MAXIMUM PRINCIPLE FOR LINEAR SECOND ORDER ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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Abstract. The maximum principle for linear second-order elliptic equations in divergence form is investigated. By means of new formulas for the first eigenvalue necessary and sufficient conditions for the validity of the maximum principle are obtained. Some qualitative properties of the first eigenvalue with respect to the coefficients of the equation are proved.

Key words. elliptic equations, maximum principle, eigenvalue problem

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1. Introduction. Let L be a linear second-order uniformly elliptic operator in divergence form

$$Lu = -\left(a_j^k(x)u_{x_k} + a_j^0(x)u\right)_{x_j} + b^j(x)u_{x_j} + b^0(x)u \quad \text{in} \quad \Omega, \tag{1.1}$$

$$a_j^k(x)\xi^j\xi^k \geq \mu|\xi|^2 \quad \text{for every} \quad x \in \overline{\Omega},\ \xi \in \mathbf{R}^n,\ \mu = const > 0. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbf{R}^n$, $\partial\Omega \in C^{1,1}$,

$$a_j^k(x),\ a_j^0(x) \in W^{1,\infty}(\Omega), \quad b^k(x),\ b^0(x) \in L^\infty(\Omega), \{a_j^k\} = \{a_k^j\} \tag{1.3}$$

and under the repeating indices the summation convention is understood.

Further on we will use also the following equivalent form of equation (1.1)

$$Lu = -(a_j^k u_{x_k} + (d^j - c^j)u)_{x_j} + (d^j + c^j)u_{x_j} + b^0 u \tag{1.4}$$

where $d^j = \dfrac{1}{2}(b^j + a_j^0)$, $c^j = \dfrac{1}{2}(b^j - a_j^0)$.

Let us recall the classical maximum principle. We say that the maximum principle holds for the operator L in Ω if for every $u \in C^2(\Omega) \cap C(\overline{\Omega})$ the inequalities $Lu \leq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ imply $u \leq 0$ in Ω.

The natural extension of the classical maximum principle for operators in divergence form is the following statement. The maximum principle in a weak sense holds for the operator L if for every $u \in H^1(\Omega)$ the inequalities $Lu \leq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ imply $\sup\limits_{\Omega} u \leq 0$. Here the function u satisfies $u \leq 0$ on $\partial\Omega$ if its positive part $u^+ = \max(u, 0) \in H_0^1(\Omega)$. The inequality $Lu \leq 0$ holds in a weak sense if the corresponding bilinear form $B_L[u,w] = \displaystyle\int_\Omega \left(a_j^k u_{x_k} w_{x_j} + a_j^0 u w_{x_j} + b^j w u_{x_j} + b^0 uw\right) dx$ of L is a nonnegative one, for all nonnegative functions $w \in C_0^1(\Omega)$.

This paper is concerned with the necessary and sufficient conditions for the validity of the maximum principle for (1.1) and (1.4) by means of new formulas for the first eigenvalue λ_L of L. The precise dependence of λ_L on the coefficients a_j^k, a_j^0, b^j, b^0 is studied in connection with the applications to the maximum principle.

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The motivation for investigations of these problems is the comparison principle for quasilinear second-order uniformly elliptic operator in divergence form

$$Q(u) = -\frac{\partial}{\partial x_j} a_j(x,u,Du) + b(x,u,Du) \ \text{ in } \Omega. \tag{1.5}$$

As it is wellknown (see [9]), the maximum principle in the linear case is crucial for the validity of the comparison principle for weak $C^1(\overline{\Omega})$ smooth sub- and supersolutions of (1.5). More precisely, if

$$a_j^k(x) = \int_0^1 \frac{\partial a_j}{\partial p_k}(x,S_t)\,dt, \ \ a_j^0(x) = \int_0^1 \frac{\partial a_j}{\partial u}(x,S_t)\,dt, \tag{1.6}$$

$$b^j(x) = \int_0^1 \frac{\partial b}{\partial p_j}(x,S_t)\,dt, \ \ b^0(x) = \int_0^1 \frac{\partial b}{\partial u}(x,S_t)\,dt,$$

where $S_t = (\nu(x) + t(u(x) - \nu(x), \nabla\nu(x) + t(\nabla u(x) - \nabla\nu(x))$, and $u, v \in C^1(\overline{\Omega})$ are weak sub-and supersolutions of (1.5), then the comparison principle for (1.5) holds if the linear equation (1.1) with the above coefficients (1.6) satisfies the maximum principle.

Another application of the maximum principle is the uniqueness and the continuous dependence on the data of the weak solutions for bvp for (1.1) and (1.5). Moreover, by means of suitable barrier functions the amplitude of the weak solutions of (1.1) and (1.5) can be estimated. These estimates are an important step in the proof of the existence of a solution with the Leray-Schauder fixed point theorem or in the applications for the numeric methods of solving of bvp for (1.1) and (1.5).

The maximum principle is important also in the investigations of the asymptotic behaviour of the solutions of linear and quasilinear parabolic equations which appear in the population dynamics modeling a population which will persist or will go extinct.

Let us recall that in the literature there are two type of conditions for the validity of the maximum principle. The first of them are necessary and sufficient and are given in [1] for linear equations in divergence form and in [4] for general nondivergence form equations. One of the main results in [1] and [4] is that the maximum principle for the operator L holds if and only if the first eigenvalue λ_L of L with zero Dirichlet data is positive. It is clear that the positiveness of the first eigenvalue λ_L is not easy checkable condition so that this result is more useful for theoretical investigations. However, there are some qualitative properties of λ_L which one uses to find out lower and upper bounds for the first eigenvalue (see, for example, [4]).

There are also second type results which are only sufficient but easy checkable conditions for wide class of equations. They are given, for example, in [7], [9], [12] (see also the references there) and guarantee the maximum principle for (1.1) if one of the following assumptions is satisfied:

$$(i) \quad b^0 - \text{ div } a^0 \geq 0 \text{ in } \Omega, \quad a^0 = (a_1^0, \cdots, a_n^0);$$

$$(ii) \quad b^0 - \text{ div } b \geq 0 \text{ in } \Omega, \quad b = (b^1, \cdots, b^n); \tag{1.7}$$

(iii) The matrix $A + A^T$ is a nonnegative one in Ω, where $\begin{pmatrix} a_j^k & b^j \\ a_k^0 & b^0 \end{pmatrix}$ and A^T is the conjugate matrix of A.

Unfortunately, conditions $(1.7)_i$, $(1.7)_{ii}$ are not useful for quasilinear equations (1.5) because the derivatives of the coefficients a_j^0, b^j given by (1.6) are not under control. That is why $(1.7)_i$, $(1.7)_{ii}$ are replaced in the nonlinear case with some additional structure assumptions guaranteeing that a_j^0 or b^j are identically equal to zero (see theorem 9.5 in [9]). By the way, $(1.7)_i$, $(1.7)_{ii}$ are not sharp even in the linear case.

As for $(1.7)_{iii}$, it seems to be the most promising general sufficient condition which is applicable in the nonlinear case but also is not sharp. Following the idea in [11] one can easily show that $(1.7)_{iii}$ is not invariant if equation (1.1) is rewritten in an equivalent way, for example

$$Lu = -\left(a_j^k u_{x_k} + (a_j^0 + f^j)u\right)_{x_j} + (b^j + f^j)u_{x_j} + (b^0 + \operatorname{div} f)u \tag{1.8}$$

for arbitrary vector $f(x)$, $f^j \in C^{0,1}(\overline{\Omega})$. Now $(1.7)_{iii}$ for equation (1.8) (or equivalently for equation (1.1) is

(1.9) The matrix $A_f + A_f^T$ is a nonnegative one in Ω,

$$\text{where } A_f = \begin{pmatrix} a_j^k & b^j + f^j \\ a_k^0 + f^k & b^0 + \operatorname{div} f \end{pmatrix}.$$

Condition (1.8) can be better than $(1.7)_{iii}$ for some special choice of f.

Starting from the idea of Protter in [11] we consider the whole class of equations (1.8) instead of (1.1) and sufficient conditions (1.9) instead of $(1.7)_{iii}$. In this way we prove in section 2 that (1.9) is also a necessary condition for the validity of the maximum principle for symmetric operators if (1.9) is taken over the set of all admissible vectors $f(x)$. Unfortunately, the same result is not true for nonsymmetric operators. The reason is that the matrix $\frac{1}{2}\left(A_f + A_f^T\right)$ corresponds only to the symmetric part $L_0 = \frac{1}{2}(L + L^*)$ of the operator L and (1.9) guarantees that the first eigenvalue of L_0 is positive. However, the first eigenvalue of L can be far from the first eigenvalue of L_0 (see theorem 4.1 and example 2 in section 4). Nevertheless, considering the set of all nondegenerate transformations of the special type $L_z u = e^{-z} L(ue^z)$, $z \in C^{0,1}(\overline{\Omega})$, which preserve the first eigenvalue of L, we get as in the previous case a necessary and sufficient condition for the maximum principle for nonsymmetric operators. In this way we prove in section 3 several equivalent formulas for the first eigenvalue λ_L for nonsymmetric operators, which are different from the well known results and in many cases are more convenient for lower and upper estimates for λ_L. Moreover λ_L is obtained as an extremum of the first eigenvalues of some explicitely given symmetric operators.

