1}^m |c_{ij}|F_j|x_i - y_i|^{p-1}|x_j - y_j|$$

$$
\begin{aligned}
&+ \sum_{j=1}^{m} |d_{ij}|\kappa_{ij} G_j |x_i - y_i|^{p-1} |x_j - y_j| \\
&\le \sum_{j=1}^{m} |c_{ij}| F_j \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right) \\
&+ \sum_{j=1}^{m} |d_{ij}|\kappa_{ij} G_j \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right), \qquad i = \overline{1,m},
\end{aligned}
$$

which can be expressed as

$$
\Xi_0(|x_1 - y_1|^p, |x_2 - y_2|^p, \ldots, |x_m - y_m|^p)^T \le 0.
$$

The assertion $x_i = y_i$, $i = \overline{1,m}$, follows by virtue of Ξ_0 being an $M-$matrix. Thus, $\mathcal{F}(x) = \mathcal{F}(y)$ implies $x = y$.

Finally, we show that $\|\mathcal{F}(x)\|_p \to \infty$ as $\|x\|_p \to \infty$. According to Lemma 3.1.2.1, $\mathcal{F}(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$. Thus, there is a unique point $x^* \in \mathbb{R}^m$ such that $\mathcal{F}(x^*) = 0$. The point represents a unique solution of the algebraic system (3.2.2.2). □

The next task is to investigate the global exponential stability characteristics of the unique equilibrium point x^* of the impulsive analogue (3.2.2.7), (3.2.2.8) for a fixed time-step $h \in (0, \theta)$. Upon introducing the translations

$$
u_i(n) = x_i(n) - x_i^*, \quad \varphi_i(\ell) = \phi_i(\ell) - x_i^*, \quad \tilde{a}_i(u_i(n)) = a_i(u_i(n) + x_i^*),
$$

$$
\tilde{b}_i(u_i(n)) = b_i(u_i(n) + x_i^*) - b_i(x_i^*), \quad \tilde{f}_j(u_j(n)) = f_j(u_j(n) + x_j^*) - f_j(x_j^*),
$$

$$
\tilde{g}_j(u_j(n)) = g_j(u_j(n) + x_j^*) - g_j(x_j^*), \quad \tilde{r}_{ik}(u_i(n_k^-)) = r_{ik}(u_i(n_k^-) + x_i^*) + u_i(n_k^-),
$$

one obtains

$$
\begin{aligned}
u_i(n+1) &= e^{-\beta_i h} u_i(n) + \psi_i(h) \Bigg\{ \beta_i u_i(n) + \tilde{a}_i(u_i(n)) \Bigg[-\tilde{b}_i(u_i(n)) \\
&+ \sum_{j=1}^{m} c_{ij} \tilde{f}_j(u_j(n - \sigma_{ij})) + \sum_{j=1}^{m} d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) \tilde{g}_j(x_j(n - \ell)) \Bigg] \Bigg\},
\end{aligned}
\tag{3.2.2.10}
$$

$$
i = \overline{1,m}, \quad n \ge n_0, \quad n \ne n_k,
$$

$$
u_i(n_k^+) = \tilde{r}_{ik}(u_i(n_k)), \quad i = \overline{1,m}, \quad k \in \mathbb{N}.
$$

This system inherits the assumptions **A3.1.2.1**–**A3.1.2.3**, **A3.2.2.1**, **A3.2.2.3** given before. In particular, $\tilde{r}_{ik}(0) = 0$ for $i = \overline{1,m}$, $k \in \mathbb{N}$ and

$$|\tilde{r}_{ik}(u)| \leq (1+\gamma_{ik})|u| \ \text{ for } \ i = \overline{1,m}, \ k \in \mathbb{N}, \ u \in \mathbb{R}, \tag{3.2.2.11}$$

where γ_{ik} denote positive real numbers.

Due to the equivalence between the systems (3.2.2.10) and (3.2.2.7), (3.2.2.8), it suffices to examine the exponential stability characteristics of the trivial equilibrium point $u^* = 0$ of the impulsive analogue (3.2.2.10). The main result is given by the following theorem.

Theorem 3.2.2.2. *Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions* **A3.1.2.1**–**A3.1.2.3***,* **A3.2.2.1***,* **A3.2.2.3** *hold. Suppose the matrix*

$$\Xi_1 = B_1 - \frac{p-1}{p}(C_1^* + D_1^*) - \frac{1}{p}(C_1 F + D_1 G) \tag{3.2.2.12}$$

is an $M-$matrix, where

$$B_1 = \operatorname{diag}\left(\underline{a}_1\underline{b}_1, \underline{a}_2\underline{b}_2, \ldots, \underline{a}_m\underline{b}_m\right),$$
$$C_1 = (\overline{a}_i|c_{ij}|)_{m\times m}, \qquad D_1 = (\overline{a}_i|d_{ij}|\kappa_{ij})_{m\times m},$$
$$C_1^* = \operatorname{diag}\left(\sum_{j=1}^m \overline{a}_1|c_{1j}|F_j, \sum_{j=1}^m \overline{a}_2|c_{2j}|F_j, \ldots, \sum_{j=1}^m \overline{a}_m|c_{mj}|F_j\right),$$
$$D_1^* = \operatorname{diag}\left(\sum_{j=1}^m \overline{a}_1|d_{1j}|\kappa_{1j}G_j, \sum_{j=1}^m \overline{a}_2|d_{2j}|\kappa_{2j}G_j, \ldots, \sum_{j=1}^m \overline{a}_m|d_{mj}|\kappa_{mj}G_j\right).$$

Suppose, further, that there exist positive numbers $\Lambda > 1$ and μ satisfying $\frac{\ln\Lambda}{\theta} < \mu < \nu$ for which

$$1 + \gamma_{ik} \leq \Lambda \ \text{ for } \ i = \overline{1,m}, \ k \in \mathbb{N}. \tag{3.2.2.13}$$

Then the impulsive analogue (3.2.2.10) *is globally exponentially stable with a Lyapunov exponent $\mu - \frac{\ln\Lambda}{\theta}$, namely,*

$$\left[\sum_{i=1}^m |u_i(n)|^p\right]^{1/p} \leq M e^{-\left(\mu - \frac{\ln\Lambda}{\theta}\right)h(n-n_0)} \left[\sum_{i=1}^m \sup_{\ell\leq n_0} |\varphi_i(\ell)|^p\right]^{1/p} \tag{3.2.2.14}$$

for $n \geq n_0$, where $M \geq 1$ denotes a constant.

Sketch of the proof. The property of the $M-$matrix Ξ_1 lends itself towards ascertaining that Ξ_0 is an $M-$matrix. Thus the existence and uniqueness of the equilibrium point x^* is assured by Theorem 3.2.2.1.

On applying the given assumptions to (3.2.2.10), we obtain

$$|u_i(n+1)| \le e^{-\beta_i h}|u_i(n)| + \psi_i(h)\left\{\sum_{j=1}^{m}\overline{a}_i|c_{ij}|F_j|u_j(n-n_{ij})| + \sum_{j=1}^{m}\overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)G_j|u_j(n-\ell)|\right\}, \quad i=\overline{1,m}, \quad n \ge n_0, \quad n \ne n_k. \tag{3.2.2.15}$$

Recall that $\beta_i = \underline{a}_i\underline{b}_i > 0$ and $\psi_i(h) = \frac{1-e^{-\beta_i h}}{\beta_i} > 0$. One has from the $M-$matrix Ξ_1 and the property (1) of Lemma 3.1.3.1 that there is a positive vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ for which

$$\xi_i\underline{a}_i\underline{b}_i - \frac{p-1}{p}\sum_{j=1}^{m}\xi_i\overline{a}_i|c_{ij}|F_j - \frac{p-1}{p}\sum_{j=1}^{m}\xi_i\overline{a}_i|d_{ij}|\kappa_{ij}G_j - \frac{1}{p}\sum_{j=1}^{m}\xi_j\overline{a}_j|c_{ji}|F_i - \frac{1}{p}\sum_{j=1}^{m}\xi_j\overline{a}_j|d_{ji}|\kappa_{ji}G_i > 0, \quad i=\overline{1,m}.$$

Let us introduce a perturbation $\mu = \mu(h)$ for a fixed value $h \in (0,\theta)$ such that $0 < \mu < \min\left\{\min_{i=\overline{1,m}}\beta_i,\, \nu\right\}$ and

$$\frac{e^{(\mu-\beta_i)h}-1}{\psi_i(h)} + \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|c_{ij}|F_j e^{\mu(\sigma_{ij}+1)h} + \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|d_{ij}|G_j\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)e^{\mu(\ell+1)h} + \frac{1}{p}\sum_{j=1}^{m}\frac{\xi_j}{\xi_i}\overline{a}_j|c_{ji}|F_i e^{\mu(\sigma_{ji}+1)h} + \frac{1}{p}\sum_{j=1}^{m}\frac{\xi_j}{\xi_i}\overline{a}_j|d_{ji}|G_i\sum_{\ell=1}^{\infty}\mathcal{K}_{ji}(\ell)e^{\mu(\ell+1)h} \le 0, \quad i=\overline{1,m}. \tag{3.2.2.16}$$

Further on, let

$$X_i(n) = e^{\mu(n-n_0)h}|u_i(n)| \quad \text{for} \quad i=\overline{1,m},\ n\in\mathbb{Z}, \tag{3.2.2.17}$$

into the system (3.2.2.15). We obtain

$$X_i(n+1) \le e^{(\mu-\beta_i)h}X_i(n) + \psi_i(h)\left\{\sum_{j=1}^{m}\overline{a}_i|c_{ij}|F_j e^{\mu(\sigma_{ij}+1)h}X_j(n-\sigma_{ij})\right.$$

$$\left.+\sum_{j=1}^{m}\overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)G_j e^{\mu(\ell+1)h}X_j(n-\ell)\right\}, \quad i=\overline{1,m}, \quad n\ge n_0, \quad n\ne n_k,$$

which with $\Delta X_i(n) = X_i(n+1) - X_i(n)$ can be rearranged as

$$\begin{aligned}\frac{\Delta X_i(n)}{\psi_i(h)} &\le \frac{e^{(\mu-\beta_i)h}-1}{\psi_i(h)}X_i(n) + \sum_{j=1}^{m}\overline{a}_i|c_{ij}|F_j e^{\mu(\sigma_{ij}+1)h}X_j(n-\sigma_{ij}) \\ &+ \sum_{j=1}^{m}\overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)G_j e^{\mu(\ell+1)h}X_j(n-\ell), \quad i=\overline{1,m}, \quad n\ge n_0, \quad n\ne n_k.\end{aligned} \tag{3.2.2.18}$$

Define a Lyapunov sequence by

$$\begin{aligned}V(n) &= \sum_{i=1}^{m}\xi_i\psi_i^{-1}(h)X_i^p(n) + \sum_{i=1}^{m}\xi_i\sum_{j=1}^{m}\overline{a}_i|c_{ij}|F_j e^{\mu(\sigma_{ij}+1)h}\sum_{\ell=n-\sigma_{ij}}^{n-1}X_j^p(\ell) \\ &+ \sum_{i=1}^{m}\xi_i\sum_{j=1}^{m}\overline{a}_i|d_{ij}|G_j\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)e^{\mu(\ell+1)h}\sum_{r=n-\ell}^{n-1}X_j^p(r), \; n\in\mathbb{Z}.\end{aligned} \tag{3.2.2.19}$$

One observes that the value

$$\begin{aligned}V(n_0) \le \max_{i=\overline{1,m}}&\left\{\xi_i\psi_i^{-1}(h)+\sum_{j=1}^{m}\xi_j\overline{a}_j|c_{ji}|F_i e^{\mu(\sigma_{ji}+1)h}\sigma_{ji}\right. \\ &\left.+\sum_{j=1}^{m}\xi_j\overline{a}_j|d_{ji}|G_i\sum_{\ell=1}^{\infty}\mathcal{K}_{ji}(\ell)e^{\mu(\ell+1)h}\ell\right\}\sum_{i=1}^{m}\sup_{r\le n_0}X_i^p(r)\end{aligned}$$

is finite since $\sum_{\ell=1}^{\infty}\mathcal{K}_{ji}(\ell)e^{\mu(\ell+1)h}\ell < \infty$ by virtue of the assumption **A3.2.2.3** in which $0<\mu<\nu$ for a fixed $h\in(0,\theta)$ and $\sup_{r\le n_0}X_i^p(r) < \infty$ due to the boundedness of the initial sequence $\varphi(r) = \phi(r) - x^*$ for $r = n_0, n_0-1, n_0-2, \ldots$

Now we estimate the forward difference $\Delta V(n) = V(n+1) - V(n)$ along the solutions of inequality (3.2.2.18), thus

$$
\begin{aligned}
&\Delta V(n) = \sum_{i=1}^{m} \xi_i \psi_i^{-1}(h) \Delta X_i^p(n) \\
+ \quad & \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} \left(X_j^p(n) - X_j^p(n-\sigma_{ij}) \right) \\
+ \quad & \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(X_j^p(n) - X_j^p(n-\ell) \right) \\
\leq \quad & \sum_{i=1}^{m} \xi_i p X_i^{p-1}(n) \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i(n) + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij}) \right. \\
& \left. + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} X_j(n-\ell) \right\} \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} \left(X_j^p(n) - X_j^p(n-\sigma_{ij}) \right) \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(X_j^p(n) - X_j^p(n-\ell) \right) \\
\leq \quad & \sum_{i=1}^{m} \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i^p(n) \right. \\
& + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n-\sigma_{ij}) \right) \\
& + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n-\ell) \right) \\
& + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n-\sigma_{ij}) \right) \\
& \left. + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n-\ell) \right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h}-1}{\psi_i(h)} + \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j e^{\mu(\sigma_{ij}+1)h} \right. \\
&+ \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |d_{ij}| G_j \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} + \frac{1}{p} \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} \overline{a}_j |c_{ji}| F_i e^{\mu(\sigma_{ji}+1)h} \\
&+ \left. \frac{1}{p} \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} \overline{a}_j |d_{ji}| G_i \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \right\} X_i^p(n)
\end{aligned}
$$

for $n \geq n_0$, $n \neq n_k$. By virtue of the condition (3.2.2.16) we find that $\Delta V(n) \leq 0$ for $n \geq n_0$, $n \neq n_k$. This means that

$$
V(n) \leq V(n_{k-1}^+) \ \text{ for } \ n_{k-1} \leq n < n_k, \ k \in \mathbb{N}. \tag{3.2.2.20}
$$

Next,

$$
X_i^p(n_{k-1}^+) \leq (1+\gamma_{i(k-1)})^p X_i^p(n_{k-1}) \leq \Lambda^p X_i^p(n_{k-1})
$$

for $k = 1, 2, 3, \ldots$, which implies

$$
V(n_{k-1}^+) \leq \Lambda^p V(n_{k-1}) \quad \text{for} \quad k \in \mathbb{N}. \tag{3.2.2.21}
$$

From the statements (3.2.2.20) and (3.2.2.21) it follows that

$$
V(n) \leq \Lambda^{(k-1)p} V(n_0) \quad \text{for} \quad n_0 \leq n < n_k, \ k \in \mathbb{N}. \tag{3.2.2.22}
$$

Since

$$
h(n-n_0) \geq (k-1)\theta \quad \text{for} \quad n_0 \leq n < n_k, \ k \in \mathbb{N},
$$

from (3.2.2.22) we derive

$$
V(n) \leq \Lambda^{p\frac{h(n-n_0)}{\theta}} V(n_0) = e^{p\frac{\ln\Lambda}{\theta} h(n-n_0)} V(n_0) \quad \text{for} \quad n \geq n_0.
$$

This estimate together with (3.2.2.17) and (3.2.2.19) leads to

$$
\sum_{i=1}^{m} \xi_i \psi_i^{-1} e^{\mu p h(n-n_0)} |u_i(n)|^p \leq V(n) \leq e^{p\frac{\ln\Lambda}{\theta} h(n-n_0)} V(n_0)
$$

for $n \geq n_0$, from which

$$
\sum_{i=1}^{m} |u_i(n)|^p \leq M_1 e^{-p\left(\mu - \frac{\ln\Lambda}{\theta}\right) h(n-n_0)} \sum_{i=1}^{m} \sup_{\ell \leq n_0} |\varphi_i(\ell)|^p
$$

for $n \geq n_0$, where the constant

$$M_1 = \max_{i=\overline{1,m}} \left\{ \xi_i \psi_i^{-1}(h) + \sum_{j=1}^{m} \xi_j \overline{a}_j |c_{ji}| F_i e^{\mu(\sigma_{ji}+1)h} \sigma_{ji} \right.$$

$$\left. + \quad \sum_{j=1}^{m} \xi_j \overline{a}_j |d_{ji}| G_i \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell \right\} \Big/ \min_{i=\overline{1,m}} \{\xi_i \psi_i^{-1}(h)\}$$

satisfies $1 \leq M_1 < \infty$. The statement (3.2.2.14) will follow subsequently and this completes the proof. □

Corollary 3.2.2.1. *Suppose the assumptions and conditions in Theorem 3.2.2.2 are satisfied with* (3.2.2.12) *being an $M-$matrix replaced by*

$$\begin{aligned} \underline{a}_i \underline{b}_i \quad &- \quad \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |c_{ij}| F_j - \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |d_{ij}| \kappa_{ij} G_j \\ &- \quad \frac{1}{p} \sum_{j=1}^{m} \overline{a}_j |c_{ji}| F_i - \frac{1}{p} \sum_{j=1}^{m} \overline{a}_j |d_{ji}| \kappa_{ji} G_i > 0, \quad i = \overline{1,m}. \end{aligned} \tag{3.2.2.23}$$

Then the impulsive analogue (3.2.2.10) *is globally exponentially stable in the sense of* (3.2.2.14).

In fact, the inequalities (3.2.2.23) imply that Ξ_1 defined in (3.2.2.12) is an $M-$matrix.

Different versions of the results of the present subsection were reported at the Third International Conference on Mathematical Sciences, Al Ain, UAE, 2008, and the Conference on Differential and Difference Equations and Applications, Strečno, Slovakia, 2008. They were published in [92, 93, 94].

3.2.3 Periodic solutions of Hopfield neural networks

In [114], the authors consider a class of Hopfield neural networks with periodic impulses and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\begin{aligned} \frac{dx_i(t)}{dt} \quad &= \quad -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{m} \int_0^{\omega} c_{ij}(s) g_j(x_j(t-s))\, ds + d_i(t), \\ & \qquad t > 0, \quad t \neq t_k, \\ & x_i(t_k + 0) = \beta_{ik} x_i(t_k), \qquad i = \overline{1,m}, \quad k \in \mathbb{N}, \end{aligned} \tag{3.2.3.1}$$

accompanied by the assumptions:

A3.2.3.1. For $j = \overline{1,m}$, $g_j(\cdot)$ is globally Lipschitz continuous with Lipschitz constant L_j:

$$|g_j(x) - g_j(y)| \le L_j|x - y| \qquad \text{for all } \; x, y \in \mathbb{R}.$$

A3.2.3.2. For $i, j = \overline{1,m}$, $c_{ij}(\cdot)$ is absolutely integrable on $[0, \omega]$.

A3.2.3.3. $0 = t_1 < t_2 < \cdots < t_p < \omega$, $t_{k+p} = t_k + \omega$, $\beta_{i,k+p} = \beta_{ik}$ for $i = \overline{1,m}$, $k \in \mathbb{N}$.

A3.2.3.4. There exist positive numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\lambda_i a_i > L_i \sum_{j=1}^{m} \lambda_j \left(|b_{ji}| + \int_0^{\omega} |c_{ji}(s)|\, ds \right), \quad i = \overline{1,m}.$$

Later in the paper [114] system (3.2.3.1) is assumed to be accompanied by the initial condition

$$x(r) = \psi(r), \qquad r \in [-\omega, 0], \tag{3.2.3.2}$$

where $\psi : [-\omega, 0] \to \mathbb{R}^m$ is piecewise continuous with discontinuities of the first kind at the points $t_k - \omega$, $k = \overline{2,p}$. Moreover, ψ is left-continuous at each discontinuity point and satisfies

$$\psi_i(t_k - \omega + 0) = \beta_{ik}\psi_i(t_k - \omega), \qquad i = \overline{1,m}, \quad k = \overline{2,p}.$$

The solution of the initial value problem (3.2.3.1), (3.2.3.2) is denoted by $x(t, \psi)$. Under the assumption that $|\beta_{ik}| \le 1$ for all $i = \overline{1,m}$ and $k = \overline{1,p}$, making use of the Contraction Mapping Principle in a suitable Banach space, in [114] it is proved that system (3.2.3.1) is globally exponentially periodic, that is, it possesses a periodic solution $x(t, \psi^*)$ and there exist positive constants α and β such that every solution $x(t, \psi)$ of (3.2.3.1) satisfies

$$\|x(t, \psi) - x(t, \psi^*)\| \le \alpha \|\psi - \psi^*\| e^{-\beta t} \quad \text{for all } \; t \ge 0.$$

Here

$$\|\psi\| = \sup_{-\omega \le r \le 0} \max_{i=\overline{1,m}} |\psi_i(r)|, \quad \|x(t, \psi)\| = \max_{i=\overline{1,m}} |x_i(t, \psi)|.$$

Now we shall formulate the discrete counterpart of problem (3.2.3.1), (3.2.3.2). For $N \in \mathbb{N}$ we choose the discretization step $h = \omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

In this case each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, for $t \geq -\omega$, $n_k = [t_k/h]$. Also by abuse of notation we write $x_i(nh) = x_i(n)$.