At the end of section 3 in proposition 3.7 we show that the symmetric condition on $\{a_j^k\}$ in (1.3) is not essential because the nonsymmetric case is transformed to the symmetric one.

Using the new expressions for λ_L we get in section 4 some qualitative properties of the first eigenvalue λ_L with respect to the coefficients a_j^0, b^j, b^0 and the matrix $\{a_j^k\}$, as monotonicity and concavity of λ_L.

2. Linear symmetric operators. In this section we will consider only the case of symmetric operators

$$L_0 u = -\left(a_j^k(x) u_{x_j} + d^j(x) u\right)_{x_j} + d^j(x) u_{x_j} + b^0(x) u \quad \text{in} \quad \Omega, \tag{2.1}$$

with coefficients a_j^k, d^j, b^0 satisfying (1.2), (1.3).

Let us recall the variational formula of the first eigenvalue for symmetric operators L_0

$$\lambda_{L_0} = \inf_{\nu} B_{L_0}[\nu, \nu], \;\; \nu \in H_0^1(\Omega), \;\; \|\nu\|_{L^2} = 1. \tag{2.2}$$

As it is wellknown (see for example [8]) the above infinum is attained for a positive function $u \in H_0^1(\Omega)$, which solves the equation

$$L_0 u = \lambda_{L_0} u \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega$$

in a weak sense. Finally, every weak solution $w \in H_0^1(\Omega)$ of the above equation is a multiple of u.

Moreover, the "max-min" representation formula for the first eigenvalue λ_{L_0}

$$\lambda_{L_0} = \sup_{\nu} \operatorname{ess} \inf_{x} (L_0 \nu / \nu), \;\; v \in W^{2,n}(\Omega), \;\; \nu > 0 \text{ in } \Omega \tag{2.3}$$

holds (see [4], [5] and [10] for more details).

Following the idea of Protter in [11] for every vector function $f(x)$ with components $f^j(x) \in C^{0,1}(\overline{\Omega})$ we rewrite (2.1) in the equivalent form

$$L_0 u = -\left(a_j^k u_{x_k} + (d^j + f^j) u\right)_{x_j} + (d^j + f^j) u_{x_j} + (b^0 + \text{ div } f) u. \tag{2.4}$$

Now from $(1.7)_{iii}$ the maximum principle for L_0 holds if for some $f^j \in C^{0,1}(\overline{\Omega})$

$$\text{The matrix } A_f + A_f^T = 2A_f \text{ is a nonnegative one in } \Omega, \text{ where} \tag{2.5}$$

$$A_f = \begin{pmatrix} a_j^k & d^j + f^j \\ d^k + f^k & b^0 + \operatorname{div} f \end{pmatrix}.$$

Fortunately, condition (2.5) taken over the set of all admissible vectors $f(x)$ is also a necessary one. To explain roughly the idea let us formulate (2.5) in the following equivalent way: condition (2.5) holds iff $\det A_f \geq 0$ or equivalently iff $\sigma_{L_0}(f) \geq 0$, where

$$\sigma_{L_0}(f) = b^0 + \operatorname{div} f - a_j^k (f^j + d_j)(f^k + d^k). \tag{2.6}$$

Here and further on we use notation $\{\alpha_j^k\} = \{\alpha_j^k\}^{-1}$.

Finally, if $\sup_{f \in F} \operatorname{ess} \inf_{x \in \Omega} \sigma_{L_0}(f) > 0$ then (2.5) holds for some $f \in F$ where F will be a suitably chosen class of functions containing the Lipschitz functions.

In order to formulate the precise result in theorem 2.1 we will need a little bit wider class of functions $f^j(x)$ than the class of Lipschitz ones. For this purpose let us introduce the following notation

$$F = \{f(x) = (f^1(x), \cdots, f^n(x)) : f^j, \text{ div } f \in L^\infty(\Omega)\}, \tag{2.7}$$

where $\operatorname{div} f$ should be understand in the distributional sense and

$$\sigma_{L_0} = \sup_f \operatorname{ess} \inf_{x\in\Omega} \sigma_{L_0}(f), \quad f \in F. \tag{2.8}$$

More precisely we have the following result.

THEOREM 2.1. *Let the operator L_0 satisfy (1.2) and (1.3). Then $\sigma_{L_0} = \lambda_{L_0}$ and the maximum principle for L_0 holds if and only if $\sigma_{L_0} > 0$.*

REMARK 1. For the special choice of f, $f = -d$, we get immediately from (2.6), (2.8) condition $(1.7)_i$ (which coincides with $(1.7)_{ii}$ in the symmetric case) and for $f = 0$, respectively, condition $(1.7)_{iii}$.

In fact σ_{L_0} gives a different expression for the first eigenvalue λ_{L_0} for symmetric operators L_0. The advantage of formula (2.8) in comparison with (2.3) is the possibility by means of an appropriate choice of a vector $f(x)$ in (2.8) (instead of the choice of a scalar function $\nu(x)$ in (2.3) one to find out a lower bound for the first eigenvalue λ_{L_0}.

Proof of theorem 1. For arbitrary vector $f(x) \in F$ we get from (2.8) the inequalities

$$\lambda_{L_0} = \inf_\nu \int_\Omega \left(a_j^k \nu_{x_j}\nu_{x_k} + 2d^j \nu\nu_{x_j} + (f^j\nu^2)_{x_j} + b^0\nu^2\right)\, dx$$

$$= \inf_\nu \int_\Omega \left\{a_j^k \left[\nu_{x_j} + \alpha_j^m(d^m + f^m)\nu\right]\left[\nu_{x_k} + \alpha_k^s(d^s+f^s)\nu\right]\right.$$

$$\left. + \left[b^0 + \operatorname{div} f - \alpha_j^k(f^j+d^j)(f^k+d^k)\right]\nu^2\right\}\, dx \geq \inf_\nu \int_\Omega \sigma_{L_0}(f)\nu^2\, dx$$

i.e.

$$\lambda_{L_0} \geq \sup_f \inf_\nu \int_\Omega \sigma_{L_0}(f)\nu^2\, dx \geq \sup_f \operatorname{ess}\inf_{x\in\Omega} \sigma_{L_0}(f) = \sigma_{L_0}, f\in F, \nu \in H_0^1(\Omega). \tag{2.9}$$

In order to prove the opposite inequality we will use a special choice of f.

Let us assume that a_j^k, $d^j \in W^{1,\infty}(\Omega_1)$, $b^0 \in L^\infty(\Omega_1)$ are extended in a wider smooth domain $\Omega_1 \supset \Omega$ preserving (1.2). For every positive constant $\delta > 0$, there exists a smooth domain Ω_δ, $\Omega_1 \supset \Omega_\delta \supset \overline{\Omega}$ such that $\lambda_{L_0}(\Omega_\delta) \geq \lambda_{L_0} - \delta$. Let u^δ be the first eigenfunction of L_0 in Ω_δ. From the Sobolev's imbedding theorems (theorem 5, sec. 5.6.2 in [8]) it follows that $a_j^k \in C(\overline{\Omega}_1)$ so that $u_\delta \in W_{loc}^{2,p}(\Omega_1)$ for every $p > 1$ and hence $u^\delta \in C^1(\overline{\Omega})$. Since $u^\delta > 0$ in $\overline{\Omega}$ and $L_0u^\delta = \lambda_{L_0}(\Omega_\delta)u^\delta$ easy calculations give us that $\overline{f} = -a_j^k u_{x_k}^\delta/u^\delta - d^j \in F$ so that we get from (2.7), (2.9) the estimates

$$\sigma_{L_0} \geq \operatorname{ess}\inf_{x\in\Omega} \sigma_{L_0}(\overline{f}) = \operatorname{ess}\inf_{x\in\Omega}\left[b^0 - \operatorname{div} d - \left((a_j^s u_{x_s}^\delta)_{x_j}/u^\delta\right)\right.$$

$$\left. + \left(a_j^k u_{x_k}^\delta u_{x_j}^\delta/(u^\delta)^2\right) - \left(\alpha_j^k(a_j^m u_{x_m}^\delta)(a_j^s u_{x_s}^\delta)/(u^\delta)^2\right)\right]$$

$$= \operatorname{ess}\inf_{x\in\Omega}\left[(-(a_j^k u_{x_k}^\delta + d^j u^\delta)_{x_j} + d^j u_{x_j}^\delta + b^0u^\delta)/u^\delta\right] = \lambda_{L_0}(\Omega_\delta) \geq \lambda_{L_0} - \delta.$$

After the limit $\delta \to 0$ the inequality $\sigma_{L_0} \geq \lambda_{L_0}$ holds which together with (2.8) proves theorem 2.1 . QED

By the way, from (2.9) we got a different formula for λ_{L_0} which in some sense is an intermediate one between (2.2) and (2.8).