Let $n \in \mathbb{N}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k. Following [95], we approximate the differential equation in (3.2.3.1) on the interval $[nh, (n+1)h]$ by

$$\frac{d}{dt}\left(x_i(t)e^{a_i t}\right) = e^{a_i t}\left\{\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu) g_j(x_j(n-\nu)) + d_i(n)\right\},$$

$i = \overline{1,m}$, where the quantities $C_{ij}(\nu)$ are suitably chosen, say, $C_{ij}(\nu) = \int_{(\nu-1)h}^{\nu h} c_{ij}(s)\, ds$ or $C_{ij}(\nu) = c_{ij}(\nu)h$. We prefer the first choice, so that $\sum\limits_{\nu=1}^{N} C_{ij}(\nu) = \int_0^1 c_{ij}(s)\, ds$ is independent of h.

We integrate this differential equation over the interval $[nh, (n+1)h]$ to obtain

$$\begin{aligned} & x_i(n+1)e^{a_i(n+1)h} - x_i(n)e^{a_i nh} \\ = & \frac{e^{a_i(n+1)h} - e^{a_i nh}}{a_i}\left\{\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu) g_j(x_j(n-\nu)) + d_i(n)\right\}. \end{aligned}$$

If we denote by $\phi_i(h)$ the positive quantities

$$\phi_i(h) = \frac{1 - e^{-a_i h}}{a_i}, \qquad i = \overline{1,m},$$

we can rewrite the last equation in the form

$$\begin{aligned} & x_i(n+1) = e^{-a_i h} x_i(n) \qquad (3.2.3.3) \\ + \quad & \phi_i(h)\left[\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu) g_j(x_j(n-\nu)) + d_i(n)\right], \\ & i = \overline{1,m}, \qquad n \in \mathbb{N}, \quad n \neq n_k. \end{aligned}$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition in (3.2.3.1) by

$$x_i(n_k + 1) = \beta_{ik} x_i(n_k), \qquad i = \overline{1,m}, \quad k \in \mathbb{N}. \tag{3.2.3.4}$$

Such approximation was used in our paper [3].

Finally, the initial condition (3.2.3.2) is replaced by

$$x(n) = \psi(n), \qquad n = -N, -N+1, \dots, 0, \tag{3.2.3.5}$$

where $\psi = (\psi_i, \dots, \psi_m) : \{-N, -N+1, \dots, 0\} \to \mathbb{R}^m$. We assume that ψ satisfies $\psi_i(-N) = \psi_i(0)$ and

$$\psi_i(n_k + 1 - N) = \beta_{ik} \psi_i(n_k - N), \qquad i = \overline{1,m}, \quad k = \overline{1,p}.$$

We can regard the initial functions ψ as elements of the vector space

$$\begin{aligned} C^* = \{\psi_i(n) \mid \ & i = \overline{1,m},\ n = -N+1, \dots, 0; \\ & \psi_i(n_k + 1 - N) = \beta_{ik} \psi_i(n_k - N),\ k = \overline{2,p}, \\ & \psi_i(-N+1) = \beta_{i1} \psi_i(0),\ i = \overline{1,m}\} \subset \mathbb{R}^{mN} \end{aligned}$$

equipped with the norm

$$\|\psi\| = \max_{-N+1 \le \nu \le 0} \max_{i=\overline{1,m}} |\psi_i(\nu)|.$$

The solution of the discrete initial value problem (3.2.3.3), (3.2.3.4), (3.2.3.5) is denoted by $x(n, \psi),\ n \in \mathbb{Z},\ n \ge -N+1$. We shall use the norm

$$\|x(n, \psi)\| = \max_{i=\overline{1,m}} |x_i(n, \psi)|.$$

Conditions **A3.2.3.3** and **A3.2.3.4** are replaced respectively by

A3.2.3.5. $0 = n_1 < n_2 < \dots < n_p < N,\ n_{k+p} = n_k + N,\ \beta_{i,k+p} = \beta_{ik}$ for $i = \overline{1,m},\ k \in \mathbb{N}$.

A3.2.3.6. There exist positive numbers $\lambda_1, \dots, \lambda_m$ such that

$$\lambda_i a_i > L_i \sum_{j=1}^{m} \lambda_j \left(|b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \right), \quad i = \overline{1,m}.$$

Our main result is the following

Theorem 3.2.3.1. *Let system* (3.2.3.3), (3.2.3.4) *satisfy the conditions* **A3.2.3.1**, **A3.3.3.5**, **A3.2.3.6**. *Then there exists a number* N_0 *such that for each integer* $N \geq N_0$ *system* (3.2.3.3), (3.2.3.4) *is globally exponentially periodic. That is, there exists an* N*-periodic solution* $x(n, \psi^*)$ *of system* (3.2.3.3), (3.2.3.4) *and positive constants* α *and* $q < 1$ *such that every solution* $x(n, \psi)$ *of* (3.2.3.3), (3.2.3.4) *satisfies*

$$\|x(n, \psi) - x(n, \psi^*)\| \leq \alpha \|\psi - \psi^*\| q^n \quad \textit{for all } \; n \in \mathbb{N}.$$

In order to prove Theorem 3.2.3.1, we need the following two lemmas.

Lemma 3.2.3.1. *Let condition* **A3.2.3.6** *hold. Then there exists a number* $\bar{\rho} > 1$ *such that for* $1 < \rho \leq \bar{\rho}$ *and* $i = \overline{1, m}$ *we have*

$$\lambda_i (1 - \rho e^{-a_i h}) - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[\rho |b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} \right] > 0. \quad (3.2.3.6)$$

In particular,

$$\lambda_i - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} > 0. \quad (3.2.3.7)$$

Proof. Let us denote by $G_i(\rho)$, $i = \overline{1, m}$, the left-hand side of inequality (3.2.3.6). The functions $G_i(\rho)$ are continuous for $\rho \geq 1$ and

$$\begin{aligned} G_i(1) &= \lambda_i (1 - e^{-a_i h}) - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[|b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \right] \\ &= \phi_i(h) \left[\lambda_i a_i - L_i \sum_{j=1}^{m} \lambda_j \left(|b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \right) \right] > 0 \end{aligned}$$

by virtue of **A3.2.3.6**. There exist numbers $\rho_i > 1$ such that $G_i(\rho) > 0$ for $\rho \in (1, \rho_i]$.

Let $\bar{\rho} = \min_{i=\overline{1,m}} \rho_i$. Then for $1 < \rho \leq \bar{\rho}$ we have $G_i(\rho) > 0$, $i = \overline{1, m}$. □

Lemma 3.2.3.2. *Let* $x(n, \psi)$, $x(n, \tilde{\psi})$ *be a pair of solutions of system* (3.2.3.3), (3.2.3.4). *If conditions* **A3.2.3.1**, **A3.3.3.5**, **A3.2.3.6** *are satisfied and* $\bar{\rho}$ *is* *given by Lemma 3.2.3.1, then for any $\rho \in (1, \bar{\rho}]$ and all $n \in \mathbb{N}$ we have*

$$\|x(n,\psi) - x(n,\tilde{\psi})\| \le K(N,\rho) \prod_{k=1}^{i(0,n-1)} (1+\rho B_k)\rho^{-n}\|\psi - \tilde{\psi}\|, \qquad (3.2.3.8)$$

where

$$K(N,\rho) = \frac{1}{\min\limits_{j=\overline{1,m}}\{\lambda_j a_j\}} \sum_{i=1}^{m} \lambda_i \left\{ \frac{1}{\phi_i(h)} + \sum_{j=1}^{m} L_j \sum_{\nu=1}^{N} |C_{ij}(\nu)| \sum_{r=1}^{\nu} \rho^r \right\} \qquad (3.2.3.9)$$

and

$$B_k = \max_{i=\overline{1,m}} |\beta_{ik}|, \qquad i(0,n-1) = \max\{k:\ n_k \le n-1\}.$$

Proof. Let us denote

$$y_i(n) = \rho^n \frac{|x_i(n,\psi) - x_i(n,\tilde{\psi})|}{\phi_i(h)}, \qquad i = \overline{1,m}.$$

Then for $n \ne n_k$ we derive

$$\begin{aligned} y_i(n+1) &\le \rho e^{-a_i h} y_i(n) + \rho \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) y_j(n) \\ &+ \sum_{j=1}^{m} \sum_{\nu=1}^{N} |C_{ij}(\nu)| L_j \phi_j(h) \rho^{\nu+1} y_j(n-\nu). \end{aligned}$$

We define a Lyapunov functional

$$V(n) = \sum_{i=1}^{m} \lambda_i \left\{ y_i(n) + \sum_{j=1}^{m} L_j \phi_j(h) \sum_{\nu=1}^{N} |C_{ij}(\nu)| \rho^{\nu+1} \sum_{r=n-\nu}^{n-1} y_j(r) \right\}.$$

We can now estimate the difference $V(n+1) - V(n)$ along the solutions of (3.2.3.3) for $n \ne n_k$ as follows:

$$\begin{aligned} V(n+1) - V(n) &\le \sum_{i=1}^{m} \lambda_i (\rho e^{-a_i h} - 1) y_i(n) \\ &+ \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m} L_j \phi_j(h) \left[\rho |b_{ij}| + \sum_{\nu=1}^{N} |C_{ij}(\nu)| \rho^{\nu+1} \right] y_j(n). \end{aligned}$$

In the second term we change the order of summation with respect to i and j, and then we replace i by j and *vice versa*. Thus we obtain

$$V(n+1) - V(n) \le \sum_{i=1}^{m} \lambda_i(\rho e^{-a_i h} - 1) y_i(n)$$

$$+ \sum_{i=1}^{m} L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[\rho |b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} \right] y_i(n)$$

$$= - \sum_{i=1}^{m} \left\{ \lambda_i (1 - \rho e^{-a_i h}) - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[\rho |b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} \right] \right\} y_i(n)$$

$$= - \sum_{i=1}^{m} G_i(\rho) y_i(n) \le 0 \qquad \text{for} \quad \rho \in (1, \bar{\rho}],$$

that is,

$$V(n+1) \le V(n) \qquad \text{for} \quad n \in \mathbb{N} \setminus \{n_1, n_2, \ldots\}.$$

Next we find successively

$$|x_i(n_k+1, \psi) - x_i(n_k+1, \tilde{\psi})| = |\beta_{ik}| |x_i(n_k, \psi) - x_i(n_k, \tilde{\psi})| \le B_k |x_i(n_k, \psi) - x_i(n_k, \tilde{\psi})|,$$

$$y_i(n_k + 1) \le \rho B_k y_i(n_k),$$

$$V(n_k+1) \le \sum_{i=1}^{m} \lambda_i \left\{ \rho B_k y_i(n_k) + \sum_{j=1}^{m} L_j \phi_j(h) \sum_{\nu=1}^{N} |C_{ij}(\nu)| \rho^{\nu+1} \sum_{r=n_k+1-\nu}^{n_k} y_j(r) \right\}.$$

Thus, by virtue of (3.2.3.7), we find

$$V(n_k + 1) - (1 + \rho B_k) V(n_k)$$

$$\le - \sum_{i=1}^{m} \left\{ \lambda_i - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} \right\} y_i(n_k)$$

$$- \rho B_k \sum_{i=1}^{m} L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} \sum_{r=n_k+1-\nu}^{n_k-1} y_i(n_k - \nu)$$

$$- (1 + \rho B_k) \sum_{i=1}^{m} L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \sum_{\nu=1}^{N} |C_{ji}(\nu)| \rho^{\nu+1} y_i(n_k - \nu) \;\le\; 0,$$

that is,

$$V(n_k+1) \le (1+\rho B_k)V(n_k) \qquad \text{for} \quad k \in \mathbb{N}.$$

Combining these estimates, we derive

$$V(n) \le V(0) \prod_{k=1}^{i(0,n-1)} (1+\rho B_k). \tag{3.2.3.10}$$

Further we notice that

$$\begin{aligned} V(n) &\ge \rho^n \sum_{i=1}^{m} \frac{\lambda_i}{\phi_i(h)} |x_i(n,\psi) - x_i(n,\tilde{\psi})| \\ &\ge \rho^n \sum_{i=1}^{m} \lambda_i a_i |x_i(n,\psi) - x_i(n,\tilde{\psi})| \\ &\ge \rho^n \min_{j=\overline{1,m}} \{\lambda_j a_j\} \sum_{i=1}^{m} |x_i(n,\psi) - x_i(n,\tilde{\psi})| \\ &\ge \rho^n \min_{j=\overline{1,m}} \{\lambda_j a_j\} \|x(n,\psi) - x(n,\tilde{\psi})\|. \end{aligned} \tag{3.2.3.11}$$

On the other hand,

$$V(0) = \sum_{i=1}^{m} \lambda_i \left\{ y_i(0) + \sum_{j=1}^{m} L_j \phi_j(h) \sum_{\nu=1}^{N} |C_{ij}(\nu)| \rho^{\nu+1} \sum_{r=-\nu}^{-1} y_j(r) \right\}$$

$$= \sum_{i=1}^{m} \lambda_i \left\{ \frac{|\psi_i(0) - \tilde{\psi}_i(0)|}{\phi_i(h)} + \sum_{j=1}^{m} L_j \sum_{\nu=1}^{N} |C_{ij}(\nu)| \rho^{\nu+1} \sum_{r=-\nu}^{-1} |\psi_j(r) - \tilde{\psi}_j(r)| \rho^r \right\}$$

$$\le \sum_{i=1}^{m} \lambda_i \left\{ \frac{1}{\phi_i(h)} + \sum_{j=1}^{m} L_j \sum_{\nu=1}^{N} |C_{ij}(\nu)| \sum_{r=1}^{\nu} \rho^r \right\} \|\psi - \tilde{\psi}\|. \tag{3.2.3.12}$$

From the inequalities (3.2.3.10), (3.2.3.11), and (3.2.3.12) we derive the assertion of Lemma 3.2.3.2 with $K(N,\rho)$ given by (3.2.3.9). □

Proof of Theorem 3.2.3.1. Let $s \in \mathbb{N} \cup \{0\}$ and $Ns+1 \le n \le N(s+1)$. Then $i(0,n-1) \le p(s+1)$ and from Lemma 3.2.3.2 we obtain

$$\|x(n,\psi) - x(n,\tilde{\psi})\| \le K(N,\rho) \prod_{k=1}^{p} (1+\rho B_k) \left[\rho^{-N} \prod_{k=1}^{p} (1+\rho B_k) \right]^s \|\psi - \tilde{\psi}\|. \tag{3.2.3.13}$$

Let $q \in (\rho^{-1}, 1)$. Then we can find N_0 such that for $N \geq N_0$ we have

$$\rho^{-N} \prod_{k=1}^{p} (1 + \rho B_k) \leq q^N.$$

Then (3.2.3.13) takes the form

$$\|x(n, \psi) - x(n, \tilde{\psi})\| \leq \tilde{K}(N, \rho) q^{sN} \|\psi - \tilde{\psi}\| \tag{3.2.3.14}$$

for $Ns + 1 \leq n \leq N(s+1)$ and $\tilde{K}(N, \rho) = K(N, \rho) \prod\limits_{k=1}^{p} (1 + \rho B_k)$.

Now we define an operator $\mathcal{P} : \ C^* \to C^*$ as follows: for $\psi = \{\psi_i(n - N); \ i = \overline{1, m}, \ n = \overline{1, N}\}$ we set

$$\mathcal{P}\psi = \{x_i(n, \psi); \ i = \overline{1, m}, \ n = \overline{1, N}\}.$$

Then

$$\mathcal{P}^{s+1}\psi = \{x_i(n, \psi); \ i = \overline{1, m}, \ n = \overline{Ns + 1, N(s+1)}\}$$

and according to (3.2.3.14) we have

$$\|\mathcal{P}^{s+1}\psi - \mathcal{P}^{s+1}\tilde{\psi}\| \leq \tilde{K}(N, \rho) q^{sN} \|\psi - \tilde{\psi}\|.$$

If we choose s so large that $\tilde{K}(N, \rho) q^{sN} \leq \tilde{q} < 1$, then $\mathcal{P}^{s+1}$ is a contraction, hence it has a unique fixed point $\psi^* : \ \mathcal{P}^{s+1}\psi^* = \psi^*$. On the other hand, $\mathcal{P}^{s+1}(\mathcal{P}\psi^*) = \mathcal{P}(\mathcal{P}^{s+1}\psi^*) = \mathcal{P}\psi^*$, *i.e.*, $\mathcal{P}\psi^*$ is also a fixed point for $\mathcal{P}^{s+1}$. These two fixed points must coincide, so

$$\mathcal{P}\psi^* = \psi^*$$

and ψ^* is a fixed point for the operator $\mathcal{P}$. This means that

$$x_i(n, \psi^*) = \psi_i(n - N) \qquad \text{for} \quad n = \overline{1, N}$$

and $x(n, \psi^*)$ is a periodic solution of problem (3.2.3.3), (3.2.3.4).

Now applying inequality (3.2.3.14) to $x(n, \psi^*)$ and an arbitrary solution $x(n, \psi)$ we have

$$\|x(n, \psi) - x(n, \psi^*)\| \leq \tilde{K}(N, \rho) q^{sN} \|\psi - \psi^*\|$$

for $Ns+1 \le n \le N(s+1)$. If we put $K_1(N,\rho) = \tilde{K}(N,\rho)q^{-N}$, then $\tilde{K}(N,\rho)q^{sN} = K_1(N,\rho)q^{(s+1)N} \le K_1(N,\rho)q^n$ for $Ns+1 \le n \le N(s+1)$. Thus we have

$$\|x(n,\psi) - x(n,\psi^*)\| \le K_1(N,\rho)q^n \|\psi - \psi^*\| \quad \text{for all } n \in \mathbb{N}.$$

This shows that any solution $x(n,\psi)$ exponentially tends to the periodic solution $x(n,\psi^*)$ as $n \to +\infty$. □

The illustrative example given here is a nontrivial modification of the examples given in [114, 115].

Consider the impulsive Hopfield neural network with finite distributed delays

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -x_1(t) + 0.5\tanh(x_1(t)) + 0.2\tanh(x_2(t)) \\
&+ \int_0^1 (1-s)\,[0.1\tanh(x_1(t-s)) + 0.3\tanh(x_2(t-s))]\,ds + \sin(2\pi t), \\
\frac{dx_2(t)}{dt} &= -x_2(t) + 0.3\tanh(x_1(t)) + 0.4\tanh(x_2(t)) \\
&+ \int_0^1 (1-s)\,[0.2\tanh(x_1(t-s)) + 0.3\tanh(x_2(t-s))]\,ds + \cos(2\pi t), \\
&\qquad t > 0, \quad t \ne t_k, \\
&x_i(t_k+0) = \beta_{ik}x_i(t_k), \qquad i = 1,2, \quad k \in \mathbb{N},
\end{aligned}$$

where

$$\begin{aligned}
&t_{3k+1} = k, \quad t_{3k+2} = k+0.3, \quad t_{3k+3} = k+0.6, \qquad k \in \mathbb{N} \cup \{0\}, \\
&\beta_{11} = 2, \quad \beta_{12} = -0.1, \quad \beta_{13} = 0.4, \quad \beta_{21} = -1.5, \quad \beta_{22} = 0.7, \quad \beta_{23} = -0.5.
\end{aligned}$$

Then $\omega = 1$, $a_1 = a_2 = 1$, $b_{11} = 0.5$, $b_{12} = 0.2$, $b_{21} = 0.3$, $b_{22} = 0.4$, $c_{11}(s) = 0.1(1-s)$, $c_{12}(s) = 0.3(1-s)$, $c_{21}(s) = 0.2(1-s)$, $c_{22}(s) = 0.3(1-s)$, $g_1(\cdot) = g_2(\cdot) = \tanh(\cdot)$, $L_1 = L_2 = 1$, $d_1(t) = \sin(2\pi t)$, $d_2(t) = \cos(2\pi t)$, $B_1 = 2$, $B_2 = 0.7$, $B_3 = 0.5$.

For $N \ge 4$ the corresponding discrete-time system is

$$
\begin{aligned}
x_1(n+1) &= e^{-h}x_1(n) + (1-e^{-h})\Big\{0.5\tanh(x_1(n)) + 0.2\tanh(x_2(n)) \qquad (3.2.3.15)\\
&+ \sum_{\nu=1}^{N}\big[C_{11}(\nu)\tanh(x_1(n-\nu)) + C_{12}(\nu)\tanh(x_2(n-\nu))\big] + \sin(2\pi nx)\Big\},\\
x_2(n+1) &= e^{-h}x_2(n) + (1-e^{-h})\Big\{0.3\tanh(x_1(n)) + 0.4\tanh(x_2(n))\\
&+ \sum_{\nu=1}^{N}\big[C_{21}(\nu)\tanh(x_1(n-\nu)) + C_{22}(\nu)\tanh(x_2(n-\nu))\big] + \cos(2\pi nx)\Big\},\\
&\qquad n \in \mathbb{N}, \quad n \neq n_k,\\
&x_i(n_k+1) = \beta_{ik}x_i(n_k), \qquad i = 1,2, \quad k \in \mathbb{N},
\end{aligned}
$$

where $h = 1/N$, $C_{11}(\nu) = 0.1\Phi(\nu)$, $C_{12}(\nu) = 0.3\Phi(\nu)$, $C_{21}(\nu) = 0.2\Phi(\nu)$, $C_{22}(\nu) = 0.3\Phi(\nu)$,

$$
\Phi(\nu) = \int_{(\nu-1)h}^{\nu h}(1-s)\,ds = h - h^2(\nu - 1/2).
$$

We have

$$
\sum_{\nu=1}^{N}\Phi(\nu) = \int_0^1 (1-s)\,ds = 0.5.
$$

Since

$$
\begin{aligned}
& L_1\left\{|b_{11}| + |b_{21}| + \sum_{\nu=1}^{N}\left[|C_{11}(\nu)| + |C_{21}(\nu)|\right]\right\}\\
= \;& 0.5 + 0.3 + (0.1+0.2)0.5 = 0.95 < 1 = a_1,\\
& L_2\left\{|b_{12}| + |b_{22}| + \sum_{\nu=1}^{N}\left[|C_{12}(\nu)| + |C_{22}(\nu)|\right]\right\}\\
= \;& 0.2 + 0.4 + (0.3+0.3)0.5 = 0.9 < 1 = a_2,
\end{aligned}
$$

condition **A3.2.3.6** is satisfied with $\lambda_1 = \lambda_2 = 1$ and Theorem 3.2.3.1 holds. More precisely, if $\rho \in (1, \bar{\rho})$, where $\bar{\rho}$ is given by Lemma 3.2.3.1, and $q \in (\rho^{-1}, 1)$, we can choose N so large that $(\rho q)^N \geq (1+2\rho)(1+0.7\rho)(1+0.5\rho)$. Thus system (3.2.3.15) has a unique N-periodic solution, which is globally exponentially stable.

Next we consider a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\left(\int_0^\omega g_{ij}(s)x_j(t-s)\,ds\right) + I_i(t), \\ &\qquad t \neq t_k, \qquad (3.2.3.16) \\ \Delta x_i(t_k) &= -\gamma_{ik}x_i(t_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\int_0^\omega c_{ij}(s)x_j(t_k-s)\,ds\right) + \alpha_{ik}, \\ &\qquad i=\overline{1,m}, \quad k\in\mathbb{Z}, \qquad (3.2.3.17)\end{aligned}$$

where γ_{ik} ($i=\overline{1,m}$, $k\in\mathbb{Z}$) are positive constants, $a_i(t)$, $b_{ij}(t)$, $I_i(t)$ are ω-periodic in t; $t_{k+p}=t_k+\omega$, $\gamma_{i,k+p}=\gamma_{ik}$, $B_{ij,k+p}=B_{ijk}$, $\alpha_{i,k+p}=\alpha_{ik}$. Without loss of generality we can assume that

$$0 < t_1 < t_2 < \cdots < t_p < \omega.$$

The Hopfield neural network (3.2.3.16) is similar to the bidirectional associative memory neural network considered in [120].

We can consider the system (3.2.3.16) for $t>0$, the impulse conditions (3.2.3.17) for $k>0$, with initial conditions

$$x_i(s)=\phi_i(s) \quad \text{for} \quad s\in[-\omega,0], \qquad i=\overline{1,m}, \qquad (3.2.3.18)$$

where the initial functions $\phi_i(s)$, $i=\overline{1,m}$, are piecewise continuous with points of discontinuity of the first kind at $t_{-p+1}, t_{-p+2},\ldots,t_{-1}, t_0$. To find an ω-periodic solution of system (3.2.3.16), (3.2.3.17) means to determine the initial functions $\phi_i(s)$ so that the solution of the initial value problem (3.2.3.16), (3.2.3.17), (3.2.3.18) is ω-periodic.

Combining some ideas of [95, 3, 120] we shall formulate the discrete counterpart of system (3.2.3.16), (3.2.3.17). For a positive integer N we choose the discretization step $h=\omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}}(t_{k+1}-t_k).$$

Then each interval $[nh,(n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k.

We approximate the integral term in (3.2.3.16) by a sum:

$$\int_0^\omega g_{ij}(s)x_j(t-s)\,ds \approx \sum_{\ell=1}^{N} g_{ij}(\ell h)x_j\left((n-\ell)h\right)\varphi(h),$$

where $\varphi(h) = h + O(h^2)$.

Next we approximate the differential equation (3.2.3.16) on the interval $[nh, (n+1)h]$ by

$$\frac{dx_i}{dt} + a_i(nh)x_i(t) = I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell h)x_j\left((n-\ell)h\right)\varphi(h)\right).$$

We multiply both sides of this equation by $\exp\left(a_i(nh)t\right)$ and integrate over the interval $[nh, (n+1)h]$. Thus we obtain

$$\begin{aligned} & x_i\left((n+1)h\right) - x_i(nh) = -\left(1 - e^{-a_i(nh)h}\right)x_i(nh) \qquad (3.2.3.19) \\ + & \frac{1 - e^{-a_i(nh)h}}{a_i(nh)}\left\{I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh)f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell h)x_j\left((n-\ell)h\right)\varphi(h)\right)\right\}. \end{aligned}$$

Henceforth by abuse of notation we write $x_i(n) = x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)$ $(i = \overline{1,m},\ n \in \mathbb{Z})$. For convenience we adopt the notations:

$$\begin{aligned} A_i(n) &= 1 - e^{-a_i(nh)h} \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}), \\ I_i(n) &= \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} I_i(nh) \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}), \\ b_{ij}(n) &= \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} b_{ij}(nh) \qquad (i,j = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}), \\ g_{ij}(\ell) &= g_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m},\ \ell = \overline{1,N}). \end{aligned}$$

Clearly, we have $0 < A_i(n) < 1$. In particular, if $a_i(t) < \frac{1}{\omega}$, then $A_i(n) < \frac{1}{N}$.

With the above notation equation (3.2.3.19) takes the form

$$\begin{aligned} \Delta x_i(n) &= -A_i(n)x_i(n) + I_i(n) + \sum_{j=1}^{m} b_{ij}(n)f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell)x_j(n-\ell)\right), \\ & i = \overline{1,m}, \quad n \neq n_k. \qquad (3.2.3.20) \end{aligned}$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (3.2.3.17) by

$$\begin{aligned}\Delta x_i(n_k) \;&=\; -\gamma_{ik}x_i(n_k) + \alpha_{ik} + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\sum_{\ell=1}^{N} c_{ij}(\ell)x_j(n_k-\ell)\right), \\ & i = \overline{1,m}, \quad k \in \mathbb{Z}, \end{aligned} \tag{3.2.3.21}$$

where

$$c_{ij}(\ell) = c_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m}, \quad \ell = \overline{1,N}).$$

For uniformity of notation we define

$$A_i(n_k) = \gamma_{ik}, \quad I_i(n_k) = \alpha_{ik} \qquad (i = \overline{1,m},\ k \in \mathbb{Z}).$$

Now the difference system (3.2.3.20), (3.2.3.21) can be written in operator form as

$$\Delta x = Hx, \tag{3.2.3.22}$$

where

$$\begin{aligned}(Hx)_i(n) &= -A_i(n)x_i(n) + I_i(n) \\ &+ \begin{cases} \sum\limits_{j=1}^{m} b_{ij}(n)f_j\left(\sum\limits_{\ell=1}^{N} g_{ij}(\ell)x_j(n-\ell)\right), & n \neq n_k, \\ \sum\limits_{j=1}^{m} B_{ijk}\Phi_j\left(\sum\limits_{\ell=1}^{N} c_{ij}(\ell)x_j(n_k-\ell)\right), & n = n_k. \end{cases}\end{aligned} \tag{3.2.3.23}$$

We can consider the system (3.2.3.22) for $n \geq 0$, with initial conditions

$$x_i(\ell) = \phi_i(\ell) \quad \text{for} \quad \ell = 0, -1, \ldots, -N, \qquad i = \overline{1,m}, \tag{3.2.3.24}$$

where $\phi(\ell) = (\phi_1(\ell), \phi_2(\ell), \ldots, \phi_m(\ell))^T$, $\ell = 0, -1, \ldots, -N$, are given initial vectors. To find an N-periodic solution of system (3.2.3.22) means to determine the initial vectors $\phi(\ell)$ so that the solution of the initial value problem (3.2.3.22), (3.2.3.24) is N-periodic.

In order to formulate our assumptions, we need some more notation:

$$I_N = \{0,\, 1,\, \ldots,\, N-1\},$$

$$\underline{A}_i = \min_{n \in I_N} A_i(n), \quad \overline{A}_i = \sum_{n=0}^{N-1} A_i(n), \qquad i = \overline{1,m}.$$

Now we introduce the following conditions:

A3.2.3.7. $A_i(n+N) = A_i(n)$, $I_i(n+N) = I_i(n)$ for $i = \overline{1,m}$, $n \in \mathbb{Z}$; $n_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}$ and $n_{k+p} = n_k + N$; $b_{ij}(n+N) = b_{ij}(n)$ $(n \neq n_k)$, $B_{ij,k+p} = B_{ijk}$ $(k \in \mathbb{Z})$ for $i, j = \overline{1,m}$.

A3.2.3.8. $\underline{A}_i > 0$, $\overline{A}_i < 1$ for $i = \overline{1,m}$.

A3.2.3.9. The functions $f_j(\cdot)$, $\Phi_j(\cdot)$ $(j = \overline{1,m})$ are Lipschitz continuous on $\mathbb{R}$, that is, there exist positive constants M_j and L_j such that

$$|f_j(x) - f_j(y)| \leq M_j|x-y|, \quad |\Phi_j(x) - \Phi_j(y)| \leq L_j|x-y|$$

for all $x, y \in \mathbb{R}$.

A3.2.3.10. $g_{ij}(\ell) \geq 0$, $c_{ij}(\ell) \geq 0$ for $i, j = \overline{1,m}$, $\ell = \overline{1,N}$.

We again introduce some notation:

$$\overline{I}_i = \max_{n \in I_N} |I_i(n)|, \qquad i = \overline{1,m},$$

$$\overline{b}_{ij} = \sup_{n \neq n_k} |b_{ij}(n)|, \quad \overline{B}_{ij} = \max_{k=\overline{1,p}} |B_{ijk}|, \qquad i, j = \overline{1,m}.$$

For an N-periodic sequence $v(n)$ we denote $\tilde{v} = \frac{1}{N}\sum\limits_{n=0}^{N-1} v(n)$; for $i = \overline{1,m}$

$$\rho_i' = \overline{I}_i + \sum_{j=1}^{m} \overline{b}_{ij}|f_j(0)|, \qquad \rho_i'' = \overline{I}_i + \sum_{j=1}^{m} \overline{B}_{ij}|\Phi_j(0)|, \tag{3.2.3.25}$$

$$\rho_i = [(N-p)\rho_i' + p\rho_i'']/N = \overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right].$$

Next we denote

$$\mathcal{M}_j = \max\{L_j,\ M_j\}, \quad j = \overline{1,m},$$

$$G_{ij} = \sum_{\ell=1}^{N} g_{ij}(\ell), \quad C_{ij} = \sum_{\ell=1}^{N} c_{ij}(\ell), \qquad i, j = \overline{1,m},$$

$$\mathcal{B}_{ij} = \max\{\overline{b}_{ij},\ \overline{B}_{ij}\}, \quad \mathcal{G}_{ij} = \max\{G_{ij},\ C_{ij}\}, \qquad i, j = \overline{1,m}.$$

We introduce the $m \times m$ matrices

$$A = \operatorname{diag}\left(\frac{\underline{A}_i\left(1 - \overline{A}_i\right)}{1 + N\underline{A}_i},\ i = \overline{1,m}\right), \qquad B = (\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij})_{i,j=1}^{m}. \tag{3.2.3.26}$$

Next we introduce the conditions

A3.2.3.11. $\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) > 0.$

A3.2.3.12. $\underline{A}_i > \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji}$ for $i = \overline{1,m}$.

A3.2.3.13. The matrix $A - B$ is an M-matrix.

Clearly, condition **A3.2.3.12** implies **A3.2.3.11** but the converse is not true. Condition **A3.2.3.13** implies that the matrix $A-B$ is nonsingular and its inverse has nonnegative entries only.

Now we can state our main results as two theorems.

Theorem 3.2.3.2. *Suppose that conditions* **A3.2.3.7**–**A3.2.3.11**, **A3.2.3.13** *hold. Then the equation* (3.2.3.22) *has at least one N-periodic solution.*

Theorem 3.2.3.3. *Suppose that conditions* **A3.2.3.7**–**A3.2.3.10**, **A3.2.3.12**, **A3.2.3.13** *hold. Then the N-periodic solution of* (3.2.3.22) *is unique and globally exponentially stable.*

Proof of the existence of a periodic solution. We shall prove Theorem 3.2.3.2 using Mawhin's continuation theorem [57, p. 40]. To state this theorem we need some preliminaries:

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \longrightarrow \mathbb{Y}$ be a linear mapping, and $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \longrightarrow \mathbb{X}$ and $Q : \mathbb{Y} \longrightarrow \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$, then the mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I-P)\mathbb{X} \longrightarrow \operatorname{Im} L$ is invertible. We denote the inverse of this mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping H will be called L-compact on $\overline{\Omega}$ if $QH(\overline{\Omega})$ is bounded and $K_P(I-Q)H : \overline{\Omega} \longrightarrow \mathbb{X}$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$.

Now Mawhin's continuation theorem can be stated as follows.

Lemma 3.2.3.3. *Let L be a Fredholm mapping of index zero, let $\Omega \subset \mathbb{X}$ be an open bounded set and let $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(a) *for each* $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Hx$;

(b) *for each* $x \in \partial\Omega \cap \operatorname{Ker} L$, $QHx \neq 0$;

(c) $\deg(JQH, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $\deg(\cdot)$ *is the Brouwer degree.*

Then the equation $Lx = Hx$ *has at least one solution in* $\overline{\Omega} \cap \operatorname{Dom} L$.

It is much easier to apply this lemma to difference equations than to differential equations since in the former case all spaces are finite dimensional.

Before we proceed further, we shall recall the definition of Brouwer degree [90].

Suppose that M and N are two oriented differentiable manifolds of dimension n (without boundary) with M compact and N connected and suppose that $f : M \to N$ is a differentiable mapping. Let $Df(x)$ denote the differential mapping at the point $x \in M$, that is the linear mapping $Df(x) : T_x(M) \to T_{f(x)}(N)$. Let $\operatorname{sign} Df(x)$ denote the sign of the determinant of $Df(x)$. That is, the sign is positive if f preserves orientation and negative if f reverses orientation.

Definition 3.2.3.1. Let $y \in N$ be a regular value, then we define the *Brouwer degree* (or just *degree*) of f by

$$\deg f := \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

It can be shown that the degree does not depend on the regular value y that we pick so that $\deg f$ is well defined.

Note that this degree coincides with the degree as defined for maps of spheres.

Let us choose $\mathbb{X} = \mathbb{Y} = \{x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T : x(n+N) = x(n),\, n \in \mathbb{Z}\}$. If we define $|x_i| = \max_{n \in I_N} |x_i(n)|$, $\|x\| = \sum_{i=1}^{m} |x_i|$, then $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$. For $x \in \mathbb{X}$, let Hx be defined by (3.2.3.23), $Lx = \Delta x$ and

$$Px = Qx = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)^T.$$

Then $\operatorname{Ker} L = \{x \in \mathbb{X} : x = h \in \mathbb{R}^m\}$ (vectors with components independent of n), $\operatorname{Im} L = \{x \in \mathbb{X} : \sum_{n=0}^{N-1} x_i(n) = 0,\, i = \overline{1, m}\}$ is a closed set in $\mathbb{X}$, and $\operatorname{codim} L = m$. Thus L is a Fredholm mapping of index zero. It is easy to see that P and Q are continuous projectors and $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, and H is L-compact on $\bar{\Omega}$ for any bounded set

$\Omega \subset \mathbb{X}$. Moreover, in condition (c) of Lemma 3.2.3.3 the isomorphism J can be taken as the identity operator I.

Now we will derive some estimates for the solutions x of the operator equation $Lx = \lambda Hx$ for $\lambda \in (0,\, 1)$, that is,

$$\Delta x_i(n) = \lambda (Hx)_i(n), \quad n \in I_N,\ i = \overline{1, m}. \tag{3.2.3.27}$$

First from (3.2.3.27) and (3.2.3.23) for $n \neq n_k$ we obtain

$$\begin{aligned}
|\Delta x_i(n)| &\leq A_i(n)|x_i(n)| + |I_i(n)| + \left| \sum_{j=1}^{m} b_{ij}(n) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right) \right| \\
&\leq A_i(n)|x_i| + \overline{I}_i + \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} g_{ij}(\ell)|x_j(n-\ell)| + \sum_{j=1}^{m} \overline{b}_{ij} |f_j(0)| \\
&\leq A_i(n)|x_i| + \rho_i' + \sum_{j=1}^{m} \overline{b}_{ij} M_j G_{ij} |x_j| \\
&\leq A_i(n)|x_i| + \rho_i' + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \mathcal{G}_{ij} |x_j|,
\end{aligned}$$

where ρ_i' was introduced in (3.2.3.25).

Similarly, for $n = n_k$ we have

$$\begin{aligned}
|\Delta x_i(n_k)| &\leq A_i(n_k)|x_i(n_k)| + |I_i(n_k)| + \left| \sum_{j=1}^{m} B_{ijk}(n) \Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k-\ell) \right) \right| \\
&\leq A_i(n_k)|x_i| + \overline{I}_i + \sum_{j=1}^{m} \overline{B}_{ij} L_j \sum_{\ell=1}^{N} c_{ij}(\ell)|x_j(n_k-\ell)| + \sum_{j=1}^{m} \overline{B}_{ij} |\Phi_j(0)| \\
&\leq A_i(n_k)|x_i| + \rho_i'' + \sum_{j=1}^{m} \overline{B}_{ij} L_j C_{ij} |x_j| \\
&\leq A_i(n_k)|x_i| + \rho_i'' + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \mathcal{G}_{ij} |x_j|.
\end{aligned}$$

From the above inequalities we obtain

$$\sum_{n=0}^{N-1} |\Delta x_i(n)| \leq \overline{A}_i |x_i| + (N-p)\rho_i' + p\rho_i'' + N \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \mathcal{G}_{ij} |x_j|$$

or

$$\sum_{n=0}^{N-1} |\Delta x_i(n)| \le \overline{A}_i |x_i| + N\rho_i + N \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|. \tag{3.2.3.28}$$

Adding together all equations of (3.2.3.27) for $n \in I_N$, we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} A_i(n)x_i(n) &= \sum_{n=0}^{N-1} I_i(n) + \sum_{j=1}^{m} \left\{ {\sum}' b_{ij}(n) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right) \right. \\ &+ \left. \sum_{k=1}^{p} B_{ijk} \Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k - \ell) \right) \right\}, \end{aligned}$$

where by definition

$$\begin{aligned} {\sum}' v(n) &= \sum_{n=0}^{N-1} v(n) - \sum_{k=1}^{p} v(n_k) \\ &= v(0) + \cdots + v(n_1 - 1) + v(n_1 + 1) + \cdots + v(n_p - 1) + v(n_p + 1) + \cdots + v(N-1). \end{aligned}$$

Then as above we obtain

$$\left| \sum_{n=0}^{N-1} A_i(n)x_i(n) \right| \le N\rho_i + N \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|. \tag{3.2.3.29}$$

Now we shall use the following lemma (see [54, 112]).

Lemma 3.2.3.4. *Let* $v : \mathbb{Z} \to \mathbb{R}$ *be* N*-periodic,* i.e., $v(n+N) = v(n)$ *for any* $n \in \mathbb{Z}$*. Then for any fixed* $\nu_1, \nu_2 \in I_N$ *and any* $n \in \mathbb{Z}$ *we have*

$$v(\nu_2) - \sum_{k=0}^{N-1} |v(k+1) - v(k)| \le v(n) \le v(\nu_1) + \sum_{k=0}^{N-1} |v(k+1) - v(k)|.$$

According to Lemma 3.2.3.4 for arbitrary $n, \nu_1, \nu_2 \in I_N$ we have

$$x_i(\nu_2) - \sum_{n=0}^{N-1} |\Delta x_i(n)| \le x_i(n) \le x_i(\nu_1) + \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

We multiply these inequalities by $A_i(n)$ and sum up over I_N to obtain

$$\overline{A}_i x_i(\nu_2) - \overline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \le \sum_{n=0}^{N-1} A_i(n) x_i(n) \le \overline{A}_i x_i(\nu_1) + \overline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

From the last two inequalities we deduce

$$
\begin{aligned}
-x_i(\nu_1) &\leq -\frac{\sum\limits_{n=0}^{N-1} A_i(n)x_i(n)}{\overline{A}_i} + \sum_{n=0}^{N-1} |\Delta x_i(n)|, \\
x_i(\nu_2) &\leq \frac{\sum\limits_{n=0}^{N-1} A_i(n)x_i(n)}{\overline{A}_i} + \sum_{n=0}^{N-1} |\Delta x_i(n)|.
\end{aligned}
$$

Let $|x_i(\nu_0)| = |x_i| \equiv \max\limits_{n \in I_N} |x_i(n)|$. If $x_i(\nu_0) \geq 0$, we choose $\nu_2 = \nu_0$. Then

$$
\begin{aligned}
\underline{A}_i|x_i| = \underline{A}_i x_i(\nu_2) &\leq \frac{\underline{A}_i}{\overline{\overline{A}}_i} \left(\sum_{n=0}^{N-1} A_i(n)x_i(n) \right) + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \\
&\leq \frac{1}{N} \left| \sum_{n=0}^{N-1} A_i(n)x_i(n) \right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.
\end{aligned}
$$

If $x_i(\nu_0) < 0$, we choose $\nu_1 = \nu_0$,

$$
\begin{aligned}
\underline{A}_i|x_i| = -\underline{A}_i x_i(\nu_1) &\leq \frac{\underline{A}_i}{\overline{\overline{A}}_i} \left(-\sum_{n=0}^{N-1} A_i(n)x_i(n) \right) + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \\
&\leq \frac{1}{N} \left| \sum_{n=0}^{N-1} A_i(n)x_i(n) \right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.
\end{aligned}
$$

Thus in both cases we have

$$
\underline{A}_i|x_i| \leq \frac{1}{N} \left| \sum_{n=0}^{N-1} A_i(n)x_i(n) \right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.
$$

Making use of the estimates (3.2.3.28) and (3.2.3.29), we obtain

$$
\underline{A}_i|x_i| \leq \frac{1}{N} \left\{ N\rho_i + N \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \right\}
$$

$$
+\underline{A}_i \left\{ \overline{A}_i|x_i| + N\rho_i + N \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \right\}
$$

$$= \underline{A}_i\overline{A}_i|x_i| + (1+N\underline{A}_i)\left(\rho_i + \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|\right)$$

or

$$\frac{\underline{A}_i(1-\overline{A}_i)}{1+N\underline{A}_i}|x_i| - \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \le \rho_i. \qquad (3.2.3.30)$$

If we introduce the vectors $|\boldsymbol{x}| = (|x_1|, \ldots, |x_m|)^T$ and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)^T$, then the system of inequalities (3.2.3.30) for $i = \overline{1,m}$ can be written in a matrix form

$$(A - B)|\boldsymbol{x}| \le \boldsymbol{\rho}, \qquad (3.2.3.31)$$

where the matrices A and B were introduced in (3.2.3.26). By virtue of condition **A3.2.3.13** the inequality (3.2.3.31) implies

$$|\boldsymbol{x}| \le (A-B)^{-1}\boldsymbol{\rho}.$$

If $(A-B)^{-1}\boldsymbol{\rho} = (C_1^*, C_2^*, \ldots, C_m^*)^T$, this means that the components of each solution of $\Delta x = \lambda Hx$ satisfy $|x_i| \le C_i^*$. If we denote $C^* = \sum_{i=1}^{m} C_i^*$, then each solution of $\Delta x = \lambda Hx$ satisfies $\|x\| \le C^*$.