COROLLARY 2.2. *Suppose the operator L_0 satisfies (1.2) and (1.3). Then the following identity is true*

$$\lambda_{L_0} = \sup_f \inf_\nu \int_\Omega \sigma_{L_0}(f)\nu^2\,dx, \quad f \in F, \quad \nu \in H_0^1(\Omega), \quad ||\nu||_{L^2} = 1. \tag{2.10}$$

We will finish this section with some comments about the regularity assumptions of the coefficients of L_0 in (2.1). As it is well-known the variational formula (2.2) is valid for $L^\infty(\Omega)$ coefficients of L_0. For more regular coefficients, for example satisfying (1.3), both of the notations λ_{L_0} and σ_{L_0} are equivalent according to theorem 2.1 However, it is not clear whether they give one and the same result for L^∞ coefficients of L_0. The answer of this question is deeply related with the continuous dependence of λ_{L_0} and σ_{L_0} with respect to the coefficients. For completeness, in the following proposition we formulate the qualitative properties of λ_{L_0} which will be used for nonsymmetric operators in section 4.

PROPOSITION 2.3. *Let L_0 satisfy (1.2) and a_j^k, d^j, $b^0 \in L^\infty(\Omega)$. Then*

i) λ_{L_0} is a continuous function of the coefficients a_j^k, d^j, b^0 and Ω in the L^∞ norm;

ii) λ_{L_0} is a monotone increasing function with respect to $\{a_j^k\}$, b^0, monotone decreasing on the domain inclusions and a concave one with respect to the coefficients a_j^k, d^j, b^0.

Proof. i) Let a_j^k, $\overline{a}_j^k$, d^j, $\overline{d}^j$, b^0, $\overline{b}^0 \in L^\infty(\Omega)$ satisfy the estimates

$$||a_j^k - \overline{a}_j^k||_{L^\infty} < \epsilon, \quad ||d^j - \overline{d}^j||_{L^\infty} < \epsilon, \quad ||b^0 - \overline{b}^0||_{L^\infty} < \epsilon, \tag{2.11}$$

$$||a_j^k||_{L^\infty},\ ||\overline{a}_j^k||_{L^\infty},\ ||d^j||_{L^\infty},\ ||\overline{d}^j||_{L^\infty},\ ||b^0||_{L^\infty},\ ||\overline{b}^0||_{L^\infty} \leq K_1$$

where ϵ is an arbitrary positive constant. For convenience we will denote with λ, ϕ and $\overline{\lambda}$, $\overline{\phi}$ the first eigenvalue and the first eigenfunction of the operator L_0, respectively, $\overline{L}_0$, where

$$\overline{L}_0 = -(\overline{a}_j^k u_{x_k} + \overline{d}^j u)_{x_j} + \overline{d}^j u_{x_j} + \overline{b}^0 u. \tag{2.12}$$

If $\psi \in H_0^1(\Omega)$ is some fixed function then from (2.2), theorem 2.1 and (2.6) with $f = 0$ we get the inequalities

$$\operatorname{ess}\inf_{x\in\Omega}(b^0 - \alpha_j^k d^j d^k) \leq \lambda \leq B_{L_0}[\psi,\psi], \quad \operatorname{ess}\inf_{x\in\Omega}(\overline{b}^0 - \overline{\alpha}_j^k \overline{d}^j \overline{d}^k) \leq \overline{\lambda} \leq B_{\overline{L}_0}[\psi,\psi],$$

i. e.

$$|\lambda|,\ |\overline{\lambda}| \leq K_2 \tag{2.13}$$

where the constant K_2 depends on K_1, Ω, ψ, n and the ellipticity constant μ but is independent of ϵ.

Easy calculations give us from (1.2), (2.11) , (2.13) and the unit L^2 norm of ϕ, $\overline{\phi}$ the estimate

$$\|\phi\|_{H_0^1(\Omega)}, \quad \|\overline{\phi}\|_{H_0^1(\Omega)} \leq K_3 \tag{2.14}$$

with K_3 depending on K_1, Ω, n and μ.

Finally, from (2.2) and (2.14) we have

$$\lambda \leq B_{L_0}[\overline{\phi}, \overline{\phi}] = B_{\overline{L}_0}[\overline{\phi}, \overline{\phi}]$$

$$+ \int_\Omega \left[(a_j^k - \overline{a}_j^k) \overline{\phi}_{x_j} \overline{\phi}_{x_k} + 2(d^j - \overline{d}^j) \overline{\phi} \overline{\phi}_{x_j} + (b^0 - \overline{b}^0) \overline{\phi}^2 \right] dx \leq \overline{\lambda} + \epsilon K_4$$

where K_4 depends only on K_i, Ω, n and μ. In the same way an estimate from below can be obtained which proves the first part of proposition 2.3.

The continuous dependence and the monotonicity of λ_{L_0} with respect to the domain Ω is well-known even under weaker assumptions and we omit the proof.

ii) The concavity of the first eigenvalue with respect to the coefficient b^0 was proved in proposition 2.1 in [4] for general nonsymmetric operators. For completeness we give here the proof.

If $0 < t < 1$ then for the operator $T = tL_0 + (1-t)\overline{L}_0$, $\overline{L}_0$ is given in (2.12), we get from (2.2) the estimates

$$\lambda_T = \inf_\nu B_T[\nu, \nu] = \inf_\nu (tB_{L_0}[\nu, \nu] + (1-t)B_{\overline{L}_0}[\nu, \nu])$$

$$\geq t \inf_\nu B_{L_0}[\nu, \nu] + (1-t) \inf_\nu B_{\overline{L}_0}[\nu, \nu] = t\lambda_{L_0} + (1-t)\lambda_{\overline{L}_0}.$$

$\nu \in H_0^1(\Omega)$, $\|\nu\|_{L_2} = 1$.

As for the monotonicity of λ_{L_0} with respect to $\{a_j^k\}$, b^0 let us suppose that $\{a_j^k\} \geq \{\hat{a}_j^k\}$, $b^0 \geq \hat{b}^0$. Then from (2.2) we have as above for the operators L_0, $\hat{L}_0$, $\hat{L}_0 u = -(\hat{a}_j^k u_{x_k} + d^j u)_{x_j} + d^j u_{x_j} + \hat{b}^0 u$, the inequalities

$$\lambda_{L_0} = \inf_\nu B_{L_0}[\nu, \nu] \geq \inf_\nu B_{\hat{L}_0}[\nu, \nu] = \lambda_{\hat{L}_0}, \quad \nu \in H_0^1(\Omega), \quad \|\nu\|_{L_2} = 1$$

which proves proposition 2.3. ◻

As a consequence of proposition 2.3 we get the following.

Corollary 2.4. *Let L_0 satisfy (1.2) and a_j^k, d^j, $b^0 \in L^\infty(\Omega)$. Then $\lambda_{L_0} = \sigma_{L_0}$ and the maximum principle for L_0 holds iff $\sigma_{L_0} > 0$.*

Proof. Since (2.9) is valid without changes for coefficients a_j^k, d^j, $b^0 \in L^\infty(\Omega)$ then the inequality $\lambda_{L_0} \geq \sigma_{L_0}$ holds. In order to prove the opposite inequality we will use the following matrix lemma.

Lemma 2.5. *Let $P(x) = \{p_j^k(x)\}$, $Q(x) = \{q_j^k(x)\}$ be strictly positive and symmetric matrices and $P > Q$. Then for all vectors $p(x) = (p^1(x), \cdots, p^n(x))$, $q(x) = (q^1(x), \cdots, q^n(x))$ the inequality*

$$< P^{-1}p, p > \leq < Q^{-1}q, q > + < (P-Q)^{-1}(p-q), p-q >$$

holds in Ω.

Proof. From the trivial inequality

$$< P^{-1}p, p > -2 < p, q > + < Pq, q > = < P^{-1}(p - Pq), p - Pq > \geq 0$$

valid for all vectors p, q we get the estimates

$$< P^{-1}p, p > = \sup_{\xi \in R^n} (2 < \xi, p > - < P\xi, \xi >)$$

$$= \sup_{\xi \in R^n} (2 < \xi, q > - < Q\xi, \xi > +2 < p - q, \xi > - < (P - Q)\xi, \xi >)$$

$$\leq \sup_{\xi \in R^n} (2 < \xi, q > - < Q\xi, \xi >) + \sup_{\xi \in R^n} (2 < p - q, \xi > - < (P - Q)\xi, \xi >)$$

$$= < Q^{-1}q, q > + < (P - Q)^{-1}(p - q), p - q > .$$

□

Now to finish the proof of corollary 2.4 we choose a smooth approximation $a_j^{k,\epsilon}$, $d^{j,\epsilon} \in W^{1,\infty}(\Omega)$ of the coefficients of L_0, $a_j^{k,\epsilon} \to a_j^k$, $d^{j,\epsilon} \to d^j$ when $\epsilon \to 0$ in the L^∞ norm . Moreover, we suppose that the following inequalities are satisfied

$$a^\epsilon = \{a_j^{k,\epsilon}\} < a = \{a_j^k\}, \ ||a - a^\epsilon||_{L^\infty} \geq K_5\epsilon, \ ||d - d^\epsilon||_{L^\infty} \leq K_5\epsilon$$

for every $\epsilon > 0$ with a constant K_5 independent of ϵ. From i) and Lemma 2.5 with $P = a$, $Q = a^\epsilon$, $p = f + d$, $q = f + d^\epsilon$ where $f \in F$ we obtain the estimates

$$\lambda_{L^\epsilon} = \sigma_{L^\epsilon} = \sup_f \operatorname{ess} \inf_{x \in \Omega} \left(b^0 + \text{ div } f - < (a^\epsilon)^{-1}(f + d^\epsilon), f + d^\epsilon >\right)$$

$$\leq \sup_f \operatorname{ess} \inf_{x \in \Omega} \left(b^0 + \text{ div } f - < \alpha(f + d), f + d > + < (a - a^\epsilon)^{-1}(d - d^\epsilon), d - d^\epsilon >\right)$$

$$\leq \sup_{x \in \Omega} < (a - a^\epsilon)^{-1}(d - d^\epsilon), d - d^\epsilon > + \sup_f \operatorname{ess} \inf_{x \in \Omega} \left(b^0 + \operatorname{div} f - < \alpha(f + d), f + d >\right)$$

$$= \sup_{x \in \Omega} < (a - a^\epsilon)^{-1}(d - d^\epsilon), d - d^\epsilon > + \sigma_{L_0} \leq K_6\epsilon + \sigma_{L_0}$$

where the constant K_6 is independent of ϵ.