Now we take $\Omega = \{x \in \mathbb{X} : \|x\| < C\}$, where $C > C^*$ will be chosen later. Obviously Ω satisfies condition (a) of Lemma 3.2.3.3.

Now let $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^m$, *i.e.*, x is a constant vector in $\mathbb{R}^m$ with $\|x\| = C$. For such x,

$$\begin{aligned} (Hx)_i(n) &= -A_i(n)x_i + I_i(n) + \sum_{j=1}^{m} b_{ij}(n)f_j(G_{ij}x_j), \quad n \ne n_k, \\ (Hx)_i(n_k) &= -A_i(n_k)x_i + I_i(n_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j(C_{ij}x_j). \end{aligned}$$

Then

$$(QHx)_i = -\tilde{A}_i x_i + \tilde{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left\{{\sum}' b_{ij}(n)f_j(G_{ij}x_j) + \sum_{k=1}^{p} B_{ijk}\Phi_j(C_{ij}x_j)\right\}$$

and

$$
\begin{aligned}
|(QHx)_i| \geq \tilde{A}_i|x_i| - |\tilde{I}_i| \quad &- \quad \frac{1}{N}\sum_{j=1}^{m}\left\{\sum{}' |b_{ij}(n)|M_jG_{ij} + \sum_{k=1}^{p}|B_{ijk}|L_jC_{ij}\right\}|x_j| \\
&- \quad \frac{1}{N}\sum_{j=1}^{m}\left\{\sum{}' |b_{ij}(n)|\cdot|f_j(0)| + \sum_{k=1}^{p}|B_{ijk}|\cdot|\Phi_j(0)|\right\} \\
\geq \tilde{A}_i|x_i| - |\tilde{I}_i| \quad &- \quad \sum_{j=1}^{m}\frac{1}{N}\left[(N-p)\overline{b}_{ij}M_jG_{ij} + p\overline{B}_{ij}L_jC_{ij}\right]|x_j| \\
&- \quad \sum_{j=1}^{m}\frac{1}{N}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right] \\
\geq \tilde{A}_i|x_i| - \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \quad &- \quad \left\{\overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right]\right\} \\
&= \quad \tilde{A}_i|x_i| - \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| - \rho_i.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|QHx\| &= \sum_{i=1}^{m}|(QHx)_i| \geq \sum_{i=1}^{m}\tilde{A}_i|x_i| - \sum_{i=1}^{m}\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| - \sum_{i=1}^{m}\rho_i \\
&= \sum_{i=1}^{m}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)|x_i| - \sum_{i=1}^{m}\rho_i \\
&\geq \min_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)\|x\| - \sum_{i=1}^{m}\rho_i \\
&= \min_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)C - \sum_{i=1}^{m}\rho_i.
\end{aligned}
$$

By condition **A3.3.3.11**

$$
\min_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right) > 0.
$$

Then we can choose $C > C^*$ so large that

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) C > \sum_{i=1}^{m} \rho_i.$$

Hence for $x \in \partial\Omega \cap \operatorname{Ker} L$ we have $\|QHx\| > 0$ and $QHx \neq 0$, that is, condition (b) of Lemma 3.2.3.3 is satisfied.

To prove (c), we define the mapping $(QH)_\mu : \operatorname{Dom} L \times [0,1] \longrightarrow \mathbb{X}$ by $(QH)_\mu = -\mu\tilde{A} + (1-\mu)QH$, where $\tilde{A}x = (\tilde{A}_1x_1, \tilde{A}_2x_2, \ldots, \tilde{A}_mx_m)^T$.

For $x \in \partial\Omega \cap \operatorname{Ker} L$ we have

$$\begin{aligned} &((QH)_\mu x)_i = -\tilde{A}_i x_i \\ + \quad &(1-\mu)\left\{ \tilde{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left({\sum}' b_{ij}(n) f_j(G_{ij}x_j) + \sum_{k=1}^{p} B_{ijk}\Phi_j(C_{ij}x_j) \right) \right\}. \end{aligned}$$

As above, we obtain

$$\|(QH)_\mu x\| \geq \min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) C - \sum_{i=1}^{m} \rho_i > 0.$$

This means that $(QH)_\mu x \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0,1]$. From the homotopy invariance of the Brouwer degree, it follows that

$$\deg(QH,\, \Omega \cap \operatorname{Ker} L,\, 0) = \deg(-\tilde{A},\, \Omega \cap \operatorname{Ker} L,\, 0) = (-1)^m \neq 0.$$

According to Lemma 3.2.3.3 the equation (3.2.3.22) has at least one N-periodic solution. This completes the proof of Theorem 3.2.3.2. □

Proof of the global exponential stability of the periodic solution. Let $\boldsymbol{g}_{ij}(\ell) = \max\{g_{ij}(\ell),\ c_{ij}(\ell)\}$, $i, j = \overline{1,m}$, $\ell = \overline{1,N}$. Clearly,

$$\sum_{\ell=1}^{N} \boldsymbol{g}_{ij}(\ell) = \mathcal{G}_{ij}, \qquad i, j = \overline{1,m}.$$

Lemma 3.2.3.5. *Assume that condition* **A3.2.3.12** *holds. Then there exists* $\overline{\lambda} > 1$ *such that for any* $i = \overline{1,m}$, $n \in I_N$ *and* $\lambda \in (1, \overline{\lambda}]$ *we have*

$$\lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ji}(\ell) - 1 \leq 0.$$

Proof. Let us introduce the functions

$$F_{in}(\lambda) = \lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ji}(\ell) - 1$$

for $\lambda \in [1, +\infty)$. It is easily seen that F_{in} are continuous and increasing on $[1, +\infty)$. From condition **A3.2.3.12** we have

$$F_{in}(1) = -A_i(n) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \leq -\left(\underline{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji}\right) < 0.$$

Since $\lim\limits_{\lambda \to +\infty} F_{in}(\lambda) = +\infty$, there exist constants $\overline{\lambda}_{in} > 1$ such that $F_{in}(\overline{\lambda}_{in}) = 0$ and $F_{in}(\lambda) \leq 0$ on $(1, \overline{\lambda}_{in}]$. If we set

$$\overline{\lambda} = \min_{i=\overline{1,m},\ n \in I_N} \overline{\lambda}_{in},$$

then for any $i = \overline{1,m}$, $n \in I_N$ we have $F_{in}(\lambda) \leq 0$ for $\lambda \in (1, \overline{\lambda}]$. This completes the proof of the lemma. □

Now let us suppose that $x^*(n) = (x_1^*(n), x_2^*(n), \ldots, x_m^*(n))^T$ is an N-periodic solution of equation (3.2.3.22), and $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ is any solution of (3.2.3.22) for $n \geq 0$, defined at least for $n \geq -N$.

From (3.2.3.22) and (3.2.3.23) for $n \in \mathbb{N} \cup \{0\}$, $n \neq n_k$ we derive

$$\begin{aligned} & x_i(n+1) - x_i^*(n+1) = (1 - A_i(n))(x_i(n) - x_i^*(n)) \\ + & \sum_{j=1}^{m} b_{ij}(n) \left\{ f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell)\right) - f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j^*(n-\ell)\right)\right\}, \end{aligned}$$

and hence,

$$\begin{aligned} & |x_i(n+1) - x_i^*(n+1)| \\ \leq & (1 - A_i(n))|x_i(n) - x_i^*(n)| + \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} g_{ij}(\ell)|x_j(n-\ell) - x_j^*(n-\ell)|, \end{aligned}$$

whereas for $n = n_k$ we have

$$\begin{aligned} & |x_i(n_k+1) - x_i^*(n_k+1)| \\ \leq & (1 - A_i(n_k))|x_i(n_k) - x_i^*(n_k)| + \sum_{j=1}^{m} \overline{B}_{ij} L_j \sum_{\ell=1}^{N} c_{ij}(\ell)|x_j(n_k-\ell) - x_j^*(n_k-\ell)|. \end{aligned}$$

Now we introduce the quantities

$$y_i(n) = \lambda^n |x_i(n) - x_i^*(n)|, \quad \lambda \in (1, \overline{\lambda}], \quad i = \overline{1,m}, \quad n \geq -N.$$

Then for $n \in \mathbb{N} \cup \{0\}$, $n \neq n_k$ we have

$$\begin{aligned} y_i(n+1) &= \lambda^{n+1}|x_i(n+1) - x_i^*(n+1)| \leq \lambda^{n+1}(1 - A_i(n))|x_i(n) - x_i^*(n)| \\ &+ \lambda^{n+1} \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} g_{ij}(\ell)|x_j(n-\ell) - x_j^*(n-\ell)| \\ &= \lambda(1 - A_i(n))y_i(n) + \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} \lambda^{\ell+1} g_{ij}(\ell) y_j(n-\ell), \end{aligned}$$

whereas for $n = n_k$

$$y_i(n_k+1) \leq \lambda(1 - A_i(n_k))y_i(n_k) + \sum_{j=1}^{m} \overline{B}_{ij} L_j \sum_{\ell=1}^{N} \lambda^{\ell+1} c_{ij}(\ell) y_j(n_k - \ell).$$

From the last two inequalities we obtain

$$y_i(n+1) \leq \lambda(1-A_i(n))y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) y_j(n-\ell), \quad (3.2.3.32)$$

$$\lambda \in (1, \overline{\lambda}], \quad i = \overline{1,m}, \quad n \in \mathbb{N} \cup \{0\}.$$

Now we consider a Lyapunov functional $V(n) = V(y_1, y_2, \ldots, y_m)(n)$ defined by

$$V(n) = \sum_{i=1}^{m} \left\{ y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) \sum_{s=n-\ell}^{n-1} y_j(s) \right\}, \quad n \in \mathbb{N} \cup \{0\}.$$

Taking into account (3.2.3.32), we estimate the difference $\Delta V(n) = V(n+1) - V(n)$ for $n \in \mathbb{N} \cup \{0\}$:

$$\Delta V(n) \leq \sum_{i=1}^{m} \left\{ \lambda(1 - A_i(n))y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) y_j(n-\ell) \right.$$

$$+\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ij}(\ell)\sum_{s=n+1-\ell}^{n}y_j(s)-y_i(n)$$

$$-\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ij}(\ell)\sum_{s=n-\ell}^{n-1}y_j(s)\Bigg\}$$

$$\begin{aligned}
&= \sum_{i=1}^{m}\left\{\lambda(1-A_i(n))y_i(n)+\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ij}(\ell)y_j(n)-y_i(n)\right\}\\
&= \sum_{i=1}^{m}\left\{\lambda(1-A_i(n))+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ji}(\ell)-1\right\}y_i(n).
\end{aligned}$$

By virtue of Lemma 3.2.3.5 we have $\Delta V(n)\le 0$ for all $n\in\mathbb{N}\cup\{0\}$, which implies that

$$V(n)\le V(0),\qquad n\in\mathbb{N}\cup\{0\}. \tag{3.2.3.33}$$

On the other hand, we have

$$V(n)\ge\sum_{i=1}^{m}y_i(n)=\sum_{i=1}^{m}\lambda^n|x_i(n)-x_i^*(n)|$$

and

$$\begin{aligned}
V(0) &= \sum_{i=1}^{m}\left\{y_i(0)+\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ij}(\ell)\sum_{s=-\ell}^{-1}y_j(s)\right\}\\
&= \sum_{i=1}^{m}\left\{y_i(0)+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ji}(\ell)\sum_{s=-\ell}^{-1}y_i(s)\right\}\\
&\le \sum_{i=1}^{m}\left\{1+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\ell\lambda^{\ell+1}\boldsymbol{g}_{ji}(\ell)\right\}\sup_{s\in I_{-N}}y_i(s)\\
&\le \sum_{i=1}^{m}\left\{1+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\ell\overline{\lambda}^{\ell+1}\boldsymbol{g}_{ji}(\ell)\right\}\max_{s\in I_{-N}}|x_i(s)-x_i^*(s)|,
\end{aligned}$$

where $I_{-N}=\{-N,-N+1,\ldots,-1,0\}$. Here we used the fact that $1<\lambda\le\overline{\lambda}$.

Thus from inequality (3.2.3.33) we obtain

$$\sum_{i=1}^{m}|x_i(n)-x_i^*(n)|\le M\lambda^{-n}\sum_{i=1}^{m}\max_{s\in I_{-N}}|x_i(s)-x_i^*(s)|,\quad n\in\mathbb{N}\cup\{0\},$$

where

$$M = \max_{i=\overline{1,m}} \left(1 + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \ell \overline{\lambda}^{\ell+1} \boldsymbol{g}_{ji}(\ell) \right).$$

This completes the proof of Theorem 3.2.3.3. □

Different versions of results of the present subsection were reported at the International Symposium on Neural Networks and Soft Computing in Structural Engineering, Cracow, Poland, 2005, the Eighth UAE University Research Conference, Al Ain, UAE, 2007, and the Sixth ISAAC (International Society for Analysis, its Applications and Computation) Congress, Ankara, Turkey, 2007. They were published in [5, 7, 17, 18, 51]. The exposition here follows closely [5] and [18].

3.2.4 Equilibrium points of Hopfield neural networks with leakage delay

Consider the impulsive continuous-time neural network consists of m elementary processing units (or neurons) whose state variables x_i $(i = \overline{1,m})$ are governed by the system

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t-\sigma) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(t-\tau_{ij})) \\ &+ \sum_{j=1}^{m} d_{ij} h_j \left(\int_0^\infty K_{ij}(s) x_j(t-s)\, ds \right) + I_i, \quad t > 0, \quad t \neq t_k, \end{aligned} \tag{3.2.4.1}$$

$$\Delta x_i(t_k) = B_{ik} x_i(t_k) + \int_{t_k-\sigma}^{t_k} \psi_{ik}(s) x_i(s)\, ds + \gamma_{ik}, \quad i = \overline{1,m}, \quad k \in \mathbb{N}, \tag{3.2.4.2}$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, 0]$. System (3.2.4.1) differs from (3.1.1.1) only by the delay $\sigma > 0$ in the stabilizing (or negative) feedback term $-a_i(x_i - \sigma)$, also called *leakage* or *forgetting term* of the unit i. The impulsive conditions (3.2.4.2) are similar to (3.1.1.2). The sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$ and $\Delta t_k = t_k - t_{k-1} \geq \theta$, where $\theta > 0$ denotes the minimum time interval between successive impulses. In other words, the value $\theta > 0$ means that the impulses do not occur too often.

The assumptions that accompany the impulsive network (3.2.4.1), (3.2.4.1) are given as follows:

A3.2.4.1. $0 < a_i < 1/\sigma$, $i = \overline{1,m}$.

A3.2.4.2. The activation functions $f_j, g_j, h_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in the sense of

$$F_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|, \qquad G_j = \sup_{x \neq y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right|,$$
$$H_j = \sup_{x \neq y} \left| \frac{h_j(x) - h_j(y)}{x - y} \right| \quad \text{for} \quad x, y \in \mathbb{R}, \quad j = \overline{1,m},$$

where F_j, G_j, H_j denote positive constants.

A3.2.4.3. $a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| > 0, \quad i = \overline{1,m}$.

A3.2.4.4. $K_{ij} : [0, \infty) \to [0, \infty)$ are bounded and piecewise continuous $(i, j = \overline{1,m})$.

A3.2.4.5. $\int_0^\infty K_{ij}(s)\, ds = 1$ $(i, j = \overline{1,m})$.

A3.2.4.6. There exists a positive number μ such that $\int_0^\infty K_{ij}(s) e^{\mu s}\, ds < \infty$ $(i, j = \overline{1,m})$.

Assumptions **A3.2.4.2–A3.2.4.6** are identical respectively to **A3.1.1.1–A3.1.1.5**.

An equilibrium point of the impulsive network (3.2.4.1), (3.2.4.2) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*) + \sum_{j=1}^{m} d_{ij} h_j(x_j^*) + I_i, \quad i = \overline{1,m}, \tag{3.2.4.3}$$

and satisfy the linear equations

$$\left(B_{ik} + \int_{t_k - \sigma}^{t_k} \psi_{ik}(s)\, ds \right) x_i^* + \gamma_{ik} = 0, \quad k \in \mathbb{N}, \ i = \overline{1,m}. \tag{3.2.4.4}$$

The algebraic system (3.2.4.3) is identical to (3.1.1.3). From Lemma 3.1.1.1 it follows that if conditions **A3.2.4.2**–**A3.2.4.5** are satisfied, the system without impulses (3.2.4.1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

Our next goal is to introduce a discrete-time counterpart of system (3.2.4.1), (3.2.4.2) without essentially changing its stability characteristics. The leakage terms $-a_i x_i(t-\sigma)$ in the right-hand side of (3.2.4.1) make difficult the application of the semi-discretization procedure described in [95, 60] and in §3.2.1. Instead, we will discretize all terms in the right-hand side of (3.2.4.1).

Suppose that $\sigma < \theta$. Let the positive integer N be sufficiently large, in particular, such that

$$\left(1 + \frac{1}{N}\right) a_i \sigma < 1, \quad i = \overline{1,m}, \qquad \left(1 + \frac{1}{N}\right) \sigma < \theta. \tag{3.2.4.5}$$

We choose a discretization step $h = \sigma/N$ and denote by $n = \left[\frac{t}{h}\right]$ the greatest integer in t/h, $\kappa_{ij} = \left[\frac{\tau_{ij}}{h}\right]$ and, for brevity, $x_i(n) = x_i(nh)$, $n \in \mathbb{Z}$. We further replace the integral term $\int_0^\infty K_{ij}(s) x_j(t-s)\, ds$ $(i,j = \overline{1,m})$ by a sum of the form $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p)$, where $n = [t/h]$, $p = [s/h]$, by an abuse of notation $\mathcal{K}_{ij}(p)$ stands for $\mathcal{K}_{ij}(ph)$ and $x_j(n-p)$ for $x_j((n-p)h)$, and the discrete kernels $\mathcal{K}_{ij}(\cdot)$, $i,j = \overline{1,m}$, satisfy the following conditions:

A3.2.4.7. $\mathcal{K}_{ij}(p) \in [0,\infty)$ and is bounded for $p \in \mathbb{N}$ $(i,j = \overline{1,m})$.

A3.2.4.8. $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p) = 1$ $(i,j = \overline{1,m})$.

A3.2.4.9. There exists a number $\nu > 1$ such that $\sum\limits_{p=1}^{\infty} \mathcal{K}_{ij}(p)\nu^p < \infty$ $(i,j = \overline{1,m})$.

Thus we obtain the following discretization of the right-hand side of (3.2.4.1):

$$\begin{aligned} & -a_i x_i(n-N) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(n-\kappa_{ij})) \\ + \quad & \sum_{j=1}^{m} d_{ij} h_j \left(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p) \right) + I_i, \quad n \in \mathbb{N}, \quad i = \overline{1,m}. \end{aligned}$$

The negative sign of the first term makes difficult the use of Lyapunov's functionals as in [95, 60] or §3.2.1. To overcome this difficulty, we eliminate this term by using a suitable approximation of the value of the derivative $\frac{dx_i}{dt}$ in the left-hand side of the equation (3.2.4.1) at the point nh for σ small enough by the expression

$$\frac{1 - Nha_i}{h} x_i(n+1) - \frac{1-(N+1)ha_i}{h} x_i(n) - a_i x_i(n-N).$$

Let us recall that $Nha_i = \sigma a_i < 1$ by condition **A3.2.4.1** and $(N+1)ha_i = \left(1 + \frac{1}{N}\right)\sigma a_i < 1$ by virtue of (3.2.4.5). Thus we obtain the following discrete-time analogue of the system without impulses (3.2.4.1):

$$\begin{aligned}(1 - Nha_i)x_i(n+1) &= (1-(N+1)ha_i)x_i(n) + h\left(\sum_{j=1}^{m} b_{ij} f_j(x_j(n))\right. \\ &+ \left.\sum_{j=1}^{m} c_{ij} g_j(x_j(n-\kappa_{ij})) + \sum_{j=1}^{m} d_{ij} h_j\left(\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) x_j(n-p)\right) + I_i\right), \\ & n \in \mathbb{N}, \quad i = \overline{1,m}, \end{aligned} \qquad (3.2.4.6)$$

with initial values of the form $x_i(-\ell) = \phi_i(-\ell)$ ($\ell \in \{0\} \cup \mathbb{N}$), where the sequences $\{\phi_i(-\ell)\}_{\ell=0}^{\infty}$ are bounded for all $i = \overline{1,m}$.