After the limit $\epsilon \to 0$ from i) we get the inequality $\lambda_{L_0} \leq \sigma_{L_0}$ which proves corollary 2.4 □

Another consequence of proposition 2.3 is the following monotonicity result of the first eigenvalue.

Corollary 2.6. *Let $L_0, \overline{L}_0$ satisfy (1.2), (1.3). If $a > \overline{a}$, $a = \{a_j^k\}$, $\overline{a} = \{\overline{a}_j^k\}$ and $b^0 \geq \overline{b}^0 - < (a - \overline{a})^{-1}(d - \overline{d}), d - \overline{d} >$ in Ω, then $\lambda_{L_0} \geq \lambda_{\overline{L}_0}$.*

Proof. From the equality $\frac{1}{2}L_0 = \frac{1}{2}\overline{L}_0 + \frac{1}{2}(L_0 - \overline{L}_0)$, the concavity of the first eigenvalue and the assumptions of 2.6 we get the chain of inequalities

$$\frac{1}{2}\lambda_{L_0} = \lambda_{\frac{1}{2}L_0} \geq \frac{1}{2}\lambda_{\overline{L}_0} + \frac{1}{2}\lambda_{L_0 - \overline{L}_0}, \text{ i.e.}$$

$$\lambda_{L_0} - \lambda_{\overline{L}_0} \geq \lambda_{L_0 - \overline{L}_0}$$

$$= \sup_{f \in F} \operatorname{ess} \inf_{x \in \Omega} \Big(b^0 - \overline{b}^0 + \operatorname{div} f - < (a - \overline{a})^{-1}(f + d - \overline{d}), f + d - \overline{d} > \Big)$$

$$\geq \operatorname{ess} \inf_{x \in \Omega} \Big(b^0 - \overline{b}^0 - < (a - \overline{a} >^{-1} (d - \overline{d}), d - \overline{d} > \Big) \geq 0.$$

□

3. Nonsymmetric operators. For general nonsymmetric operators (1.1) an equivalent definition of the first eigenvalue of L by means of (2.8) as in theorem 2.1 is not more possible. The corresponding expression for σ_L is a little bit more complicated. The idea is one to consider all operators L_z derived by L with a nondegenerate transformation $L_z u = e^{-z/2} L(u e^{z/2})$ for every $z \in C^{0,1}(\overline{\Omega})$ which preserve the first eigenvalue of L, i.e. $\lambda_L = \lambda_{L_z}$. There exists a transformation with the extreme property that the new transformed nonsymmetric operator L_z has the same first eigenvalue as its symmetric part $L_{z,0} = \frac{1}{2}(L_z + L_z^*)$ (see theorems 3.1 and 4.1). Thus theorem 2.1 is applicable for $L_{z,0}$ as well as for L_z.

Now for every Lipschitz function $z \in C^{0,1}(\overline{\Omega})$ we have

$$L_{z,0} = \frac{1}{2} \Big[(e^{-z/2} L(u e^{z/2}) + e^{z/2} L^*(u e^{-z/2}) \Big] \tag{3.1}$$

$$= -(a_j^k u_{x_k} + d^j u)_{x_j} + d^j u_{x_j} + (b^0 + c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k}) u$$

$$= L_0 u + (c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k}) u$$

and the natural definition of σ_L for nonsymmetric operators L is by means of $\sigma_{L_{z,0}}$ for the symmetric operator $L_{z,0}$. For this purpose for every $z \in C^{0,1}(\overline{\Omega})$, $f \in F$ we introduce the notations

$$\sigma_L(f, z) = b^0 + \operatorname{div} f - \alpha_j^k (f^j + d^j)(f^k + d^k) + c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \tag{3.2}$$

and

$$\sigma_L = \sup_{z,f} \operatorname{ess} \inf_{x \in \Omega} \sigma_L(f, z), \; z \in C^{0,1}(\overline{\Omega}), \; f \in F. \tag{3.3}$$

The following theorem gives the relation between the first eigenvalue λ_L of the nonsymmetric operator L and the first eigenvalues of the family of symmetric operators $L_{z,0}$ defined in (3.1).

THEOREM 3.1. *Let the nonsymmetric operator L satisfies (1.2) and (1.3). Then $\sigma_L = \lambda_L$ and hence the maximum principle for L holds if and only if $\sigma_L > 0$. Moreover, the identity*

$$\sigma_L = \sup_z \lambda_{L_{z,0}} = \sup_z \inf_\nu B_{L_{z,0}}[\nu, \nu], \; z \in C^{0,1}(\overline{\Omega}), \; \nu \in H_0^1(\Omega), \; ||\nu||_{L^2} = 1 \tag{3.4}$$

is satisfied.

Proof. Let ϕ be the first eigenfunction of L in Ω, $\phi = 0$ on $\partial\Omega$. For arbitrary $z \in C^{0,1}(\overline{\Omega})$, $w = Ke^{-z/2}\phi$, $K = \left(\int_\Omega e^{-z}\phi^2\,dx\right)^{-1/2}$ we have from the variational formula (2.2) for the symmetric operator $L_{z,0}$ the inequalities

$$\lambda_{L_{z,0}} = \inf_\nu B_{L_{z,0}}[\nu,\nu] \le B_{L_{z,0}}[w,w] = \lambda_L,\ \nu \in H_0^1(\Omega),\ \|\nu\|_{L^2} = 1.$$

Hence, from theorem 2.1 and (3.2), (3.3) the estimate

$$\lambda_L \ge \sigma_L = \sup_z \lambda_{L_{z,0}},\ z \in C^{0,1}(\overline{\Omega}) \tag{3.5}$$

holds.

In order to prove the opposite inequality suppose that the coefficients of L are extended in a small neighborhood of Ω satisfying (1.2) and (1.3). Let us consider a sequence Ω_j of C^∞ smooth domains, $\Omega_j \supset \overline{\Omega}$, $\Omega_j \supset \Omega_{j+1}$, $\lambda_L(\Omega) = \lim_{j\to\infty} \lambda_L(\Omega_j)$, where $\lambda_L(\Omega_j)$ are the first eigenvalues of L in Ω_j. If $\phi^j > 0$, $\psi^j > 0$ are the first eigenfunctions of L and L^* , respectively, in Ω_j we consider the functions

$$z^j = \log(\phi^j/\psi^j) \quad \text{for } x \in \Omega,\ j = 1,2,\cdots. \tag{3.6}$$

Since $\phi^j, \psi^j \in W^{2,p}_{loc}(\Omega_j)$ for every $1 < p < \infty$ it follows that $z^j \in C^{0,1}(\overline{\Omega})$ as well as $\nu^j = (\phi^j\psi^j)^{1/2} \in W^{2,p}(\Omega)$.

Simple computations give us from (3.1) the identity

$$L_{z^j,0}\nu^j = \lambda_L(\Omega_j)\nu^j \text{ in } \Omega,\quad j = 1,2,\cdots. \tag{3.7}$$

Since $\nu^j > 0$ in $\overline{\Omega}$ it follows from corollary 2.1 in [4] that, $\lambda_{L_{z^j,0}}(\Omega) \ge \lambda_L(\Omega_j)$ where $\lambda_{L_{z^j,0}}(\Omega)$ is the first eigenvalue of $L_{z^j,0}$ in Ω. Hence $\sup_z \lambda_{L_{z,0}}(\Omega) \ge \lambda_L(\Omega_j)$ for every $j = 1,2,\cdots$, which after the limit $j \to \infty$ proves theorem 3.1 $\square$

Using theorem 3.1 we will give some new expressions for σ_L or equivalently of λ_L which are useful for the investigations of the qualitative properties of λ_L in section 4.

PROPOSITION 3.2. *Let the operator L satisfy (1.2), (1.3) and $b^j \in W^{1,\infty}(\Omega)$. Then the identity*

$$\lambda_L = \sigma_L = \sup_z \mathit{ess}\inf_{x\in\Omega} \left[b^0 + \ \mathit{div}\ f + \alpha_j^k c^j c^k - \alpha_j^k(f^j + d^j)(f^k + d^k)\right],\ z \in Z \tag{3.8}$$

holds, where $f^j = \pm c^j - d^j + a_j^k z_{x_k}$, $Z = \{z \in C^{0,1}(\overline{\Omega}); (\pm c^j - d^j + a_j^k z_{x_k})_{x_j} \in L^\infty(\Omega)\}$.

REMARK 2. For the special choice of f and z in (3.8), $f^j = c^j - d^j = -a_j^0$, $z = 0$ we get immediately from (3.8) sufficient condition $(1.7)_i$ and for $f^j = -c^j - d^j = -b^j$, $z = 0$, respectively, condition $(1.7)_{ii}$.

Proof. of proposition 3.2 Let $u \in C^\infty(\overline{\Omega})$ be an arbitrary positive in $\overline{\Omega}$ function. For $z \in -\log u$ and $f^j = \pm c^j - d^j + a_j^k z_{x_k}$ we get from (3.2) and (2.3) the equalities

$$\sup_z \operatorname{ess}\inf_{x\in\Omega} \left[b^0 + \ \operatorname{div} f - \alpha_j^k(f^j + d^j)(f^k + d^k) + \alpha_j^k c^j c^k\right]$$

$$= \sup_u \operatorname{ess}\inf_{x\in\Omega} \Bigg[b^0 - \operatorname{div} d - \frac{1}{u}(a_j^k u_{x_k})_{x_j} + \frac{1}{u^2} a_j^k u_{x_j} u_{x_k} \pm \operatorname{div} c$$

$$-\alpha_j^k c^j c^k \pm \frac{2}{u} c^j u_{x_j} - \frac{1}{u^2} a_j^k u_{x_j} u_{x_k} + \alpha_j^k c^j c^k\Big]$$

$$= \sup_u \operatorname{ess} \inf_{x\in\Omega} \left[(a_j^k u_{x_k} + (d^j \mp c^j)u)_{x_j} + (d^j \pm c^j)u_{x_j} + b^0 u\right] \frac{1}{u}$$

$$\leq \sup_w \operatorname{ess} \inf_{x\in\Omega} \left[(a_j^k w_{x_k} + (d^j \mp c^j)w)_{x_j} + (d^j \pm c^j)w_{x_j} + b^0 w\right] \frac{1}{w} = \lambda_L, \;\; w \in W^{2,n}(\Omega)$$

because $-\left[a_j^k u_{x_k} + (d^j \mp c^j)u\right]_{x_j} + (d^j \pm c^j)u_{x_j} + b^0 u$ is equal to Lu or $L^* u$, respectively.

In order to prove the opposite inequality we will use the notations in the proof of theorem 3.1 If $\phi^i, \lambda_L(\Omega_i)$ are the first eigenfunction and the first eigenvalue, respectively, of L in Ω_i, $\Omega_i \supset \overline{\Omega}$, $\lambda_L(\Omega_i) \to \lambda_L(\Omega)$, then $z^i = -\log \phi^i \in Z$.

Repeating the above calculations for $\overline{f}^j = c^j - d^j - a_j^k (\log \phi^i)_{x_k}$ we get the inequality

$$\lambda_L(\Omega_i) = b^0 + \operatorname{div}\overline{f} - \alpha_j^k(\overline{f}^j + d^j)(\overline{f}^k + d^k) + \alpha_j^k c^j c^k$$

$$\leq \sup_z \operatorname{ess} \inf_{x\in\Omega} \left[b^0 + \operatorname{div} f - \alpha_j^k (f^j + d^j)(f^k + d^k) + \alpha_j^k c^j c^k\right], \;\; z \in Z$$

and after the limit $i \to \infty$ we obtain

$$\lambda_L(\Omega) \leq \sup_z \operatorname{ess} \inf_{x\in\Omega} \left[b^0 + \operatorname{div} f - \alpha_j^k (f^j + d^j)(f^k + d^k) + \alpha_j^k c^j c^k\right], \;\; z \in Z$$

which proves proposition 3.2. □

For the last formula for λ_L in proposition 3.4 we need an information about the boundary behaviour of the first eigenfunction of an arbitrary uniformly elliptic operator L. For completeness the following lemma is formulated

LEMMA 3.3. *Suppose l satisfies (1.2) , (1.3). Then for every function $\nu(x) \in C^1(\overline{\Omega})$, $\nu = 0$ on $\partial\Omega$, there exists a constant $K_0 > 0$ such that $\nu \leq K_0 \phi$ in $\overline{\Omega}$, where ϕ is the first eigenfunction of L with zero data on $\partial\Omega$.*

Sketch of the proof. From the boundary regularity of the weak solutions (see ch. 9 in [9]) it follows that $\phi \in W^{2,p}(\Omega)$ for every $p > 1$ and from the Sobolev's imbedding theorems $\phi \in C^1(\overline{\Omega})$.

The next step is to prove that

$$\frac{\partial \phi}{\partial l} \leq -K_7, \quad \text{on } \partial\Omega, \quad K_7 > 0, \tag{3.9}$$

where l is the unit outer normal to $\partial\Omega$. For this purpose one can easily check that ϕ is a positive supersolution of the operator $L + K$, where K is a sufficiently large positive constant such that $K + b^0 - \operatorname{div} a^0 \geq 0$, $K + \lambda_L \geq 0$, i.e. $(L + K)\phi = (\lambda_L + K)\phi \geq 0$ in Ω. The rest of the proof of (3.9) follows as in the proof of the Hopf's maximum principle for classical supersolutions (see for example [2], [9], [12])using the weak maximum principle, th. 8.1 in [9] or the strong maximum principle, th. 8.19 in [9], (see also the results in [5], [10]).

Since the function $K_0\phi - \nu$, for K_0 large enough, is positive in every compact subdomain of Ω and $\dfrac{\partial(K_0\phi-\nu)}{\partial l} < 0$ in a neighborhood of $\partial\Omega$, $K_0\phi-\nu = 0$ on $\partial\Omega$, lemma (3.3) is proved. QED

PROPOSITION 3.4. *Let the operator L satisfy (1.2), (1.3). Then the identity*

$$\lambda_L = \inf_{\nu,h} B_{L^h}[\nu,\nu],\ \nu\in H_0^1(\Omega),\ \|\nu\|_{L^2}=1,\ h\in H_\nu \tag{3.10}$$

holds, where $H_\nu = \{(h^1,\cdots,h^n); h^j\nu^2\in H_0^1(\Omega),$ *and* $div(h\nu^2)=0$*for a.e.* $x\in\Omega\}$, $L^h = L_0 + \alpha_j^k(c^j-h^j)(c^k-h^k)$. *Moreover, the infimum in (3.10) is attained for* $\overline{\nu} = K(\phi\psi)^{1/2}$, $K = \left(\int_\Omega \phi\psi\,dx\right)^{-1/2}$, $\overline{h}^j = c^j - \frac{1}{2}a_j^k\overline{z}_{x_k}$, $\overline{z} = \log(\phi/\psi)$ *where* ϕ, ψ *are the first eigenfunctions of* L *and* L^*, *respectively.*

Proof. From the regularity theory (see for example ch. 9 in [9]) and the Sobolev's imbedding theorems it follows that $\phi,\psi\in W^{2,p}(\Omega)\cap C^1(\overline{\Omega})$ for every $1<p<\infty$ so that $\overline{\nu}\in H_0^1(\Omega)$ because from lemma 3.3, $0<k_1\le\phi/\psi\le k_2<\infty$ in Ω, $k_i = const.$ Analogously for $\overline{h}^j = c^j - \dfrac{1}{2}a_j^k\overline{z}_{x_k}$ we obtain that $\overline{h}^j\overline{\nu}^2\in H_0^1(\Omega)$. Moreover, simple computations give us the following identities in Ω

$$\begin{aligned}\operatorname{div}(\overline{h}\overline{\nu}^2) &= \left[c^j\phi\psi - \frac{1}{2}\alpha_j^k(\psi\phi_{x_k}-\phi\psi_{x_k})\right]_{x_j}\\ &= -\frac{1}{2}\psi(a_j^k\phi_{x_k})_{x_j} + \frac{1}{2}\phi(a_j^k\psi_{x_k})_{x_j} + (c^j\phi\psi)_{x_j}\\ &= \frac{1}{2}\psi\left[(\lambda_L - b^0 + \operatorname{div}d)\phi - \frac{1}{\phi}\operatorname{div}(c\phi^2)\right]\\ +\frac{1}{2}\phi&\left[(\lambda_L - b^0+\operatorname{div}d)\psi - \frac{1}{\psi}\operatorname{div}(c\psi^2)\right] + (c^j\phi\psi)_{x_j} = 0.\end{aligned} \tag{3.11}$$

Hence, $\overline{h}^j = c^j - \dfrac{1}{2}a_j^k\overline{z}_{x_k}\in H_{\overline{\nu}}$, $\overline{\nu} = K(\phi\psi)^{1/2}\in H_0^1(\Omega)$, $\|\overline{\nu}\|_{L^2}=1$ so that we get the inequalities

$$\begin{aligned}\inf_{\nu,h} B_{L^h}[\nu,\nu] &\le B_{L^{\overline{h}}}[\overline{\nu},\overline{\nu}] = B_{L_0}[\overline{\nu},\overline{\nu}] + \frac{K^2}{4}\int_\Omega a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\phi\psi\,dx\\ = B_{L_0}[\overline{\nu},\overline{\nu}] &+ K^2\int_\Omega\left[c^j\overline{z}_{x_j} - \frac{1}{4}a_j^k\overline{z}_{x_j}\overline{z}_{x_k} - c^j\overline{z}_{x_j} + \frac{1}{2}a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\right]\phi\psi\,dx\\ = B_{L_0}[\overline{\nu},\overline{\nu}] &+ K^2\int_\Omega\left[c^j\overline{z}_{x_j} - \frac{1}{4}a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\right]\phi\psi\,dx = B_{L_{\overline{z},0}}[\overline{\nu},\overline{\nu}] = \lambda_L.\end{aligned} \tag{3.12}$$

Here the operator $L_{z,0}$ is defined in (3.1) and as in (3.7) one can check that $L_{\overline{z},0}\overline{\nu} = \lambda_L\overline{\nu}$ in Ω. Moreover, in the above calculations we used the equality

$$\int_\Omega\left[-c^j + \frac{1}{2}a_j^k\overline{z}_{x_k}\right]\phi\psi\overline{z}_{x_j}\,dx = -\int_\Omega\left[\left(-c^j+\frac{1}{2}a_j^k\overline{z}_{x_k}\right)\phi\psi\right]_{x_j}\overline{z}\,dx = 0$$

which follows from (3).