Next we discretize the impulse conditions (3.2.4.2). If we denote $n_k = \left[\frac{t_k}{h}\right]$, $k \in \mathbb{N}$, we obtain a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ satisfying $0 < n_1 < n_2 < \cdots < n_k \to \infty$ as $k \to \infty$ and $\Delta n_k = n_k - n_{k-1} \geq \left[\frac{\theta}{h}\right] - 1$. With each such integer n_k we associate two values of the solution $x(n)$, namely, $x(n_k)$ which can be regarded as the value of the solution before the impulse effect and whose components are evaluated by equations (3.2.4.6), and $x^+(n_k)$ which can be regarded as the value of the solution after the impulse effect and whose components are evaluated by the equations

$$x_i^+(n_k) - x_i(n_k) = \sum_{\ell=n_k-N}^{n_k} B_{ik\ell} x_i(\ell) + \gamma_{ik}, \qquad i = \overline{1,m}, \quad k \in \mathbb{N}, \quad (3.2.4.7)$$

where $B_{ik\ell}$ are suitably chosen constants.

From condition (3.2.4.5) it follows that none of the values of $x(n)$ in the right-hand side of (3.2.4.7) are evaluated at members of the sequence $\{n_k\}_{k\in\mathbb{N}}$. On the other hand, if a value of $x(n)$ in the right-hand side of

(3.2.4.6) must be evaluated at a member of the sequence $\{n_k\}_{k\in\mathbb{N}}$, we take $x^+(n_k)$ evaluated from (3.2.4.7). If we want to give a formal description of the discrete-time analogue of the impulsive system (3.2.4.1), (3.2.4.2), we should write

$$\begin{aligned} & (1 - Nha_i)x_i^-(n+1) = (1-(N+1)ha_i)x_i^+(n) \\ + \quad & h\left(\sum_{j=1}^m b_{ij}f_j(x_j^+(n)) + \sum_{j=1}^m c_{ij}g_j(x_j^+(n-\kappa_{ij}))\right. \\ + \quad & \left.\sum_{j=1}^m d_{ij}h_j\left(\sum_{p=1}^\infty \mathcal{K}_{ij}(p)x_j^+(n-p)\right) + I_i\right), \quad n \in \mathbb{N}, \\ x_i^+(n) \quad = \quad & \begin{cases} x_i^-(n) & \text{for } n \neq n_k, \\ x_i^-(n) + \sum\limits_{\ell=n-N}^{n} B_{ik\ell}x_i^-(\ell) + \gamma_{ik} & \text{for } n = n_k, \end{cases} \quad i = \overline{1,m}. \end{aligned}$$

A similar discretization of the impulse condition was given in §3.2.1.

Further on we will call system (3.2.4.6), (3.2.4.7) the discrete-time analogue of the system with impulses (3.2.4.1), (3.2.4.2).

The components of an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.2.4.6), (3.2.4.7) must satisfy the equations (3.2.4.3) and

$$\sum_{\ell=n_k-N}^{n_k} B_{ik\ell}x_i^* + \gamma_{ik} = 0. \tag{3.2.4.8}$$

Systems (3.2.4.1), (3.2.4.2) and (3.2.4.6), (3.2.4.7) have the same equilibrium points if any. To ensure this we choose the constants $B_{ik\ell}$ so that

$$\sum_{\ell=n_k-N}^{n_k} B_{ik\ell} = B_{ik} + \int_{t_k-\sigma}^{t_k} \psi_{ik}(s)\,ds, \qquad i = \overline{1,m}, \quad k \in \mathbb{N}.$$

Definition 3.2.4.1. The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.2.4.6), (3.2.4.7) is said to be globally exponentially stable with a multiplier ρ if there exist constants $M > 1$ and $\rho \in (0,1)$ and any other solution $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ of system (3.2.4.6), (3.2.4.7) is defined for all $n \in \mathbb{N}$ and satisfies the estimate

$$\sum_{i=1}^m |x_i(n) - x_i^*| \leq M\rho^n \sum_{i=1}^m \sup_{\ell\in\{0\}\cup\mathbb{N}} |x_i(-\ell) - x_i^*|. \tag{3.2.4.9}$$

Our main result in the present subsection is the following

Theorem 3.2.4.1. *Let system* (3.2.4.6), (3.2.4.7) *satisfy conditions* **A3.2.4.1**–**A3.2.4.3**, **A3.2.4.7**–**A3.2.4.9**, (3.2.4.5) *and the components of the unique equilibrium point* $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *of system* (3.2.4.6) *satisfy* (3.2.4.8). *Then there exist constants* $M' > 1$ *and* $\lambda \in (1, \nu]$ *such that any other solution* $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ *of system* (3.2.4.6), (3.2.4.7) *is defined for all* $n \in \mathbb{N}$ *and satisfies the estimate*

$$\sum_{i=1}^{m} |x_i(n) - x_i^*| \le M' \lambda^{-n} \prod_{k=1}^{i(1,n)} B_k' \sum_{i=1}^{m} \sup_{\ell \in \{0\} \cup \mathbb{N}} |x_i(-\ell) - x_i^*|, \qquad (3.2.4.10)$$

$$i(1,n) = \begin{cases} 0, & n \le n_1, \\ \max\{k \in \mathbb{N} : n_k < n\}, & n > n_1, \end{cases} \quad B_k' = B_k \left(1 + \sigma \max_{i=\overline{1,m}} (1 - \sigma a_i)^{-1}\right)$$

and $B_k = \max_{i=\overline{1,m}} \max \left\{ |1 + B_{ikn_k}|, \max_{n_k - N \le \ell \le n_k - 1} |B_{ik\ell}| \right\}, \quad k \in \mathbb{N}.$

Proof. From the conditions of the theorem it follows that system (3.2.4.6), (3.2.4.7) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$. For any $n \in \mathbb{N} \cup \{0\}$, from equations (3.2.4.6) and (3.2.4.3), by virtue of condition **A3.2.4.2** we obtain the inequalities

$$\begin{aligned} &(1 - Nha_i)|x_i(n+1) - x_i^*| \le (1 - (N+1)ha_i)|x_i(n) - x_i^*| \\ + \quad & h \sum_{j=1}^{m} \Big\{ |b_{ij}|\, F_j\, |x_j(n) - x_j^*| + |c_{ij}|\, G_j\, |x_j(n - \kappa_{ij}) - x_j^*| \\ & + |d_{ij}|\, H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) |x_j(n-p) - x_j^*| \Big\}, \quad i = \overline{1,m}. \end{aligned}$$

For $\lambda \in [1, \nu]$ let us denote $y_i(n) = \lambda^n |x_i(n) - x_i^*|$, $n \in \mathbb{Z}$. Then

$$\begin{aligned} &(1 - Nha_i) y_i(n+1) \le (1 - (N+1)ha_i)\lambda y_i(n) \\ + \quad & h \sum_{j=1}^{m} \Big\{ |b_{ij}|\, F_j\, \lambda y_j(n) + |c_{ij}|\, G_j\, \lambda^{\kappa_{ij}+1} y_j(n - \kappa_{ij}) \\ & + |d_{ij}|\, H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) \lambda^{p+1} y_j(n-p) \Big\}, \quad i = \overline{1,m}. \end{aligned}$$

From here,

$$
\begin{aligned}
&(1-\sigma a_i)\left(y_i(n+1)-y_i(n)\right) \le \left[(\lambda-1)(1-\sigma a_i)-ha_i\lambda\right] y_i(n) \\
+\quad & h\sum_{j=1}^{m}\Bigg\{\left|b_{ij}\right| F_j\,\lambda y_j(n)+\left|c_{ij}\right| G_j\,\lambda^{\kappa_{ij}+1}y_j(n-\kappa_{ij}) \qquad (3.2.4.11)\\
&+\left|d_{ij}\right| H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1}y_j(n-p)\Bigg\}, \quad i=\overline{1,m}.
\end{aligned}
$$

We define a Lyapunov functional $V(\cdot)$ by

$$
\begin{aligned}
V(n) \;=\; & \sum_{i=1}^{m}\Bigg\{(1-\sigma a_i)y_i(n)+h\sum_{j=1}^{m}\Bigg[|c_{ij}|G_j\lambda^{\kappa_{ij}+1}\sum_{\ell=n-\kappa_{ij}}^{n-1}y_j(\ell) \\
&+\;|d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1}\sum_{\ell=n-p}^{n-1}y_j(\ell)\Bigg]\Bigg\}. \qquad (3.2.4.12)
\end{aligned}
$$

It is easy to see that $V(n)\ge 0$ for $n\in\mathbb{N}\cup\{0\}$ and $V(0)<\infty$ by **A3.2.4.9**. More precisely,

$$
\begin{aligned}
V(0) \;=\; & \sum_{i=1}^{m}\Bigg\{(1-\sigma a_i)y_i(0)+h\sum_{j=1}^{m}\Bigg[|c_{ij}|G_j\lambda^{\kappa_{ij}+1}\sum_{\ell=1}^{\kappa_{ij}}y_j(-\ell) \\
&+\;|d_{ij}|H_j\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\lambda^{p+1}\sum_{\ell=1}^{p}y_j(-\ell)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\le\sum_{i=1}^{m}\Bigg\{1-\sigma a_i+h\Bigg[G_i\sum_{j=1}^{m}|c_{ji}|\lambda^{\kappa_{ji}+1}+H_i\sum_{j=1}^{m}|d_{ji}|\sum_{p=1}^{\infty}\mathcal{K}_{ji}(p)\lambda^{p+1}\Bigg]\Bigg\}\sup_{\ell\in\mathbb{N}\cup\{0\}}y_i(-\ell).
$$

If we denote

$$
M=\max_{i=\overline{1,m}}\Bigg\{1-\sigma a_i+h\Bigg[G_i\sum_{j=1}^{m}|c_{ji}|\lambda^{\kappa_{ji}+1}+H_i\sum_{j=1}^{m}|d_{ji}|\sum_{p=1}^{\infty}\mathcal{K}_{ji}(p)\lambda^{p+1}\Bigg]\Bigg\},
$$

then we have

$$
V(0)\le M\sum_{i=1}^{m}\sup_{\ell\in\mathbb{N}\cup\{0\}}y_i(-\ell)=M\sum_{i=1}^{m}\sup_{\ell\in\mathbb{N}\cup\{0\}}\lambda^{-\ell}|x_i(-\ell)-x_i^*|
$$

and, finally,

$$V(0) \le M \sum_{i=1}^{m} \sup_{\ell \in \mathbb{N}\cup\{0\}} |x_i(-\ell) - x_i^*|. \tag{3.2.4.13}$$

Further on, using (3.2.4.11) and (3.2.4.12), we obtain

$$\begin{aligned} V(n+1) - V(n) \le \sum_{i=1}^{m} \Bigg\{ & [(\lambda-1)(1-\sigma a_i) - h a_i \lambda] y_i(n) \qquad (3.2.4.14) \\ + \ h \sum_{j=1}^{m} & \left[|b_{ij}| F_j \lambda + |c_{ij}| G_j \lambda^{\kappa_{ij}+1} + |d_{ij}| H_j \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) \lambda^{p+1} \right] y_j(n) \Bigg\} \\ = \ - \sum_{i=1}^{m} \Bigg\{ h\lambda \Bigg(& a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| \lambda^{\kappa_{ji}} \\ & - H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \lambda^{p} \Bigg) - (\lambda-1)(1-\sigma a_i) \Bigg\} y_i(n). \end{aligned}$$

If we denote

$$\begin{aligned} \Psi_i(\lambda) \ = \ h\lambda \Bigg(& a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| \lambda^{\kappa_{ji}} \\ & - H_i \sum_{j=1}^{m} |d_{ji}| \sum_{p=1}^{\infty} \mathcal{K}_{ji}(p) \lambda^{p} \Bigg) - (\lambda-1)(1-\sigma a_i), \end{aligned}$$

then inequality (3.2.4.14) can be written as

$$V(n+1) - V(n) \le - \sum_{i=1}^{m} \Psi_i(\lambda) y_i(n).$$

By condition **A3.2.4.9** the functions $\Psi_i(\lambda)$ $(i = \overline{1,m})$ are well-defined and continuous for $\lambda \in [1, \nu]$. Moreover,

$$\Psi_i(1) = h \left(a_i - F_i \sum_{j=1}^{m} |b_{ji}| - G_i \sum_{j=1}^{m} |c_{ji}| - H_i \sum_{j=1}^{m} |d_{ji}| \right) > 0, \quad i = \overline{1,m},$$

by virtue of **A3.2.4.8** and **A3.2.4.3**. By continuity, for each $i = \overline{1,m}$ there exists a number $\lambda_i \in (1, \nu]$ such that $\Psi_i(\lambda) \ge 0$ for $\lambda \in (1, \lambda_i]$. If we denote

$\lambda_0 = \min_{i=\overline{1,m}} \lambda_i$, then $\lambda_0 > 1$ and $\Psi_i(\lambda) \geq 0$ for $\lambda \in (1, \lambda_0]$ and $i = \overline{1,m}$. This implies $V(n+1) \leq V(n)$ for $n \neq n_k$ and $V(n_k+1) \leq V^+(n_k)$, where $V^+(n_k)$ contains $|x_i^+(n_k) - x_i^*|$ instead of $|x_i(n_k) - x_i^*|$. The above inequalities yield

$$V(n) \leq \begin{cases} V^+(n_k) & \text{for } \; n_k < n \leq n_{k+1}, \\ V(0) & \text{for } \; 0 < n \leq n_1. \end{cases} \tag{3.2.4.15}$$

From equalities (3.2.4.7) and (3.2.4.8) we find successively

$$\begin{aligned} |x_i^+(n_k) - x_i^*| \; &\leq \; |1 + B_{ikn_k}| \, |x_i(n_k) - x_i^*| + \sum_{\ell=n_k-N}^{n_k-1} |B_{ik\ell}| \, |x_i(\ell) - x_i^*| \\ &\leq \; B_k \sum_{\ell=n_k-N}^{n_k} |x_i(\ell) - x_i^*|, \end{aligned}$$

where the constants B_k were introduced in the statement of Theorem 3.2.4.1,

$$y_i^+(n_k) \leq B_k \sum_{\ell=n_k-N}^{n_k} y_i(\ell)\lambda^{\ell-n_k}$$

and, finally,

$$\begin{aligned} V^+(n_k) \; &\leq \; B_k \left(V(n_k) + h \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1} \sum_{\ell=n_k-N}^{n_k-1} V(\ell)\lambda^{\ell-n_k} \right) \\ &\leq \; B_k V^+(n_{k-1}) \left[1 + h \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1} \sum_{\ell=1}^{N} \lambda^{-\ell} \right] \\ &\leq \; B_k V^+(n_{k-1}) \left[1 + Nh \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1} \right] \\ &= \; B_k V^+(n_{k-1}) \left[1 + \sigma \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1} \right] \equiv B_k' V^+(n_{k-1}) \end{aligned}$$

for $k \geq 2$ and, similarly, $V^+(n_1) \leq B_1' V(0)$.

Combining the last inequalities and (3.2.4.15), we derive the estimate

$$V(n) \leq \prod_{k=1}^{i(1,n)} B_k' V(0). \tag{3.2.4.16}$$

Finally, from the inequalities

$$\sum_{i=1}^{m} |x_i(n) - x_i^*| \le \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1} \lambda^{-n} V(n),$$

(3.2.4.16) and (3.2.4.13) we deduce (3.2.4.10) with $M' = M \max_{i=\overline{1,m}} (1-\sigma a_i)^{-1}$ and any $\lambda \in (1, \lambda_0]$. □

For three sets of additional assumptions we will show that inequality (3.2.4.10) implies global exponential stability of the equilibrium point x^* of the discrete-time system (3.2.4.6), (3.2.4.7).

Corollary 3.2.4.1. *Let all conditions of Theorem 3.2.4.1 hold. Suppose that* $B'_k \le 1$ *for all sufficiently large values of* $k \in \mathbb{N}$*. Then the equilibrium point* x^* *of the discrete-time system* (3.2.4.6), (3.2.4.7) *is globally exponentially stable with multiplier* $1/\lambda_0$*.*

Corollary 3.2.4.2. *Let all conditions of Theorem 3.2.4.1 hold and*

$$\limsup_{n\to\infty} \frac{i(1,n)}{n} = p < +\infty.$$

Let there exist a positive constant B *such that* $B'_k \le B$ *for all sufficiently large values of* $k \in \mathbb{N}$ *and* $B^p < \lambda_0$*. Then for any* $\rho \in \left(\frac{B^p}{\lambda_0}, 1\right)$ *the equilibrium point* x^* *of the discrete-time system* (3.2.4.6), (3.2.4.7) *is globally exponentially stable with multiplier* ρ*.*

Corollary 3.2.4.3. *Let all conditions of Theorem 3.2.4.1 hold. Suppose that there exists a constant* $\mu \in (1, \lambda_0)$ *such that*

$$B'_k \le \mu^{n_k - n_{k-1}}$$

for all sufficiently large values of $k \in \mathbb{N}$*. Then the equilibrium point* x^* *of the discrete-time system* (3.2.4.6), (3.2.4.7) *is globally exponentially stable with multiplier* μ/λ_0*.*

The results of the present subsection was reported at the International Conference on Differential & Difference Equations and Applications, Ponta Delgada, Portugal, 2011, and appeared in its proceedings [13].

3.3 Asymptotic Stability of Equilibrium Points of Cohen-Grossberg Neural Networks of Neutral Type

Sufficient conditions for the existence and global asymptotic stability of a unique equilibrium point of a continuous-time impulsive Cohen-Grossberg neural network of neutral type and its discrete-time counterpart are obtained. Examples are given.

3.3.1 Continuous-time neural networks of neutral type

We consider a Cohen-Grossberg neural network of neutral type consisting of $m \geq 2$ elementary processing units (or neurons) whose state variables x_i $(i = \overline{1,m})$ are governed by the system

$$\begin{aligned} \dot{x}_i(t) &+ \sum_{j=1}^{m} e_{ij}\dot{x}_j(t-\tau_j) = a_i(x_i(t))\Bigg[-b_i(x_i(t)) \\ &+ \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} d_{ij} g_j(x_j(t-\tau_j)) + I_i\Bigg], \quad (3.3.1.1) \\ & i = \overline{1,m}, \quad t > 0, \end{aligned}$$

with initial values prescribed by continuous functions $x_i(s) = \phi_i(s)$ for $s \in [-\tau, 0]$, $\tau = \max\limits_{j=\overline{1,m}}\{\tau_j\}$. In (3.3.1.1), $a_i(x_i)$ denotes an amplification function; $b_i(x_i)$ denotes an appropriate function which supports the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i; $f_j(x_j)$, $g_j(x_j)$ denote activation functions; the parameters c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the $j-$th unit and the $i-$th unit, respectively without and with time delays τ_j; the real numbers e_{ij} show how the state velocities of the neurons are delay feed-forward connected in the network; the real constant I_i represents an input signal introduced from outside the network to the $i-$th unit.

Let E be the unit $(m \times m)$-matrix. Denote by $\mathcal{E}$ and $|\mathcal{E}|$ the $(m \times m)$-matrices with entries e_{ij} and $|e_{ij}|$, respectively.

The assumptions that accompany system (3.3.1.1) are given as follows:

A3.3.1.1. The amplification functions $a_i : \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded in the sense that

$$0 < \underline{a}_i \le a_i(x) \le \overline{a}_i \quad \text{for } \ x \in \mathbb{R},\ i = \overline{1,m},$$

for some constants $\underline{a}_i,\ \overline{a}_i$.

A3.3.1.2. The stabilizing functions $b_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and monotone increasing, namely,

$$0 < \underline{b}_i \le \frac{b_i(x) - b_i(y)}{x - y} \le \overline{b}_i \quad \text{for } \ x \ne y,\ x, y \in \mathbb{R},\ i = \overline{1,m},$$

for some constants $\underline{b}_i,\ \overline{b}_i$.

A3.3.1.3. The activation functions $f_j, g_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in the sense of

$$F_j = \sup_{x \ne y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|,\ G_j = \sup_{x \ne y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right|$$

for $x, y \in \mathbb{R},\ j = \overline{1,m}$, where F_j, G_j denote positive constants.

A3.3.1.4. $\|\mathcal{E}\| < 1$, where $\|\cdot\|$ is the spectral matrix norm.

A3.3.1.5. $E - |\mathcal{E}|$ is an M-matrix.

The "stability condition" $\|\mathcal{E}\| < 1$ guarantees the existence and uniqueness of the solution of the Cauchy problem. Since $E - |\mathcal{E}|$ is an M-matrix, it is nonsingular and its inverse has nonnegative entries only.

Under these assumptions and the given initial conditions, there is a unique solution of system (3.3.1.1). The solution is a vector $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ in which $x_i(t)$ are continuously differentiable for $t \in (0, \beta)$, where β is some positive number, possibly ∞. An equilibrium point of system (3.3.1.1) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ where the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^{m} c_{ij} f_j(x_j^*) + \sum_{j=1}^{m} d_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1,m}. \tag{3.3.1.2}$$

Definition 3.3.1.1. The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of system (3.3.1.1) is said to be *globally asymptotically stable* if any other solution $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ of system (3.3.1.1) is defined for all $t > 0$ and satisfies

$$\lim_{t\to\infty} x(t) = x^*.$$

Our first task is to prove the existence and uniqueness of the solution x^* of the algebraic system (3.3.1.2).