In order to prove the opposite inequality we will use (3.4) and theorem (3.1), i.e. $\lambda_L = \sup\limits_z \lambda_{L_{z,0}} = \sup\limits_z\inf\limits_\nu B_{L_{z,0}}[\nu,\nu]$, $z\in C^{0,1}(\overline{\Omega})$, $\nu\in H_0^1(\Omega)$, $\|\nu\|_{L^2}=1$.

From the trivial inequality $c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \le h^j z_{x_j} + \alpha_j^k (h^j - c^j)(h^k - c^k)$ after multiplication by ν^2 and integration in Ω we get the inequality

$$\int_\Omega \left(c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \right) \nu^2 \, dx \le \int_\Omega h^j z_{x_j} \nu^2 \, dx + \int_\Omega \alpha_j^k (h^j - c^j)(h^k - c^k) \nu^2 \, dx$$

$$= \int_\Omega \alpha_j^k (h^j - c^j)(h^k - c^k) \nu^2 \, dx$$

if $h \in H$, because

$$\int_\Omega h^j \nu^2 z_{x_j} \, dx = - \int_\Omega z \, \operatorname{div}(h \nu^2) \, dx = 0.$$

Since the above estimate is valid for every $z \in C^{0,1}(\overline{\Omega})$ and every $h \in H_\nu$ it follows that

$$\sup_z \int_\Omega \left(c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \right) \nu^2 \, dx \le \inf_h \int_\Omega \alpha_j^k (h^j - c^j)(h^k - c^k) \nu^2 \, dx.$$

Finally, from the chain of inequalities

$$\lambda_L = \sup_z \inf_\nu B_{L_{z,0}}[\nu, \nu] \le \inf_\nu \left[B_{L_0}[\nu, \nu] + \sup_z \int_\Omega \left(c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \right) \nu^2 \, dx \right]$$

$$\le \inf_{\nu, h} \left(B_{L_0}[\nu, \nu] + \int_\Omega \alpha_j^k (h^j - c^j)(h^k - c^k) \nu^2 \, dx \right) = \inf_{\nu, h} B_{L^h}[\nu, \nu]$$

we obtain the estimate $\lambda_L \le \inf\limits_{\nu, h} B_{L^h}[\nu, \nu]$ and with (3.12) the desired result in (3.5).

The last statement of proposition 3.4 follows immediately from (3.10) since $\lambda_L = \inf\limits_{\nu, h} B_{L^h}[\nu, \nu]$. □

By the way, the last chain of inequalities in the proof of proposition 3.4 gives another formula for λ_L which can be combine with (33) in the following way (see also the results in [6]).

Corollary 3.5. *Let the operator satisfies (1.2), (1.3). Then the equality*

$$\lambda_L = \inf_\nu \left(B_{L_0}[\nu, \nu] + \beta(\nu^2) \right) \quad \nu \in H_0^1(\Omega), \ \|\nu\|_{L^2} = 1 \quad \textit{holds} \ ,$$

$$\textit{where} \qquad \beta(\nu^2) = \sup_z \int_\Omega \left(c^j z_{x_j} - \frac{1}{4} a_j^k z_{x_j} z_{x_k} \right) \nu^2 \, dx, \ \ z \in C^{0,1}(\overline{\Omega})$$

$$\textit{or} \qquad \beta(\nu^2) = \inf_h \int_\Omega \alpha_j^k (h^j - c^j)(h^k - c^k) \nu^2 \, dx, \ \ h \in H_\nu.$$

Repeating the same argument as in the proof of proposition 3.4 one can easily find out the functions ν and z for which the extremums in formula (3.4) are attained.

COROLLARY 3.6. *Under the assumptions of proposition 3.4 the infimum and the supremum in (3.4) is attained for* $\overline{\nu} = K(\phi\psi)^{1/2}$, $K = \left(\int_\Omega \phi\psi \, dx\right)^{-1/2}$, $\overline{z} = \log(\phi/\psi)$, *where* ϕ, ψ *are the first eigenfunctions of* L *and* L^*, *respectively.*

Note that $\overline{\nu} \in H_0^1(\Omega)$, $\overline{z} \notin C^{0,1}(\overline{\Omega})$ but formula (3.4) is true for $\nu = \overline{\nu}$, $z = \overline{z}$.

We will finish this section with a simple but useful remark about the case of operators L with a nonsymmetric principle symbol. More precisely, we consider the operator

$$\overline{L}u = -\left(\overline{a}_j^k u_{x_k} + a_j^0 u\right)_{x_j} + b^j u_{x_j} + b^0 u \tag{3.13}$$

with a nonsymmetric matrix $\overline{a}_j^k(x)$, $\overline{a}_j^k(x) \neq \overline{a}_k^j(x)$ for some $j \neq k$.

Let us introduce the following notations $a_j^k = \frac{1}{2}\left(\overline{a}_j^k + \overline{a}_k^j\right)$, $t_j^k = \frac{1}{2}\left(\overline{a}_j^k - \overline{a}_k^j\right)$ so that $a_j^k = a_k^j$, $t_j^k = -t_k^j$ for $j, k = 1, 2, \cdots, n$.

Using the identities $\sum_{j,k=1}^n t_j^k u_{x_j x_k} = 0$, $\sum_{j,k=1}^n (t_j^k)_{x_j x_k} = 0$ we can rewrite (3.13) in the following way

$$\overline{L}u = -\left(a_j^k u_{x_k} + t_j^k u_{x_k} + a_j^0 u\right)_{x_j} + b^j u_{x_j} + b^0 u$$

$$= -\left(a_j^k u_{x_k} + d^j u\right)_{x_j} + d^j u_{x_j} + \frac{1}{u}\operatorname{div}(cu^2) - (t_j^k)_{x_j} u_{x_k} + b^0 u$$

$$= -\left(a_j^k u_{x_k} + d^j u\right)_{x_j} + d^j u_{x_j} + b^0 u + \frac{1}{u}\left[\left(c^k - \frac{1}{2}(t_j^k)_{x_j}\right) u^2\right]_{x_k}, \text{ i. e.}$$

$$\overline{L} = -\left[a_j^k u_{x_j} + \left(a_j^0 + \frac{1}{2}(t_m^j)_{x_m}\right) u\right]_{x_j} + \left[b^j - \frac{1}{2}(t_m^j)_{x_m}\right] u_{x_j} + b^0 u. \tag{3.14}$$

Thus we have the following simple proposition.

PROPOSITION 3.7. *Suppose* $\overline{a}_j^k \in W^{2,\infty}(\Omega)$, $a_j^0 \in W^{1,\infty}(\Omega)$, $b^j, b^0 \in L^\infty(\Omega)$ *and* $\overline{a}_j^k$ *satisfies the uniform ellipticity condition (1.2). Then the first eigenvalue of* $\overline{L}$ *can be defined by means of formulas (3.3), (3.8), (3.10) for the operator (3.14).*

4. Properties of the first eigenvalue. In this section we will give some applications of theorems 2.1, 3.1 and propositions 2.3, 3.2, 3.4, 3.7 for the qualitative properties of λ_L. For this purpose let us recall the well-known monotonicity and concavity properties of λ_L with respect to b^0. More precisely, λ_L is a concave function of b^0 and when b^0 increases the first eigenvalue λ_L increases, too (see for example proposition 2.1 in [4]).

For the time being it is not known whether a similar monotonicity result for λ_L is true with respect to the matrix $\{a_j^k\}$ or coefficients a_j^0, b^j, respectively, d^j, c^j. To give some partial answer of these questions we will need the following properties of λ_L.

THEOREM 4.1. *Let the operator* L *satisfies (1.2) and (1.3). Then the inequalities*

$$\lambda_{L_0} \leq \lambda_L \leq \lambda_{L^0} \tag{4.1}$$

hold, where $L^0 = L_0 + \alpha_j^k c^j c^k$. *Moreover, if* $b^j \in W^{1,\infty}(\Omega)$ *then*

(i) $\lambda_L = \lambda_{L_0} \iff \phi = \phi_0$ *in* $\Omega \iff div(c\phi_0^2) = 0$ *in* Ω, *where* ϕ, ϕ_0 *are the first*

(4.2) *eigenfunctions of* L *and* L_0 *respectively;*

(ii) $\lambda_L = \lambda_{L_0} \iff c^j = a_j^k p_{x_k}$ *for some* $p \in W^{2,\infty}(\Omega)$ *and more precisely,* $p = \dfrac{1}{2}\log(\phi/\psi)$ *where* ψ *is the first eigenfunction of* L^*.