Theorem 3.3.1.1. *Let the assumptions* **A3.3.1.2**, **A3.3.1.3** *hold. Suppose, further, that the following inequalities are valid:*

$$\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m}(|c_{ij}|F_j + |c_{ji}|F_i) - \frac{1}{2}\sum_{j=1}^{m}(|d_{ij}|G_j + |d_{ji}|G_i) > 0, \quad i = \overline{1,m}. \tag{3.3.1.3}$$

Then system (3.3.1.1) *has a unique equilibrium point* $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

Proof. Let us define a mapping $\Phi:\ \mathbb{R}^m \to \mathbb{R}^m$ by $\Phi(x) = (\Phi_1(x), \Phi_2(x), \dots, \Phi_m(x))^T$ for $x \in \mathbb{R}^m$, where

$$\Phi_i(x) = b_i(x_i) - \sum_{j=1}^{m} c_{ij} f_j(x_j) - \sum_{j=1}^{m} d_{ij} g_j(x_j) - I_i, \quad i = \overline{1,m}.$$

The space $\mathbb{R}^m$ is endowed with the Euclidean norm $\|x\| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$. We denote by $\langle\cdot,\cdot\rangle$ the respective inner product. Under the assumptions **A3.3.1.2**, **A3.3.1.3**, $\Phi(x) \in C^0$. It is known that if $\Phi(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$, then there is a unique point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T \in \mathbb{R}^m$ such that $\Phi(x^*) = 0$, that is, $\Phi_i(x^*) = 0,\ i = \overline{1,m}$. The last equalities are, in fact, (3.3.1.2), so $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ is the equilibrium point we are looking for.

To demonstrate the one-to-one property of $\Phi(x)$, we take arbitrary vectors $x, y \in \mathbb{R}^m$ and assume that $\Phi(x) = \Phi(y)$. We multiply the equalities

$$b_i(x_i) - b_i(y_i) = \sum_{j=1}^{m} c_{ij}\left(f_j(x_j) - f_j(y_j)\right) + \sum_{j=1}^{m} d_{ij}\left(g_j(x_j) - g_j(y_j)\right), \quad i = \overline{1,m},$$

respectively by $x_i - y_i$ and add them together to obtain

$$\begin{aligned}\sum_{i=1}^{m} \left(b_i(x_i) - b_i(y_i)\right)(x_i - y_i) &= \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left(f_j(x_j) - f_j(y_j)\right)(x_i - y_i) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij}\left(g_j(x_j) - g_j(y_j)\right)(x_i - y_i).\end{aligned}$$

According to the assumptions **A3.3.1.2**, **A3.3.1.3** we derive

$$\begin{aligned}&\sum_{i=1}^{m} \underline{b}_i (x_i - y_i)^2 \\ \leq\ & \sum_{i=1}^{m}\sum_{j=1}^{m} |c_{ij}|\, F_j\, |x_j - y_j|\, |x_i - y_i| + \sum_{i=1}^{m}\sum_{j=1}^{m} |d_{ij}|\, G_j\, |x_j - y_j|\, |x_i - y_i| \\ \leq\ & \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} |c_{ij}|\, F_j\, \left[(x_j - y_j)^2 + (x_i - y_i)^2\right] \\ +\ & \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} |d_{ij}|\, G_j\, \left[(x_j - y_j)^2 + (x_i - y_i)^2\right] \\ =\ & \sum_{i=1}^{m}\left\{\frac{1}{2}\sum_{j=1}^{m}\left(|c_{ij}|F_j + |c_{ji}|F_i\right) + \frac{1}{2}\sum_{j=1}^{m}\left(|d_{ij}|G_j + |d_{ji}|G_i\right)\right\}(x_i - y_i)^2,\end{aligned}$$

that is,

$$\sum_{i=1}^{m}\left\{\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m}\left(|c_{ij}|F_j + |c_{ji}|F_i\right) - \frac{1}{2}\sum_{j=1}^{m}\left(|d_{ij}|G_j + |d_{ji}|G_i\right)\right\}(x_i - y_i)^2 \leq 0.$$

Now the assertion $x_i = y_i$, $i = \overline{1,m}$, follows by virtue of inequalities (3.3.1.3). Thus, $\Phi(x) = \Phi(y)$ implies $x = y$.

Next we show that $\|\Phi(x)\| \to \infty$ as $\|x\| \to \infty$. It suffices to show that $\|\tilde{\Phi}(x)\| \to \infty$, where $\tilde{\Phi}(x) = \Phi(x) - \Phi(0)$. We have $\tilde{\Phi}(x) = (\tilde{\Phi}_1(x), \tilde{\Phi}_2(x), \ldots,$ $\tilde{\Phi}_m(x))^T$, where

$$\tilde{\Phi}_i(x) = (b_i(x_i) - b_i(0)) - \sum_{j=1}^{m} c_{ij}(f_j(x_j) - f_j(0)) - \sum_{j=1}^{m} d_{ij}(g_j(x_j) - g_j(0)).$$

Then

$$\langle\tilde{\Phi}(x), x\rangle = \sum_{i=1}^{m} \tilde{\Phi}_i(x)x_i = \sum_{i=1}^{m} \Bigg\{ (b_i(x_i) - b_i(0))x_i$$

$$- \quad \sum_{j=1}^{m} c_{ij}(f_j(x_j) - f_j(0))x_i - \sum_{j=1}^{m} d_{ij}(g_j(x_j) - g_j(0))x_i \Bigg\},$$

from which by virtue of the assumptions **A3.3.1.2**, **A3.3.1.3** we derive

$$\left|\langle\tilde{\Phi}(x), x\rangle\right| \geq \sum_{i=1}^{m} \left\{ \underline{b}_i x_i^2 - \sum_{j=1}^{m} |c_{ij}|\, F_j\, |x_j|\, |x_i| - \sum_{j=1}^{m} |d_{ij}|\, G_j\, |x_j|\, |x_i| \right\}$$

$$\geq \sum_{i=1}^{m} \left\{ \underline{b}_i x_i^2 - \frac{1}{2} \sum_{j=1}^{m} |c_{ij}|\, F_j\, (x_j^2 + x_i^2) - \frac{1}{2} \sum_{j=1}^{m} |d_{ij}|\, G_j\, (x_j^2 + x_i^2) \right\}$$

$$= \sum_{i=1}^{m} \left\{ \underline{b}_i - \frac{1}{2} \sum_{j=1}^{m} (|c_{ij}|F_j + |c_{ji}|F_i) - \frac{1}{2} \sum_{j=1}^{m} (|d_{ij}|G_j + |d_{ji}|G_i) \right\} x_i^2.$$

According to inequalities (3.3.1.3) there exists a number $\mu > 0$ such that

$$\underline{b}_i - \frac{1}{2} \sum_{j=1}^{m} (|c_{ij}|F_j + |c_{ji}|F_i) - \frac{1}{2} \sum_{j=1}^{m} (|d_{ij}|G_j + |d_{ji}|G_i) \geq \mu, \qquad i = \overline{1, m}.$$

Then $\|\tilde{\Phi}(x)\| \cdot \|x\| \geq |\langle\tilde{\Phi}(x), x\rangle| \geq \mu\|x\|^2$ and $\|\tilde{\Phi}(x)\| \geq \mu\|x\| \to \infty$ as $\|x\| \to \infty$.

According to Lemma 3.1.2.1, $\Phi(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$. Thus, there is a unique point $x^* \in \mathbb{R}^m$ such that $\Phi(x^*) = 0$. The point represents a unique solution of the algebraic system (3.3.1.2). □

Theorem 3.3.1.2. *Let the assumptions* **A3.3.1.1**–**A3.3.1.5** *hold. Suppose, further, that the inequalities*

$$\underline{a}_i \underline{b}_i - \frac{1}{2} \sum_{j=1}^{m} (\overline{a}_i |c_{ij}| F_j + \overline{a}_j |c_{ji}| F_i) - \frac{1}{2} \sum_{j=1}^{m} (\overline{a}_i |d_{ij}| G_j + \overline{a}_j |d_{ji}| G_i)$$

$$- \quad \frac{1}{2} \sum_{j=1}^{m} \left(\overline{a}_i \overline{b}_i |e_{ij}| + \overline{a}_j \overline{b}_j |e_{ji}| \right) - \frac{1}{2} \sum_{j=1}^{m} \overline{a}_j \sum_{k=1}^{m} (|c_{ji}|\, |e_{jk}|\, F_i + |c_{jk}|\, |e_{ji}|\, F_k)$$

$$- \quad \frac{1}{2} \sum_{j=1}^{m} \overline{a}_j \sum_{k=1}^{m} (|d_{ji}|\, |e_{jk}|\, G_i + |d_{jk}|\, |e_{ji}|\, G_k) > 0, \quad i = \overline{1, m}, \qquad (3.3.1.4)$$

are valid and system (3.3.1.1) *has an equilibrium point* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ *whose components satisfy* (3.3.1.2). *Then the equilibrium point* x^* *is globally asymptotically stable.*

Remark 3.3.1.1. Inequalities (3.3.1.3) can be deduced from (3.3.1.4) for $\underline{a}_i = \overline{a}_i = 1$, $e_{ij} = 0$ for $i, j = \overline{1, m}$. However, in general inequalities (3.3.1.4) do not imply (3.3.1.3).

Remark 3.3.1.2. In [42] it is assumed that $g_j = f_j$, the functions $b_i(x_i)$ and $b_i^{-1}(x_i)$ are assumed to be continuosly differentiable and $b_i'(x_i)$ are bounded both below and above by positive constants. Instead of the m inequalities (3.3.1.4) a single inequality is presented, which in our notation can be written as

$$\min_{i=\overline{1,m}}(\underline{a}_i\underline{b}_i) - \max_{i=\overline{1,m}}(\overline{a}_i\overline{b}_i)\|\mathcal{E}\| - \max_{i=\overline{1,m}}\overline{a}_i\left(\max_{i=\overline{1,m}}F_i\|C\| + \max_{i=\overline{1,m}}G_i\|D\|\right)(1+\|\mathcal{E}\|) > 0, \tag{3.3.1.5}$$

where C and D are $(m \times m)$-matrices with entries c_{ij} and d_{ij}, respectively.

Though condition (3.3.1.5) seems much simpler than (3.3.1.4), in our opinion it is much less precise since the individual lower and upper bounds, Lipschitz constants, and matrix entries are replaced by their minima or maxima, and matrix norms. Below we shall give an example of a system satisfying conditions (3.3.1.4) but not (3.3.1.5).

Proof. Upon introducing the translations

$$u_i(t) = x_i(t) - x_i^*, \qquad \varphi_i(s) = \phi_i(s) - x_i^*$$

we derive the system

$$\begin{aligned} \dot{u}_i(t) \;&+\; \sum_{j=1}^m e_{ij}\dot{u}_j(t-\tau_j) = \tilde{a}_i(u_i(t))\Bigg[-\tilde{b}_i(u_i(t)) \\ &+\; \sum_{j=1}^m c_{ij}\tilde{f}_j(u_j(t)) + \sum_{j=1}^m d_{ij}\tilde{g}_j(u_j(t-\tau_j))\Bigg], \quad t > 0, \\ & u_i(s) = \varphi_i(s), \qquad s \in [-\tau, 0], \quad i = \overline{1, m}, \end{aligned} \tag{3.3.1.6}$$

where

$$\begin{aligned} \tilde{a}_i(u_i) = a_i(u_i + x_i^*), &\qquad \tilde{b}_i(u_i) = b_i(u_i + x_i^*) - b_i(x_i^*), \\ \tilde{f}_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), &\qquad \tilde{g}_j(u_j) = g_j(u_j + x_j^*) - g_j(x_j^*). \end{aligned}$$

This system inherits the assumptions **A3.3.1.1**–**A3.3.1.5** given before. It suffices to examine the stability characteristics of the trivial equilibrium point $u^* = 0$ of system (3.3.1.6).

We define a Lyapunov functional $V(t)$ by

$$V(t) = \frac{1}{2}\sum_{i=1}^{m}\left\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]^2 + \omega_i\int_{t-\tau_i}^{t} u_i^2(s)\,ds\right\},$$

where the positive constants ω_i, $i = \overline{1,m}$, will be determined later. First we notice that the value

$$V(0) = \frac{1}{2}\sum_{i=1}^{m}\left\{\left[\varphi_i(0) + \sum_{j=1}^{m} e_{ij}\varphi_j(-\tau_j)\right]^2 + \omega_i\int_{-\tau_i}^{0} \varphi_i^2(s)\,ds\right\}$$

is completely determined from the initial values of the system. Then, calculating the rate of change of $V(t)$ along the solutions of (3.3.1.6), we successively find

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{m}\Bigg\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]\left[\dot{u}_i(t) + \sum_{j=1}^{m} e_{ij}\dot{u}_j(t-\tau_j)\right] \\
&\qquad + \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\Bigg\} \\
&= \sum_{i=1}^{m}\Bigg\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]\tilde{a}_i(u_i(t))\Bigg[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) \\
&\qquad + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t-\tau_j))\Bigg] + \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\Bigg\} \\
&= \sum_{i=1}^{m}\Bigg\{ - \tilde{a}_i(u_i(t))\tilde{b}_i(u_i(t))u_i(t) \\
&\qquad + \tilde{a}_i(u_i(t))u_i(t)\left[\sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t-\tau_j))\right]
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \tilde{a}_i(u_i(t)) \sum_{j=1}^{m} e_{ij} u_j(t-\tau_j) \Bigg[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij} \tilde{f}_j(u_j(t)) \\
& \qquad + \sum_{j=1}^{m} d_{ij} \tilde{g}_j(u_j(t-\tau_j)) \Bigg] + \frac{\omega_i}{2} \left(u_i^2(t) - u_i^2(t-\tau_i) \right) \Bigg\} \\
\leq \quad & \sum_{i=1}^{m} \Bigg\{ -\underline{a}_i \underline{b}_i u_i^2(t) + \overline{a}_i |u_i(t)| \left[\sum_{j=1}^{m} |c_{ij}| \, F_j \, |u_j(t)| + \sum_{j=1}^{m} |d_{ij}| \, G_j \, |u_j(t-\tau_j)| \right] \\
& \quad + \overline{a}_i \sum_{j=1}^{m} |e_{ij}| \, |u_j(t-\tau_j)| \Bigg[\overline{b}_i \, |u_i(t)| + \sum_{j=1}^{m} |c_{ij}| \, F_j \, |u_j(t)| \\
& \qquad + \sum_{j=1}^{m} |d_{ij}| \, G_j \, |u_j(t-\tau_j)| \Bigg] + \frac{\omega_i}{2} \left(u_i^2(t) - u_i^2(t-\tau_i) \right) \Bigg\} \\
\leq \quad & \sum_{i=1}^{m} \Bigg\{ -\underline{a}_i \underline{b}_i u_i^2(t) + \frac{\overline{a}_i}{2} \sum_{j=1}^{m} |c_{ij}| \, F_j \left(u_i^2(t) + u_j^2(t) \right) \\
& \qquad\qquad\qquad + \frac{\overline{a}_i}{2} \sum_{j=1}^{m} |d_{ij}| \, G_j \left(u_i^2(t) + u_j^2(t-\tau_j) \right) \\
+ \quad & \frac{\overline{a}_i \overline{b}_i}{2} \sum_{j=1}^{m} |e_{ij}| \left(u_i^2(t) + u_j^2(t-\tau_j) \right) + \frac{\overline{a}_i}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} |e_{ij}| \, |c_{ik}| \, F_k \left(u_k^2(t) + u_j^2(t-\tau_j) \right) \\
+ \quad & \frac{\overline{a}_i}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} |e_{ij}| \, |d_{ik}| \, G_k \left(u_j^2(t-\tau_j) + u_k^2(t-\tau_k) \right) + \frac{\omega_i}{2} \left(u_i^2(t) - u_i^2(t-\tau_i) \right) \Bigg\} \\
= \quad & \sum_{i=1}^{m} \Bigg\{ - \Bigg[\underline{a}_i \underline{b}_i - \frac{1}{2} \left(\overline{a}_i \sum_{j=1}^{m} |c_{ij}| F_j + F_i \sum_{j=1}^{m} |c_{ji}| \overline{a}_j \right) - \frac{a_i}{2} \sum_{j=1}^{m} |d_{ij}| G_j \\
& - \frac{\overline{a}_i \overline{b}_i}{2} \sum_{j=1}^{m} |e_{ij}| - \frac{F_i}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} |c_{ki}| \, |e_{kj}| \overline{a}_k - \frac{\omega_i}{2} \Bigg] u_i^2(t) \\
& + \frac{1}{2} \Bigg[G_i \sum_{j=1}^{m} |d_{ji}| \overline{a}_j + \sum_{j=1}^{m} |e_{ji}| \overline{a}_j \overline{b}_j + \sum_{j=1}^{m} \sum_{k=1}^{m} |e_{ji}| \, |c_{jk}| \overline{a}_j F_k \\
& + \sum_{j=1}^{m} \sum_{k=1}^{m} \left(|e_{ji}| \, |d_{jk}| \overline{a}_j G_k + |e_{kj}| \, |d_{ki}| \overline{a}_k G_i \right) - \omega_i \Bigg] u_i^2(t-\tau_i) \Bigg\} .
\end{aligned}
$$

Choose

$$\begin{aligned}\omega_i &= G_i\sum_{j=1}^m |d_{ji}|\overline{a}_j + \sum_{j=1}^m |e_{ji}|\overline{a}_j\overline{b}_j + \sum_{j=1}^m\sum_{k=1}^m |e_{ji}|\,|c_{jk}|\overline{a}_j F_k \\ &+ \sum_{j=1}^m\sum_{k=1}^m (|e_{ji}|\,|d_{jk}|\overline{a}_j G_k + |e_{kj}|\,|d_{ki}|\overline{a}_k G_i) > 0,\end{aligned}$$

then after some simplifications we obtain

$$\begin{aligned}\dot{V}(t) \le& -\sum_{i=1}^m \Bigg\{\underline{a}_i\underline{b}_i - \frac{1}{2}\sum_{j=1}^m (\overline{a}_i|c_{ij}|F_j + \overline{a}_j|c_{ji}|F_i) \\ &-\frac{1}{2}\sum_{j=1}^m (\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i) - \frac{1}{2}\sum_{j=1}^m \left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) \\ &-\frac{1}{2}\sum_{j=1}^m \overline{a}_j\sum_{k=1}^m (|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k) \\ &-\frac{1}{2}\sum_{j=1}^m \overline{a}_j\sum_{k=1}^m (|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k)\Bigg\} u_i^2(t).\end{aligned}$$

According to inequalities (3.3.1.4) there exists $\tilde{\mu} > 0$ such that

$$\begin{aligned}\tilde{\mu} =& \min_{i=\overline{1,m}}\Bigg\{\underline{a}_i\underline{b}_i - \frac{1}{2}\sum_{j=1}^m (\overline{a}_i|c_{ij}|F_j + \overline{a}_j|c_{ji}|F_i) - \frac{1}{2}\sum_{j=1}^m (\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i) \\ &- \frac{1}{2}\sum_{j=1}^m \left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) - \frac{1}{2}\sum_{j=1}^m \overline{a}_j\sum_{k=1}^m (|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k) \\ &- \frac{1}{2}\sum_{j=1}^m \overline{a}_j\sum_{k=1}^m (|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k)\Bigg\},\end{aligned}$$

then

$$\dot{V}(t) \le -\tilde{\mu}\|u(t)\|^2, \quad t > 0. \tag{3.3.1.7}$$

Inequality (3.3.1.7) shows that for any solution $u(t)$ of system (3.3.1.6) the function $V(t)$ is monotone decreasing and it is bounded below by 0. Thus there exists the limit $L = \lim_{t\to\infty} V(t) \ge 0$.

Let us integrate inequality (3.3.1.7) from 0 to t:

$$V(t) - V(0) \le -\tilde{\mu} \int_0^t \|u(s)\|^2 \, ds$$

for all $t > 0$, that is,

$$\int_0^t \|u(s)\|^2 \, ds \le (V(0) - V(t))/\tilde{\mu}.$$

The last inequality and $L = \lim_{t\to\infty} V(t) \ge 0$ show that

$$\int_0^\infty \|u(t)\|^2 \, dt < \infty. \tag{3.3.1.8}$$

Below we show that the zero solution of system (3.3.1.6) is stable and (3.3.1.8) implies $\lim_{t\to\infty} \|u(t)\| = 0$, that is, $\lim_{t\to\infty} \|x(t) - x^*\| = 0$. This means that the equilibrium point x^* of system (3.3.1.1) is globally asymptotically stable.

We complete the proof by arguments using fragments from the proofs of Theorems 1.1, 1.3 and 1.4 in [81, Chapter 8]. In the sequel for a vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$ we shall also use the norm

$$|v| = \max_{i=\overline{1,m}} |v_i|.$$

First we shall prove that for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if

$$|u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij})| \le \delta_1 \text{ for } t \ge 0, \quad i = \overline{1,m}, \text{ and } \sup_{s\in[-\tau,0]} |\varphi(s)| \le \delta_1,$$

then $|u(t)| \le \varepsilon$ for $t \ge 0$.

Let T be an arbitrary positive number. For $0 \le t \le T$ we have

$$\begin{aligned} |u_i(t)| &\le |u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij})| + |\sum_{i=1}^m e_{ij} u_j(t - \tau_{ij})| \\ &\le \delta_1 + \sum_{i=1}^m |e_{ij}| \, |u_j(t - \tau_{ij})| \le \delta_1 + \sum_{i=1}^m |e_{ij}| \sup_{-\tau\le t\le T} |u_j(t)| \\ &\le \delta_1 + \sum_{i=1}^m |e_{ij}| \left(\sup_{0\le t\le T} |u_j(t)| + \sup_{-\tau\le s\le 0} |\varphi_j(s)| \right) \\ &\le \sum_{i=1}^m |e_{ij}| \sup_{0\le t\le T} |u_j(t)| + \delta_1 \left(1 + \sum_{i=1}^m |e_{ij}| \right), \end{aligned}$$

thus

$$\sup_{0\le t\le T}|u_i(t)| \le \sum_{i=1}^m |e_{ij}| \sup_{0\le t\le T}|u_j(t)| + \delta_1\left(1+\sum_{i=1}^m |e_{ij}|\right)$$

or

$$\sup_{0\le t\le T}|u_i(t)| - \sum_{i=1}^m |e_{ij}| \sup_{0\le t\le T}|u_j(t)| \le \delta_1\left(1+\sum_{i=1}^m |e_{ij}|\right) \quad \text{for} \quad i=\overline{1,m}.$$

If we introduce the vectors

$$U(T)=\left(\sup_{0\le t\le T}|u_1(t)|,\dots,\sup_{0\le t\le T}|u_m(t)|\right)^T \quad \text{and} \quad \mathbf{e}=(1,1,\dots,1)^T,$$

we can write the last inequalities in a matrix form as

$$(E-|\mathcal{E}|)U(T)\le \delta_1(E+|\mathcal{E}|)\mathbf{e},$$

meaning inequalities between the respective components of the vectors. Since by condition **A3.3.1.5** $E-|\mathcal{E}|$ is an M-matrix, we obtain

$$U(T)\le \delta_1(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}.$$

We have

$$\begin{aligned}\sup_{0\le t\le T}|u(t)| &= \sup_{0\le t\le T}\max_{i=\overline{1,m}}|u_i(t)| = \max_{i=\overline{1,m}}\sup_{0\le t\le T}|u_i(t)| \\ &= |U(T)| \le \delta_1|(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}|.\end{aligned}$$

If we choose $\delta_1>0$ so small that $\delta_1|(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}|<\varepsilon$, then $|u(t)|\le\varepsilon$ for $0\le t\le T$, where T was an arbitrary positive number. Thus, $|u(t)|\le\varepsilon$ for $t\ge 0$.

Next we shall show that the zero solution of system (3.3.1.6) is stable, that is, for any $\varepsilon>0$ there exists $\delta>0$ such that if $\sup_{s\in[-\tau,0]}|\varphi(s)|\le\delta$, then $|u(t)|\le\varepsilon$ for $t\ge 0$. For any $t\ge 0$ we have

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^m\left[u_i(t)+\sum_{j=1}^m e_{ij}u_j(t-\tau_j)\right]^2 \le V(t)\le V(0) \\ =\ &\frac{1}{2}\sum_{i=1}^m\left\{\left[\varphi_i(0)+\sum_{j=1}^m e_{ij}\varphi_j(-\tau_j)\right]^2+\omega_i\int_{-\tau_i}^0\varphi_i^2(s)\,ds\right\} \\ \le\ &\frac{\delta^2}{2}\sum_{i=1}^m\left\{\left[1+\sum_{j=1}^m|e_{ij}|\right]^2+\omega_i\tau_i\right\}.\end{aligned}$$

If we choose $\delta \in (0, \delta_1)$ so small that

$$\delta^2 \sum_{i=1}^{m} \left\{ \left[1 + \sum_{j=1}^{m} |e_{ij}| \right]^2 + \omega_i \tau_i \right\} \leq \delta_1^2,$$

then

$$\sum_{i=1}^{m} \left[u_i(t) + \sum_{j=1}^{m} e_{ij} u_j(t - \tau_j) \right]^2 \leq \delta_1^2,$$

which implies that

$$|u_i(t) + \sum_{i=1}^{m} e_{ij} u_j(t - \tau_{ij})| \leq \delta_1 \quad \text{for} \quad t \geq 0, \quad i = \overline{1, m},$$

and, consequently, $|u(t)| \leq \varepsilon$ for $t \geq 0$.

Because of the stability of the zero solution of system (3.3.1.6) we can assume that $|u(t)| \leq h$ for some positive constant h when $\sup_{s \in [-\tau, 0]} |\varphi(s)| \leq \delta$. Suppose that $\lim_{t \to \infty} u(t) = 0$ is not true. In this case there exists a number $\nu > 0$ and an increasing sequence $\{t_k\}$ such that $t_k \to \infty$ and $|x(t_k)| \geq \nu$ for $k \in \mathbb{N}$. For the sake of brevity we write system (3.3.1.6) in the form

$$\frac{d}{dt} \left(u_i(t) + \sum_{j=1}^{m} e_{ij} u_j(t - \tau_j) \right) = \mathcal{F}_i(u(t), u(t - \tau)), \quad i = \overline{1, m}, \quad t > 0, \tag{3.3.1.9}$$

where

$$\mathcal{F}_i(u, \bar{u}) := \tilde{a}_i(u_i) \left[-\tilde{b}_i(u_i) + \sum_{j=1}^{m} c_{ij} \tilde{f}_j(u_j) + \sum_{j=1}^{m} d_{ij} \tilde{g}_j(\bar{u}_j) \right], \qquad i = \overline{1, m}.$$

We denote

$$C_i = \sup_{|u|, |\bar{u}| \leq h} |\mathcal{F}_i|, \qquad i = \overline{1, m}.$$

For $t \geq 0$ and $\Delta > 0$ integrate equation (3.3.1.9) from t to $t + \Delta$ to obtain

$$u_i(t+\Delta) - u_i(t) = -\sum_{j=1}^{m} e_{ij}(u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)) + \int_t^{t+\Delta} \mathcal{F}_i(u(s), u(s-\tau))\, ds,$$

hence

$$
\begin{aligned}
& |u_i(t+\Delta) - u_i(t)| \\
\leq & \sum_{j=1}^{m} |e_{ij}||u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)| + \int_t^{t+\Delta} |\mathcal{F}_i(u(s), u(s-\tau))|\, ds \\
\leq & \sum_{j=1}^{m} |e_{ij}| \sup_{t \geq -\tau} |u_j(t+\Delta) - u_j(t)| + C_i\Delta \\
\leq & \sum_{j=1}^{m} |e_{ij}| \left(\sup_{t\geq 0} |u_j(t+\Delta) - u_j(t)| + \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)| \right) + C_i\Delta,
\end{aligned}
$$

thus

$$
\begin{aligned}
& \sup_{t\geq 0} |u_i(t+\Delta) - u_i(t)| \\
\leq & \sum_{j=1}^{m} |e_{ij}| \left(\sup_{t\geq 0} |u_j(t+\Delta) - u_j(t)| + \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)| \right) + C_i\Delta
\end{aligned}
$$

or

$$
\begin{aligned}
& \sup_{t\geq 0} |u_i(t+\Delta) - u_i(t)| - \sum_{j=1}^{m} |e_{ij}| \sup_{t\geq 0} |u_j(t+\Delta) - u_j(t)| \\
\leq & \sum_{j=1}^{m} |e_{ij}| \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)| + C_i\Delta, \quad i = \overline{1,m}.
\end{aligned}
$$

If we introduce the vectors

$$
\begin{aligned}
\rho(\Delta) &= \left(\sup_{t\geq 0} |u_1(t+\Delta) - u_1(t)|, \ldots, \sup_{t\geq 0} |u_m(t+\Delta) - u_m(t)| \right)^T, \\
\sigma(\Delta) &= \left(\sup_{s\in[-\tau,0]} |u_1(s+\Delta) - u_1(s)|, , \ldots, \sup_{s\in[-\tau,0]} |u_m(s+\Delta) - u_m(s)| \right)^T
\end{aligned}
$$

and $\mathbf{C} = (C_1, C_2, \ldots, C_m)^T$, we can write the last inequalities in a matrix form as

$$(E - |\mathcal{E}|)\rho(\Delta) \leq |\mathcal{E}|\sigma(\Delta) + \Delta\mathbf{C}.$$

From here as above we obtain

$$\rho(\Delta) \leq (E - |\mathcal{E}|)^{-1}(|\mathcal{E}|\sigma(\Delta) + \Delta\mathbf{C})$$

and

$$\sup_{t\geq 0}|u(t+\Delta)-u(t)| \leq |(E-|\mathcal{E}|)^{-1}(|\mathcal{E}|\sigma(\Delta)+\Delta\mathbf{C})|. \qquad (3.3.1.10)$$

Let $\eta > 0$ and $\Delta \leq \eta$. Since $u(t)$ is uniformly continuous on the interval $[-\tau, \eta]$, the right-hand side of (3.3.1.10) can be made arbitrarily small for sufficiently small values of Δ. Thus we can choose $\eta > 0$ so that $|u(t+\Delta) - u(t)| \leq \nu/2$ for all $t \geq 0$ and $\Delta \in [0, \eta]$. In particular,

$$|u(t_k)+\Delta| \geq |u(t_k)| - |u(t_k+\Delta)-u(t_k)| \geq \nu - \frac{\nu}{2} = \frac{\nu}{2}$$

or

$$|u(t)| \geq \frac{\nu}{2} \quad \text{and} \quad \|u(t)\|^2 \geq \frac{\nu^2}{4} \quad \text{for} \quad t \in [t_k, t_k+\eta], \quad k \in \mathbb{N}.$$

Without loss of generality we can assume that the intervals $[t_k, t_k+\eta]$ are disjoint (otherwise we choose a subsequence). Then

$$\int_0^\infty \|u(t)\|^2\,dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k+\eta} \|u(t)\|^2\,dt \geq \sum_{k=1}^\infty \eta\frac{\nu^2}{4} = \infty,$$

which contradicts (3.3.1.8). Thus $\lim\limits_{t\to\infty} u(t) = 0$ is true and the proof is complete. □

Example. Consider the system

$$\begin{aligned}
& \dot{x}_1(t) + 0.1\dot{x}_1(t-\tau_1) + 0.15\dot{x}_2(t-\tau_2) \qquad (3.3.1.11)\\
= \; & (2+0.01\sin x_1(t))\left[-2x_1(t) + 0.1\arctan x_1(t) + 0.15\arctan x_2(t)\right.\\
& \left. + 0.1\arctan x_1(t-\tau_1) + 0.15\arctan x_2(t-\tau_2) + 1\right],\\
& \dot{x}_2(t) - 0.2\dot{x}_1(t-\tau_1) + 0.1\dot{x}_2(t-\tau_2)\\
= \; & (3-0.02\sin x_2(t))\left[-3x_2(t) + 0.15\arctan x_1(t) - 0.2\arctan x_2(t)\right.\\
& \left. + 0.1\arctan x_1(t-\tau_1) - 0.2\arctan x_2(t-\tau_2) + 1\right], \quad t > 0,
\end{aligned}$$

with arbitrary delays τ_1, τ_2 and initial conditions $x_i(s) = \phi(s)$, $i = 1, 2$, $s \in [-\max\{\tau_1, \tau_2\}, 0]$.

System (3.3.1.11) has the form (3.3.1.1). It satisfies assumptions **A3.3.1.1**–**A3.3.1.5** with $\underline{a}_1 = 1.99$, $\overline{a}_1 = 2.01$, $\underline{a}_2 = 2.98$, $\overline{a}_2 = 3.02$, $\underline{b}_1 = \overline{b}_1 = 2$, $\underline{b}_2 = \overline{b}_2 = 3$, $F_1 = F_2 = G_1 = G_2 = 1$, $\|\mathcal{E}\| = 0.2863903109$ and

$$E - |\mathcal{E}| = \begin{bmatrix} 0.9 & -0.15 \\ -0.2 & 0.8 \end{bmatrix}$$

is an M-matrix.

It is easy to see that system (3.3.1.11) satisfies inequalities (3.3.1.3). In fact, the left-hand sides of these inequalities are equal respectively to 1.525 and 2.325 for $i = 1$ and 2. Thus system (3.3.1.11) has a unique equilibrium point x^*. We can find that $x^* = (0.6027869379,\ 0.3353919007)^T$.

Further on, system (3.3.1.11) satisfies the assumptions of Theorem 3.3.1.2. In fact, the left-hand sides of inequalities (3.3.1.4) are equal respectively to 0.5415 and 4.449 for $i = 1$ and 2. Thus the equilibrium point x^* of system (3.3.1.11) is globally asymptotically stable.

On the other side, system (3.3.1.11) does not satisfy condition (3.3.1.5) since the left-hand side of the inequality equals -0.608775797.

The results of the present subsection were reported at the International Conference: Mathematical Science and Applications, Abu Dhabi, UAE, 2012, and the Bogolyubov Readings DIF-2013: Differential Equations, Theory of Functions and Their Applications, Sevastopol, Ukraine, 2013. They were accepted for publication as [14].

3.3.2 Discrete-time impulsive neural networks of neutral type

Now let us consider again the continuous-time Cohen-Grossberg neural network (3.3.1.1) for $t > t_0 = 0$, $t \neq t_k$, satisfying the assumptions **A3.3.1.1**–**A3.3.1.4** and provided with the impulse conditions

$$\Delta x_i(t_k) = \gamma_{ik} x_i(t_k) + \sum_{j=1}^{m} \delta_{ijk} x_j(t_k - \tau_j) + \zeta_{ik}, \qquad i = \overline{1,m}, \quad k \in \mathbb{N}, \tag{3.3.2.1}$$

and with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ with discontinuities of the first kind for $s \in [-\tau, 0]$. In (3.3.2.1) $\Delta x_i(t_k)$ denote impulsive state displacements at fixed moments of time t_k, $k \in \mathbb{N}$, involving time delays τ_j. Here it is assumed that the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$.

Recall that the components x_i^* of an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.3.1.1), (3.3.2.1) are governed by the algebraic system (3.3.1.2). Theorem 3.3.1.1 provides a sufficient condition for the system without impulses (3.3.1.1) to have a unique equilibrium point x^*. In order for

$x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ to be an equilibrium point of the impulsive system (3.3.1.1), (3.3.2.1), its components must also satisfy the linear equations

$$\gamma_{ik} x_i^* + \sum_{j=1}^{m} \delta_{ijk} x_j^* + \zeta_{ik} = 0, \quad k \in \mathbb{N},\ i = \overline{1,m}. \tag{3.3.2.2}$$

Till recently, the semi-discretization model had not been exploited for obtaining a discrete-time analogue of Cohen-Grossberg neural network mainly due to the nonlinearity of the feedback terms $-a_i(x_i)b_i(x_i)$. An appropriate extension of the method was presented in [93] (see §3.2.2). Exploiting the same idea, we start by rewriting the differential system (3.3.1.1) as

$$\begin{aligned} \dot{x}_i(t) \quad &+ \quad \beta_i x_i(t) + \sum_{j=1}^{m} e_{ij} \left(\dot{x}_j(t-\tau_j) + \beta_i x_j(t-\tau_j) \right) \qquad (3.3.2.3) \\ &= \quad \beta_i x_i(t) + \sum_{j=1}^{m} e_{ij} \beta_i x_j(t-\tau_j) + a_i(x_i(t)) \Bigg[-b_i(x_i(t)) \\ &\quad + \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} g_j(x_j(t-\tau_{ij})) + I_i \Bigg], \\ &\qquad i = \overline{1,m}, \quad t > 0,\ t \neq t_k, \end{aligned}$$

where $\beta_i = \underline{a}_i \underline{b}_i > 0$. Let the value $h > 0$ of the discretization step be fixed, and $n = [t/h]$, $\sigma_j = [\tau_j/h]$, where $[r]$ denotes the greatest integer contained in the real number r. On any interval $[nh, (n+1)h)$ not containing a moment of impulse effect t_k we multiply equation (3.3.2.3) by $e^{\beta_i t}$ and approximate it by an equation with constant arguments of the form

$$\begin{aligned} &\frac{d}{dt} \left(x_i(t) e^{\beta_i t} + \sum_{j=1}^{m} e_{ij} x_j(t - \sigma_j h) e^{\beta_i t} \right) \qquad (3.3.2.4) \\ &= \quad \beta_i e^{\beta_i t} \left(x_i(nh) + \sum_{j=1}^{m} e_{ij} x_j((n-\sigma_j)h) \right) \\ &+ \quad e^{\beta_i t} a_i(x_i(nh)) \Bigg[-b_i(x_i(nh)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(nh)) \\ &+ \quad \sum_{j=1}^{m} d_{ij} g_j(x_j((n-\sigma_j)h) + I_i \Bigg],\ i = \overline{1,m}, \end{aligned}$$

with $[t/h]h = nh \to t$, $[\tau_j/h]h = \sigma_j h \to \tau_j$ for a fixed time t as $h \to 0$. Upon integrating (3.3.2.4) over the interval $[nh, (n+1)h)$, one obtains a discrete analogue of the differential system (3.3.1.1) given by

$$\begin{aligned} & x_i(n+1)e^{\beta_i(n+1)h} - x_i(n)e^{\beta_i nh} \\ + & \sum_{j=1}^{m} e_{ij}\left(x_j(n+1-\sigma_j)e^{\beta_i(n+1)h} - x_j(n-\sigma_j)e^{\beta_i nh}\right) \\ = & \left(e^{\beta_i(n+1)h} - e^{\beta_i nh}\right)\left(x_i(n) + \sum_{j=1}^{m} e_{ij}x_j(n-\sigma_j)\right) \\ + & \frac{e^{\beta_i(n+1)h} - e^{\beta_i nh}}{\beta_i} a_i(x_i(n)) \Bigg[-b_i(x_i(n)) \\ + & \sum_{j=1}^{m} c_{ij}f_j(x_j(n)) + \sum_{j=1}^{m} d_{ij}g_j(x_j(n-\sigma_j)) + I_i\Bigg] \end{aligned}$$

for $i = \overline{1,m}$, $n \in \{0\} \cup \mathbb{N}$, wherein the notation $w(n) \equiv w(nh)$ has been adopted for simplicity. We multiply the i-th equation of this system by $e^{-\beta_i(n+1)h}$ and obtain the difference system

$$\begin{aligned} & x_i(n+1) = x_i(n) + \sum_{j=1}^{m} e_{ij}(x_j(n-\sigma_j) - x_j(n+1-\sigma_j)) \\ + & \psi_i(h)a_i(x_i(n))\Bigg[-b_i(x_i(n)) + \sum_{j=1}^{m} c_{ij}f_j(x_j(n)) \\ + & \sum_{j=1}^{m} d_{ij}g_j(x_j(n-\sigma_j)) + I_i\Bigg], \qquad i = \overline{1,m}, \quad n \in \{0\} \cup \mathbb{N}, \end{aligned} \tag{3.3.2.5}$$

where we have denoted $\psi_i(h) = \frac{1-e^{-\beta_i h}}{\beta_i}$. Observe that $0 < \psi_i(h) < \frac{1}{\beta_i}$ for $h > 0$ and $\psi_i(h) = h + O(h^2)$ for small $h > 0$.

The analogue (3.3.2.5) is supplemented with an initial vector sequence $\phi(\ell) = (\phi_1(\ell), \phi_2(\ell), \ldots, \phi_m(\ell))^T$ for $\ell = \overline{-\sigma, 0}$, $\sigma = \max_{j=\overline{1,m}} \sigma_j$. Next we discretize the impulse conditions (3.3.2.1). If we denote $n_k = \left[\frac{t_k}{h}\right]$, we obtain a sequence of positive integers $\{n_k\}$ satisfying $0 < n_1 < n_2 < \cdots < n_k \to \infty$ as $k \to \infty$. With each such integer n_k we associate two values of the solution $x(n)$, namely, $x(n_k)$ which can be regarded as the value of the solution before the impulse effect and whose components are evaluated by equations (3.3.2.5), and $x^+(n_k)$ which can be regarded as the value of the solution after the impulse effect and whose components are evaluated by the equations

$$\begin{aligned} x_i^+(n_k) &= (1+\gamma_{ik})x_i(n_k) + \sum_{j=1}^{m} \delta_{ijk} x_j(n_k - \sigma_j) + \zeta_{ik}, \qquad (3.3.2.6) \\ & i = \overline{1,m}, \quad k \in \mathbb{N}. \end{aligned}$$

If a value of $x(n)$ in the right-hand side of (3.3.2.5) or (3.3.2.6) must be evaluated at a member of the sequence $\{n_k\}_{k\in\mathbb{N}}$, we take $x^+(n_k)$ evaluated from (3.3.2.6). The existence of a unique solution $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ of the impulsive analogue (3.3.2.5), (3.3.2.6) for $n \in \{0\} \cup \mathbb{N}$ is therefore justified.

If we want to give a formal description of the discrete-time analogue of the impulsive system (3.3.1.1), (3.3.2.1), we should write

$$\begin{aligned} x_i^-(n+1) &= x_i^+(n) + \sum_{j=1}^{m} e_{ij}(x_j^+(n-\sigma_j) - x_j^+(n+1-\sigma_j)) \\ &+ \psi_i(h) a_i(x_i^+(n)) \left[-b_i(x_i^+(n)) + \sum_{j=1}^{m} c_{ij} f_j(x_j^+(n)) \right. \\ &+ \left. \sum_{j=1}^{m} d_{ij} g_j(x_j^+(n-\sigma_j)) + I_i \right], \qquad n \in \{0\} \cup \mathbb{N}, \end{aligned}$$

$$x_i^+(n) = \begin{cases} x_i^-(n) & \text{for } n \neq n_k, \\ (1+\gamma_{ik})x_i^-(n_k) + \sum\limits_{j=1}^{m} \delta_{ijk} x_j^-(n_k - \sigma_j) + \zeta_{ik} & \text{for } n = n_k, \end{cases}$$

$i = \overline{1,m}$. Systems (3.3.1.1), (3.3.2.1) and (3.3.2.5), (3.3.2.6) have the same equilibrium points if any. Their components must satisfy (3.3.1.2), (3.3.2.2).

Definition 3.3.2.1. The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (3.3.2.5), (3.3.2.6) is said to be *globally asymptotically stable* if any other solution $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ of system (3.3.2.5), (3.3.2.6) is defined for all $n \in \mathbb{N}$ and satisfies

$$\lim_{n\to\infty} x(n) = x^*.$$

Our main result in the present subsection is the following

Theorem 3.3.2.1. *Let the assumptions* **A3.3.1.1**–**A3.3.1.4** *hold. Suppose, further, that the inequalities* (3.3.1.4) *and the conditions*

$$\delta_{ijk} = \gamma_{ik} e_{ij}, \quad -2 \le \gamma_{ik} \le 0, \qquad i,j = \overline{1,m}, \quad k \in \mathbb{N}, \tag{3.3.2.7}$$

are valid and the system (3.3.2.5), (3.3.2.6) *has an equilibrium point* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ *whose components satisfy* (3.3.1.2), (3.3.2.2). *Then the equilibrium point* x^* *is globally asymptotically stable for all sufficiently small values of* $h > 0$.