REMARK 3. Since $\lambda_L = \lambda_{L^*}$ condition $(4.2)_i$ can be extended in the following way

$$\lambda_L = \lambda_{L_0} \iff \phi = \phi_0 \iff \operatorname{div}(c\phi_0^2) = 0 \iff \phi = \psi \iff \operatorname{div}(c\phi^2) = 0$$

$$(4.2)_i' \qquad \iff \operatorname{div}(c\psi^2) = 0 \iff \lambda_L = \inf_\nu B_L[\nu,\nu], \ \nu \in H_0^1(\Omega), \ \|\nu\|_{L^2} = 1.$$

The following example illustrate the situation described in $(4.2)_i$:

Example 1. Consider the operator

$$Lu = -\Delta u + 2K\theta_y u_x - 2K\theta_x u_y = 0$$

in Ω, where $\Omega \subset R^2$ is a bounded smooth domain and θ is the first eigenfunction of the Laplacian, $K = const$. Since $c^1 = K\theta_y$, $c^2 = -K\theta_x$, $d^1 = d^2 = 0$ simple computations give us the identity $\operatorname{div}(c\theta^2) = K(\theta_y\theta^2)_x - K(\theta_x\theta^2)_y \equiv 0$ in Ω and hence $\lambda_L = \lambda_{L_0} = \lambda_{-\Delta}$, $\phi = \psi = \theta$ where ϕ, ψ are the first eigenfunctions of L and L^*, respectively. Note that the norm of the coefficients c^j, $|c| = K|\nabla\theta|$ increases to infinity when $K \to \infty$, but the first eigenvalue λ_L of L does not change at all. By the way, the divergence of the coefficients c^j stays constant for every K, i.e. $\operatorname{div} c = 0$ in Ω.

Proof of theorem 4.1. By integration by parts we get immediately the estimate

$$\lambda_L = B_{L_0}[\phi,\phi] \geq \inf_\nu B_{L_0}[\nu,\nu] = \lambda_{L_0}, \ \nu \in H_0^1(\Omega), \ \|\nu\| = 1.$$

Since $c^j z_{x_j} - \dfrac{1}{4}a_j^k z_{x_j} z_{x_k} \leq \alpha_j^k c^j c^k$ for every $z \in C^{0,1}(\overline{\Omega})$ we have from (27), and proposition 2.3 the inequalities $\lambda_L = \sup_z \inf_\nu B_{L_{z,0}}[\nu,\nu] \leq \inf_\nu B_{L^0}[\nu,\nu] = \lambda_{L^0}$, $z \in C^{0,1}(\overline{\Omega})$, $\nu \in H_0^1(\Omega)$, $\|\nu\|_{L^2} = 1$.

Now let us suppose that $\operatorname{div}(c\phi_0^2) = 0$ in Ω and for simplicity let us denote $\lambda_{L_0} = \lambda_0$. Since $Lu = L_0 + \dfrac{1}{u}\operatorname{div}(cu^2)$ it follows that $L\phi_0 = \lambda_0\phi_0$, $\phi_0 = 0$ on $\partial\Omega$, $\phi_0 > 0$ in Ω, i.e. from corollary 2.1 in [4] ϕ_0 is the first eigenfunction of L, $\phi = \phi_0$ and $\lambda_L = \lambda_0$.

Suppose that $\phi = \phi_0$ in Ω. Easy calculations give us the identity

$$\lambda_L\phi_0 = \lambda_L\phi = L\phi = L\phi_0 = L_0\phi_0 + \frac{1}{\phi_0}\operatorname{div}(c\phi_0^2) = \lambda_0\phi_0 + \frac{1}{\phi_0}\operatorname{div}(c\phi_0^2)$$

i.e. $(\lambda_L - \lambda_0)\phi_0^2 = \operatorname{div}\,(c\phi_0^2)$ in Ω. Integrating the above expression in Ω we get immediately that $\lambda_L = \lambda_0$ and $\operatorname{div}(c\phi_0^2) = 0$ in Ω.

Finally, let us suppose that $\lambda_L = \lambda_0$. By integration by parts we have $B_{L_0}[\phi_0,\phi_0] = \lambda_0 = \lambda_L = B_L[\phi,\phi] = B_{L_0}[\phi,\phi]$ and from theorem 2 in section 6.5 in [8] it follows that $\phi = \phi_0$.

To prove $(4.2)_{ii}$ let us suppose that $c^j = a^k_j p_{x_k}$ for some $p \in W^{2,\infty}(\Omega)$. Since the operator $e^{-p}L(ue^p) = L^0 u$ has the same first eigenvalue as the operator L we have $\lambda_L = \lambda_{L^0}$. Moreover, if ϕ^0 is the first eigenfunction of L^0 then $\phi = e^p \phi^0$, $\psi = e^{-p}\phi^0$ and $p = \dfrac{1}{2}\log(\phi/\psi)$.

Now let us suppose that $\lambda_L = \lambda_{L^0}$. From corollary 3.6 it follows that $\lambda_L = B_{L_{\overline{z},0}}[\overline{\nu},\overline{\nu}]$, where $\overline{\nu} = K(\phi\psi)^{1/2}$, $K = \left(\int_\Omega \phi\psi\, dx\right)^{-1/2}$, $\overline{z} = \log(\phi/\psi)$, so that we get the inequalities

$$\lambda_{L^0} = \lambda_L = \sup_z \inf_\nu B_{L_{z,0}}[\nu,\nu] = B_{L_{\overline{z},0}}[\overline{\nu},\overline{\nu}] \le B_{L_{\overline{z},0}}[\phi^0,\phi^0] = B_{L^0}[\phi^0,\phi^0]$$

$$-\int_\Omega \alpha^k_j (c^j - \frac{1}{2}a^m_j \overline{z}_{x_m})(c^k - \frac{1}{2}a^s_k \overline{z}_{x_s})(\phi^0)^2\, dx$$

$$= \lambda_{L^0} - \int_\Omega \alpha^k_j (c^j - \frac{1}{2}a^m_j \overline{z}_{x_m})(c^k - \frac{1}{2}a^s_k \overline{z}_{x_s})(\phi^0)^2\, dx$$

i.e.

$$0 = \int_\Omega \alpha^k_j \left(c^j - \frac{1}{2}a^m_j \overline{z}_{x_m}\right)\left(c^k - \frac{1}{2}a^s_k \overline{z}_{x_s}\right)(\phi^0)^2\, dx.$$

The positiveness of ϕ_0 and (1.2) gives the desired result $c^j = \dfrac{1}{2}a^m_j \overline{z}_{x_m}$ in Ω. QED

An open question is to characterize the conditions guaranteeing when λ_L coincides with some strictly interior point of the interval $(\lambda_{L_0}, \lambda_{L^0})$. However, the following example illustrates by means of a family of equations having one and the same "maximal operator" L^0 that the first eigenvalue λ_L covers the whole interval.

Example 2. Consider the operator

$$Lu = -\Delta u + 2(p\theta_x - q\theta_y)u_x + 2(p\theta_y + q\theta_x)u_y + \lambda_0 p u = 0$$

in Ω, where $\Omega \subset R^2$ is a bounded domain with a smooth boundary, λ_0, θ are the first eigenvalue and the first eigenfunction of the Laplacian and $p, q = const$, $p^2 + q^2 = 1$.

Since $c^1 = p\theta_x - q\theta_y$, $c^2 = p\theta_y + q\theta_x$, $d^1 = d^2 = 0$ we have $\alpha^k_j c^j c^k = |c|^2 = |\nabla\theta|^2$, i.e. the "maximal operator" $L^0 = L_0 + \alpha^k_j c^j c^k = -\Delta + |\nabla\theta|^2$ is independent of the parameters p, q. We will use also the following notations $L_t u = -\Delta u + t|\nabla\theta|^2 u$, $0 \le t \le 1$, so that $L^0 = L_1$.

For all p, q on the unit circle from $(4.1)_i$ we get the estimate $\lambda_L \le \lambda_1$. In order to obtain an estimate from below for λ_L we will use theorem 3.1 and (3.2), (3.3) i.e.

$$\lambda_L = \sigma_L = \sup_{z,f} \operatorname{ess\,inf}_x (\operatorname{div} f - |\nabla f|^2 + (p\theta_x - q\theta_y)z_x + (p\theta_y + q\theta_x)z_y - \frac{1}{4}|\nabla z|^2)$$

$$\ge \sup_f \operatorname{ess\,inf}_x (\operatorname{div} f - |\nabla f|^2 + (p\theta_x - q\theta_y)2p\theta_x + (p\theta_y + q\theta_y)2p\theta_y - p^2|\nabla\theta|^2)$$

$$= \sup_f \operatorname{ess\,inf}_x (\operatorname{div} f - |\nabla f|^2 + p^2|\nabla\theta|^2) = \lambda_{p^2},$$

where in the above inequalities the function z was replaced with $2p\theta$ and λ_t is the first eigenvalue of L_t.