Proof. Upon introducing the translations

$$u_i(n) = x_i(n) - x_i^*, \qquad \varphi_i(\ell) = \phi_i(\ell) - x_i^*$$

we derive the system

$$u_i(n+1) + \sum_{j=1}^m e_{ij} u_j(n+1-\sigma_j) = u_i(n) + \sum_{j=1}^m e_{ij} u_j(n-\sigma_j) \tag{3.3.2.8}$$

$$+ \quad \psi_i(h)\tilde{a}_i(u_i(n)) \left[-\tilde{b}_i(u_i(n)) + \sum_{j=1}^m c_{ij} \tilde{f}_j(u_j(n)) + \sum_{j=1}^m d_{ij} \tilde{g}_j(u_j(n-\sigma_j)) \right],$$

$$i = \overline{1,m}, \quad n \in \mathbb{N},$$

$$u_i^+(n_k) = (1+\gamma_{ik}) u_i(n_k) + \sum_{j=1}^m \delta_{ijk} u_j(n_k - \sigma_j), \tag{3.3.2.9}$$

$$i = \overline{1,m}, \quad k \in \mathbb{N},$$

$$u_i(\ell) = \varphi_i(\ell), \qquad i = \overline{1,m}, \quad \ell = \overline{-\sigma, 0},$$

where

$$\tilde{a}_i(u_i) = a_i(u_i + x_i^*), \qquad \tilde{b}_i(u_i) = b_i(u_i + x_i^*) - b_i(x_i^*),$$
$$\tilde{f}_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), \qquad \tilde{g}_j(u_j) = g_j(u_j + x_j^*) - g_j(x_j^*).$$

This system inherits the assumptions **A3.3.1.1**–**A3.3.1.4** given before. It suffices to examine the stability characteristics of the trivial equilibrium point $u^* = 0$ of system (3.3.2.8), (3.3.2.9).

We define a Lyapunov sequence $\{V(n)\}_{n=0}^{\infty}$ by

$$V(n) = \frac{1}{2} \sum_{i=1}^m \left[u_i(n) + \sum_{j=1}^m e_{ij} u_j(n-\sigma_j) \right]^2 + h \sum_{i=1}^m \omega_i \sum_{\ell=n-\sigma_i}^{n-1} u_i^2(\ell),$$

where ω_i, $i = \overline{1,m}$, will be determined later. First we notice that the value $V(0)$ is completely determined from the initial values of the system. Then we successively find

$$\begin{aligned} V(n+1) &= \frac{1}{2}\sum_{i=1}^{m}\left[u_i(n+1)+\sum_{j=1}^{m}e_{ij}u_j(n+1-\sigma_j)\right]^2 + h\sum_{i=1}^{m}\omega_i\sum_{\ell=n+1-\sigma_i}^{n}u_i^2(\ell) \\ &= \frac{1}{2}\sum_{i=1}^{m}\left\{\left[u_i(n)+\sum_{j=1}^{m}e_{ij}u_j(n-\sigma_j)\right] + \psi_i(h)\tilde{a}_i(u_i(n))\left[-\tilde{b}_i(u_i(n))\right.\right. \\ &+ \left.\left.\sum_{j=1}^{m}c_{ij}\tilde{f}_j(u_j(n)) + \sum_{j=1}^{m}d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\right]\right\}^2 + h\sum_{i=1}^{m}\omega_i\sum_{\ell=n+1-\sigma_i}^{n}u_i^2(\ell), \end{aligned}$$

$$\begin{aligned} & V(n+1)-V(n) = \sum_{i=1}^{m}\psi_i(h)\tilde{a}_i(u_i(n))\left[u_i(n)+\sum_{j=1}^{m}e_{ij}u_j(n-\sigma_j)\right] \\ & \times\left[-\tilde{b}_i(u_i(n))+\sum_{j=1}^{m}c_{ij}\tilde{f}_j(u_j(n))+\sum_{j=1}^{m}d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\right] \\ & +\frac{1}{2}\sum_{i=1}^{m}\psi_i^2(h)\tilde{a}_i^2(u_i(n))\left[-\tilde{b}_i(u_i(n))+\sum_{j=1}^{m}c_{ij}\tilde{f}_j(u_j(n))\right. \\ & \left.+\sum_{j=1}^{m}d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\right]^2 + h\sum_{i=1}^{m}\omega_i\left(u_i^2(n)-u_i^2(n-\sigma_i)\right) \\ & = h\sum_{i=1}^{m}\left\{\tilde{a}_i(u_i(n))\left[u_i(n)+\sum_{j=1}^{m}e_{ij}u_j(n-\sigma_j)\right]\right. \\ & \times\left[-\tilde{b}_i(u_i(n))+\sum_{j=1}^{m}c_{ij}\tilde{f}_j(u_j(n))+\sum_{j=1}^{m}d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\right] \\ & \quad \left.+\left(C_i(h)+\omega_i\right)u_i^2(n)+\left(D_i(h)-\omega_i\right)u_i^2(n-\sigma_i)\right\}, \end{aligned}$$

where $C_i(h)$, $D_i(h) = O(h)$ for $i = \overline{1,m}$. For the sake of brevity we do not write in details the terms of order $O(h^2)$. Then

$$
\begin{aligned}
& V(n+1)-V(n) = h\sum_{i=1}^{m}\Bigg\{-\tilde{a}_i(u_i(n))\tilde{b}_i(u_i(n))u_i(n) \\
+ \quad & \tilde{a}_i(u_i(n))u_i(n)\left[\sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(n)) + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\right] \\
+ \quad & \tilde{a}_i(u_i(n))\sum_{j=1}^{m} e_{ij}u_j(n-\sigma_j)\Bigg[-\tilde{b}_i(u_i(n)) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(n)) \\
+ \quad & \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(n-\sigma_j))\Bigg] + (C_i(h)+\omega_i)\,u_i^2(n) + (D_i(h)-\omega_i)\,u_i^2(n-\sigma_i)\Bigg\} \\
\leq \quad & h\sum_{i=1}^{m}\Bigg\{-\underline{a}_i\underline{b}_i u_i^2(n) + \overline{a}_i|u_i(n)|\Bigg[\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(n)| \\
& +\sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(n-\sigma_j)|\Bigg] + \overline{a}_i\sum_{j=1}^{m}|e_{ij}|\,|u_j(n-\sigma_j)|\Bigg[\overline{b}_i\,|u_i(n)| \\
& +\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(n)| + \sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(n-\sigma_j)|\Bigg] \\
& +(C_i(h)+\omega_i)\,u_i^2(n) + (D_i(h)-\omega_i)\,u_i^2(n-\sigma_i)\Bigg\} \\
\leq \quad & h\sum_{i=1}^{m}\Bigg\{-\underline{a}_i\underline{b}_i u_i^2(n) + \overline{a}_i\frac{1}{2}\sum_{j=1}^{m}|c_{ij}|\,F_j\left(u_i^2(n)+u_j^2(n)\right) \\
& +\overline{a}_i\frac{1}{2}\sum_{j=1}^{m}|d_{ij}|\,G_j\left(u_i^2(n)+u_j^2(n-\sigma_j)\right) + \overline{a}_i\overline{b}_i\frac{1}{2}\sum_{j=1}^{m}|e_{ij}|\left(u_i^2(n)+u_j^2(n-\sigma_j)\right) \\
& +\overline{a}_i\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ij}|\,|c_{ik}|\,F_k\left(u_k^2(n)+u_j^2(n-\sigma_j)\right) \\
& +\overline{a}_i\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ij}|\,|d_{ik}|\,G_k\left(u_j^2(n-\sigma_j)+u_k^2(n-\sigma_k)\right) \\
& +(C_i(h)+\omega_i)\,u_i^2(n) + (D_i(h)-\omega_i)\,u_i^2(n-\sigma_i)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= h\sum_{i=1}^{m}\Bigg\{-\Bigg[\underline{a}_i\underline{b}_i-\frac{1}{2}\left(\overline{a}_i\sum_{j=1}^{m}|c_{ij}|F_j+F_i\sum_{j=1}^{m}|c_{ji}|\overline{a}_j\right)-\frac{a_i}{2}\sum_{j=1}^{m}|d_{ij}|G_j\\
&- \frac{\overline{a}_i\overline{b}_i}{2}\sum_{j=1}^{m}|e_{ij}|-\frac{F_i}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|c_{ki}|\,|e_{kj}|\overline{a}_k-C_i(h)-\omega_i\Bigg]u_i^2(n)\\
&+ \Bigg[\frac{G_i}{2}\sum_{j=1}^{m}|d_{ji}|\overline{a}_j+\frac{1}{2}\sum_{j=1}^{m}|e_{ji}|\overline{a}_j\overline{b}_j+\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ji}|\,|c_{jk}|\overline{a}_jF_k\\
&+ \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\left(|e_{ji}|\,|d_{jk}|\overline{a}_jG_k+|e_{kj}|\,|d_{ki}|\overline{a}_kG_i\right)+D_i(h)-\omega_i\Bigg]u_i^2(n-\sigma_i)\Bigg\}.
\end{aligned}
$$

Choose

$$
\begin{aligned}
\omega_i &= \frac{G_i}{2}\sum_{j=1}^{m}|d_{ji}|\overline{a}_j+\frac{1}{2}\sum_{j=1}^{m}|e_{ji}|\overline{a}_j\overline{b}_j+\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ji}|\,|c_{jk}|\overline{a}_jF_k\\
&+ \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\left(|e_{ji}|\,|d_{jk}|\overline{a}_jG_k+|e_{kj}|\,|d_{ki}|\overline{a}_kG_i\right)+D_i(h),
\end{aligned}
$$

then after some simplifications we obtain

$$
\begin{aligned}
&V(n+1)-V(n)\le -h\sum_{i=1}^{m}\Bigg\{\underline{a}_i\underline{b}_i-\frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|c_{ij}|F_j+\overline{a}_j|c_{ji}|F_i\right)\\
&- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|d_{ij}|G_j+\overline{a}_j|d_{ji}|G_i\right)-\frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}|+\overline{a}_j\overline{b}_j|e_{ji}|\right)\\
&- \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|c_{ji}|\,|e_{jk}|\,F_i+|c_{jk}|\,|e_{ji}|\,F_k\right)\\
&- \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|d_{ji}|\,|e_{jk}|\,G_i+|d_{jk}|\,|e_{ji}|\,G_k\right)-C_i(h)-D_i(h)\Bigg\}u_i^2(n).
\end{aligned}
$$

According to inequalities (3.3.1.4) there exists $\tilde{\mu}>0$ such that

$$
\tilde{\mu}=\min_{i=\overline{1,m}}\Bigg\{\underline{a}_i\underline{b}_i-\frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|c_{ij}|F_j+\overline{a}_j|c_{ji}|F_i\right)-\frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|d_{ij}|G_j+\overline{a}_j|d_{ji}|G_i\right)
$$

$$- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k\right)$$
$$\left. - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k\right)\right\}.$$

Further on, we can choose the discretization step h so small that $|C_i(h) + D_i(h)| < \tilde{\mu}/2$, then

$$V(n+1) - V(n) \le -\frac{h\tilde{\mu}}{2}\|u(n)\|^2, \quad n \in \{0\} \cup \mathbb{N}. \tag{3.3.2.10}$$

In case $n = n_k$, in the above inequality instead of $V(n)$ we should take the value $V^+(n)$ evaluated for $u_i^+(n_k)$ given by (3.3.2.9). Thus

$$\begin{aligned} V^+(n_k) - V(n_k) &= \frac{1}{2}\sum_{i=1}^{m}\left\{\left[(1+\gamma_{ik})u_i(n_k) + \sum_{j=1}^{m}(e_{ij} + \delta_{ijk})u_j(n_k - \sigma_j)\right]^2\right. \\ &\qquad \left. - \left[u_i(n_k) + \sum_{j=1}^{m}e_{ij}u_j(n_k - \sigma_j)\right]^2\right\} \\ &= \frac{1}{2}\sum_{i=1}^{m}\left[\gamma_{ik}u_i(n_k) + \sum_{j=1}^{m}\delta_{ijk}u_j(n_k - \sigma_j)\right] \\ &\times \left[(2+\gamma_{ik})u_i(n_k) + \sum_{j=1}^{m}(2e_{ij} + \delta_{ijk})u_j(n_k - \sigma_j)\right]. \end{aligned}$$

According to conditions (3.3.2.7) we have

$$V^+(n_k) - V(n_k) = \frac{1}{2}\sum_{i=1}^{m}\gamma_{ik}(2+\gamma_{ik})\left[u_i(n_k) + \sum_{j=1}^{m}e_{ij}u_j(n_k - \sigma_j)\right]^2 \le 0,$$

which implies the validity of (3.3.2.10) also for $n = n_k$ and $V(n_k)$ evaluated for $u_i(n_k)$.

The inequalities (3.3.2.10) show that for any solution $u(n)$ of system (3.3.2.8), (3.3.2.9) the sequence $\{V(n)\}_{n=0}^{\infty}$ is monotone decreasing and it is bounded below by 0. Thus there exists the limit $\lim_{n\to\infty} V(n) \ge 0$. Passing to the limit as $n \to \infty$ in (3.3.2.10), we find that $\lim_{n\to\infty}\|u(n)\| = 0$, that is,

$\lim_{n\to\infty} \|x(n) - x^*\| = 0$. This means that the equilibrium point x^* of system (3.3.2.5), (3.3.2.6) is globally asymptotically stable. □

Example. Consider the system (3.3.1.11) for $t > 0$, $t \neq t_k$, provided with the impulse conditions

$$\begin{aligned}\Delta x_1(t_k) &= -1.1x_1(t_k) - 0.11x_1(t_k - \tau_1) - 0.165x_2(t_k - \tau_2) + 0.7847118585, \\ && (3.3.2.11) \\ \Delta x_2(t_k) &= -0.9x_2(t_k) + 0.18x_1(t_k - \tau_1) - 0.18\ x_2(t_k - \tau_2) + 0.253721604,\end{aligned}$$

$k \in \mathbb{N}$, and with arbitarary initial conditions $x_i(s) = \phi(s)$, $i = 1, 2$, $s \in [-\max\{\tau_1, \tau_2\}, 0]$. Its discrete-time counterpart is

$$\begin{aligned}
& x_1(n+1) = x_1(n) + 0.1\left[x_1(n-\sigma_1) - x_1(n+1-\sigma_1)\right] \qquad (3.3.2.12)\\
+ \quad & 0.15\left[x_2(n-\sigma_2) - x_2(n+1-\sigma_2)\right] + \frac{1-e^{-3.98h}}{3.98}\left(2 + 0.01\sin x_1(n)\right)\\
\times \quad & \left[-2x_1(n) + 0.1\arctan x_1(n) + 0.15\arctan x_2(n)\right.\\
& \left.+ 0.1\arctan x_1(n-\sigma_1) + 0.15\arctan x_2(n-\sigma_2) + 1\right],\\
& x_2(n+1) = x_2(n) - 0.2\left[x_1(n-\sigma_1) - x_1(n+1-\sigma_1)\right]\\
+ \quad & 0.2\left[x_2(n-\sigma_2) - x_2(n+1-\sigma_2)\right] + \frac{1-e^{-8.94h}}{8.94}\left(3 - 0.02\sin x_2(n)\right)\\
\times \quad & \left[-3x_2(n) + 0.15\arctan x_1(n) - 0.2\arctan x_2(n)\right.\\
& \left.+ 0.1\arctan x_1(n-\sigma_1) - 0.2\arctan x_2(n-\sigma_2) + 1\right], \quad n \in \{0\} \cup \mathbb{N},
\end{aligned}$$

$$\begin{aligned}
x_1^+(n_k) &= -0.1x_1(n_k) - 0.11x_1(n_k - \sigma_1) \qquad (3.3.2.13)\\
&\quad - 0.165x_2(n_k - \sigma_2) + 0.7847118585,\\
x_2^+(n_k) &= 0.1x_2(t_k) + 0.18x_1(n_k - \sigma_1)\\
&\quad - 0.18x_2(n_k - \sigma_2) + 0.253721604, \quad k \in \mathbb{N},
\end{aligned}$$

with initial conditions $x_i(\ell) = \phi(\ell)$, $\ell = \overline{-\sigma, 0}$, $\sigma = \max\{\sigma_1, \sigma_2\}$ and $\sigma_i = [\tau_i/h]$, $i = 1, 2$.

The unique equilibrium point $x^* = (0.6027869379,\ 0.3353919007)^T$ of the continuous-time system (3.3.1.11) is also the unique equilibrium point of its discrete-time impulsive analogue (3.3.2.12), (3.3.2.13).

Further on, system (3.3.2.12), (3.3.2.13) satisfies the assumptions of Theorem 3.3.2.1. Thus, its equilibrium point x^* is globally asymptotically stable for sufficiently small values of the discretization step $h > 0$.

The results of the present subsection was reported at the International Conference on Differential & Difference Equations and Applications, Ponta Delgada, Portugal, 2011. They were published in a complete form (without references to the continuous-time case) in [48].

CONCLUSION

In the thesis we have presented the following results:

- For an impulsive system with a constant delay it is proved that if the corresponding system without delay has an isolated ω–periodic solution, then in any neighbourhood of this orbit the system considered also has an ω–periodic solution if the delay is small enough.
- For a neutral impulsive system with a constant delay it is proved that if the corresponding system without delay has an isolated ω–periodic solution, then in any neighbourhood of this orbit the system considered also has an ω–periodic solution if the delay is small enough.
- For an impulsive differential-difference system such that the corresponding system without delay is linear and has an r–parametric family of ω–periodic solutions an equation for the generating amplitudes (a necessary condition for the existence of ω–periodic solutions) is derived, and sufficient conditions are obtained for the existence of ω–periodic solutions in the critical cases of first (simple root of the equation for the generating amplitudes) and second order.
- For an age-dependent model with a dominant age class the problem of existence of a periodic regime in the presence of impulsive perturbations is reduced to operator systems solvable by a convergent simple iteration method for both noncritical case and critical case of first order.
- For a nonlinear boundary value problem for an impulsive system of ordinary differential equations with concentrated delays in the general case when the number of the boundary conditions does not coincide with the order of the system, under the assumption that the corresponding boundary value problem without delay is linear and has an r–parametric family of solutions, the equation for the generating amplitudes is derived, and sufficient conditions for the existence and an iteration algorithm for the construction of a solution of the initial problem are obtained in the critical case of first order if the delays are sufficiently small.
- For an impulsive system with delay which differs from a constant by a small-amplitude ω–periodic perturbation such that the corresponding

system with constant delay has an isolated ω–periodic solution, under a nondegeneracy assumption it is proved that in any sufficiently small neighbourhood of this orbit the perturbed system also has a unique ω–periodic solution.

- For an impulsive system with delay which differs from a constant by a small-amplitude periodic perturbation such that the corresponding system with constant delay has an isolated ω–periodic solution, if the period of the delay is rationally independent with ω, under a nondegeneracy assumption it is proved that in any sufficiently small neighbourhood of this orbit the perturbed system has a unique almost periodic solution.

- The last two results are extended to the case of a neutral impulsive system with a small delay of the argument of the derivative and another delay which differs from a constant by a small-amplitude periodic perturbation.

- For impulsive continuous-time neural networks of Hopfield type with both constant and infinite distributed delays sufficient conditions are found for the existence of a unique equilibrium point and its global exponential stability.

- For impulsive continuous-time Cohen-Grossberg neural networks with finite S-type distributed delays (given by Lebesgue-Stieltjes integrals) sufficient conditions are found for the existence of a unique equilibrium point and its global exponential stability.

- For impulsive continuous-time Cohen-Grossberg neural networks with time-varying and infinite S-type distributed delays and reaction-diffusion terms of spatial dimension $n \geq 3$, by using Hardy-Poincaré inequality improved stability estimates are obtained for the system with zero Dirichlet boundary conditions.

- Sufficient conditions in terms of minimal Lipschitz constants and nonlinear measures are obtained for the existence of a unique equilibrium point and its exponential stability for impulsive neural networks which are generalizations of Cohen-Grossberg neural networks, with time-varying delays.

- For impulsive continuous-time neural networks of Hopfield type with both constant and infinite distributed delays discrete-time counterparts are formulated by the semi-discretization method, and sufficient conditions are found for the global exponential stability of the unique equilibrium point.

- For impulsive continuous-time Cohen-Grossberg neural networks with constant and infinite distributed delays discrete-time counterparts are formulated by an extension of the semi-discretization method, and sufficient conditions are found for the global exponential stability of the unique equilibrium point.

- For two different classes of Hopfield-type with periodic impulses and finite distributed delays discrete-time counterparts are introduced. Using different methods, sufficient conditions for the existence and global exponential stability of a unique periodic solution of the discrete systems considered are found.

- A discrete-time counterpart of a class of Hopfield neural networks with impulses and concentrated and infinite distributed delays as well as a small delay in the leakage terms is introduced. Sufficient conditions for the existence and global exponential stability of a unique equilibrium point of the discrete-time system considered are found.

- For a Cohen-Grossberg neural network of neutral type sufficient conditions for the existence and global asymptotic stability of a unique equilibrium point are obtained.

- For an impulsive Cohen-Grossberg neural network of neutral type a discrete-time counterpart is obtained by an extension of the semi-discretization method. For sufficiently small values of the discretization step it is shown that the sufficient conditions for the global asymptotic stability of the unique equilibrium point of the continuous-time system are such for the discrete-time counterpart too.

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