Thus we proved the final estimate

$$\lambda_{p^2} \le \lambda_L \le \lambda_1 \quad \text{for every} \quad p, q, p^2 + q^2 = 1. \tag{4.3}$$

From example 1 when $K = -q$ we know that $\lambda_L = \lambda_0$ for $p = 0$, $q^2 = 1$. Using the continuous dependence of the first eigenvalue λ_L with respect to the coefficients c^j (see proposition 5 in [4]) and (4.3) we obtain that λ_L covers the whole interval $[\lambda_0, \lambda_1]$ when p^2 increases from zero to one.

Note that in this example, in a contrast to example 1, the norm of the coefficients c^j, $|c| = |\nabla\theta|^2$, stays constant for all p, q, while the divergence of c^j, $\operatorname{div} c = p\Delta\theta = -\lambda_0 p\theta$, strictly decreases when p increases from -1 to 1.

Using theorems (3.1), (4.1) we will give some partial results about the monotonicity of λ_L with respect to the matrix $\{a_j^k\}$. For this purpose we introduce the operator

$$Mu = -(m_j^k u_{x_k} + (d^j - c^j)u)_{x_j} + (d^j + c^j)u_{x_j} + b^0 u \quad \text{in} \quad \Omega.$$

Proposition 4.2. *Let the operators L and M satisfy (1.2) and (1.3). Suppose that $\{a_j^k\} \ge \{m_j^k\}$ and one of the following assumptions is satisfied:*

i) $\lambda_M = \lambda_{M_0}$. $M_0 = \dfrac{1}{2}(M + M^*)$;

$$ii) c^j = a_j^k p_{x_k} \text{ for some } p \in W^{2,\infty}(\Omega) \text{ and } \mu_j^k c^j c^k = \alpha_j^k c^j c^k, \{\mu_j^k\} = \{m_j^k\}^{-1}; \tag{4.4}$$

iii) $\{a_j^k\} \ge \{m_j^k\} + rI$, $r = const > 0$, I is the unit matrix and $\mu_j^k c^j c^k \le r\,(\omega_n/|\Omega|)^{2/n}$, where ω_n is the volume of the unit ball in R^n and $|\Omega| =$ mes Ω.

Then the inequality $\lambda_L \ge \lambda_M$ holds.

Proof. i) The proof follows immediately from (4.1) and ii) in proposition 2.3 because $\lambda_L \ge \lambda_{L_0} \ge \lambda_{M_0} = \lambda_M$.

ii) Since from $(4.2)_{ii}$ $\lambda_L = \lambda_{L^0}$ we get from ii) in proposition 2.3 and (4.1) the inequalities $\lambda_L = \lambda_{L^0} \ge \lambda_{M^0} \ge \lambda_M$ where $L^0 = L_0 + \alpha_j^k c^j c^k$, $M^0 = M_0 + \mu_j^k c^j c^k$.

iii) From (4.1) and the Poincare inequality we get the estimates

$$\lambda_M \le \lambda_{M^0} = \inf_\nu B_{M^0}[\nu, \nu]$$

$$\le \inf_\nu \int_\Omega \left(a_j^k \nu_{x_j} \nu_{x_k} + 2d^j \nu \nu_{x_j} + b^0 \nu^2 + r\,(\omega_n/|\Omega|)^{2/n}\, \nu^2 - r|\nabla\nu|^2 \right) dx$$

$$\le \lambda_{L_0} \le \lambda_L, \quad \nu \in H_0^1(\Omega), \ ||\nu||_L^2 = 1.$$

□

The following example illustrates that, without additional assumptions (4.4) only with condition $\{a_j^k\} \ge \{m_j^k\}$ the result in proposition 4.2 is not true.

Example 3. Consider in Ω the operators

$$Lu = -\Delta u + 2\sqrt{\lambda_0} u_{x_n} = 0, \ Mu = -\beta\Delta u + 2\sqrt{\lambda_0} u_{x_n} = 0,$$

where $\Omega \subset R^n$ is a bounded domain with a smooth boundary, $0 < \beta < 1$, $\beta =$ *const* and λ_0 is the first eigenvalue of the Laplacian with zero Dirichlet data. Since $L_z u = e^{-z} L(ue^z) = -\Delta u + \lambda_0 u$, $M_{z/\beta} = e^{-z/\beta} M(ue^{z/\beta}) = -\beta \Delta u + (\lambda_0/\beta) u$ and $\lambda_L = \lambda_{L_z} = 2\lambda_0$, $\lambda_M = \lambda_{M_z} = (\beta + (1/\beta))\lambda_0 > 2\lambda_0$, where $\sqrt{\lambda_0} x_n = z$, it follows that $\lambda_L < \lambda_M$.

As for the monotonicity of λ_L with respect to d^j and c^j, it is trivially to prove that λ_L increases when div$d$ decreases. However, the monotonicity of λ_L with respect to c^j is not clear. For convenience we will denote with λ_c, ϕ_c, ψ_c the first eigenvalue and the first eigenfunctions of L, L^* respectively, when the coefficients a_j^k, d^j, b^0 are fixed and c^j vary.

Proposition 4.3. *Let the operator L satisfies (1.2), (1.3). Then :*

$$\text{(4.5)} \qquad i) \quad \lambda_{ct} \text{ is a concave monotone increasing function of } t^2;$$

ii) $\lambda_{ct} = \lambda_c$ *for some* t, $|t| \neq 1 \iff \lambda_{ct} = \lambda_0$ *for every* $t \in R$.

Proof. i) For arbitrary $t_1, t_2 \in R$ and $0 < s < 1$ we define t from the equality $t^2 = (1-s)t_1^2 + st_2^2$. From (3.10) we get the inequality

$$\lambda_{ct} = \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + \int_\Omega \alpha_j^k (tc^j - h^j)(tc^k - h^k)\nu^2\, dx \right]$$

$$= \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + t^2 \int_\Omega \alpha_j^k (c^j - h^j)(c^k - h^k)\nu^2\, dx \right]$$

$$\geq (1-s) \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + t_1^2 \int_\Omega \alpha_j^k (c^j - h^j)(c^k - h^k)\nu^2\, dx \right]$$

$$+ s \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + t_2^2 \int_\Omega \alpha_j^k (c^j - h^j)(c^k - h^k)\nu^2\, dx \right] = (1-s)\lambda_{ct_1} + s\lambda_{ct_2}$$

where the infinimum is taken over the functions $\nu \in H_0^1(\Omega)$, $||\nu||_{L^2} = 1$ and $h \in H_\nu$.

As for the monotonicity of λ_{ct} , as in the proof above, the inequality

$$\lambda_{ct} = \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + t^2 \int_\Omega \alpha_j^k (c^j - h^j)(c^k - h^k)\nu^2\, dx \right]$$

$$\geq \inf_{h,\nu} \left[B_{L_0}[\nu,\nu] + \int_\Omega \alpha_j^k (c^j - h^j)(c^k - h^k)\nu^2\, dx \right] = \lambda_c$$

holds for every $t^2 > 1$.

(ii) Suppose that $\lambda_{ct} = \lambda_c$ for some $|t| > 1$. From proposition 3.4 we have the equality

$$\lambda_{ct} = B_{L_0}[\overline{\nu},\overline{\nu}] + \int_\Omega \alpha_j^k (tc^j - \overline{h}^j)(tc^k - \overline{h}^k)\overline{\nu}^2\, dx$$

where, $\overline{\nu} = K(\phi_{ct}\psi_{ct})^{1/2}$, $K = \left(\int_\Omega \phi_{ct}\psi_{ct}\,dx\right)^{-1/2}$, $\overline{h}^j = tc^j - \frac{1}{2}a_j^k\overline{z}_{x_k}$, $\overline{z} = \log(\phi_{ct}/\psi_{ct})$. Hence

$$\lambda_{ct} = B_{L_0}[\overline{\nu},\overline{\nu}] + \frac{1}{t^2}\int_\Omega \alpha_j^k(tc^j - \overline{h}^j)(tc^k - \overline{h}^k)\overline{\nu}^2\,dx + K^2\left(1-\frac{1}{t^2}\right)\int_\Omega a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\phi_{ct}\psi_{ct}\,dx$$

$$\geq \inf_{h,\nu}\left[B_{L_0}[\nu,\nu] + \frac{1}{t^2}\int_\Omega \alpha_j^k(tc^j - h^j)(tc^k - h^k)\nu^2\,dx\right]$$

$$+K^2\left(1-\frac{1}{t^2}\right)\int_\Omega a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\phi_{ct}\psi_{ct}\,dx = \lambda_c + K^2\left(1-\frac{1}{t^2}\right)\int_\Omega a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\phi_{ct}\psi_{ct}\,dx$$

and from $\lambda_{ct} = \lambda_c$ we get the identity $\int_\Omega a_j^k\overline{z}_{x_j}\overline{z}_{x_k}\phi_{ct}\psi_{ct}\,dx = 0$.

From the positiveness of ϕ_{ct}, ψ_{ct} and (1.2) it follows that $\nabla\overline{z} \equiv 0$ in Ω i.e. $\phi_{ct} = \psi_{ct}$ in Ω. Applying theorem (4.1) (and more precisely, $(4.2)_i'$ in remark 3) we get immideately that div $(tc\phi_0^2) = 0$ in Ω. Hence, from $(4.2)_i$ it follows that $\lambda_{ct} = \lambda_0$ for every $t \in R$. □

As for the concavity of the first eigenvalue λ_L with respect to different coefficients c, $\overline{c}$, $c \neq \overline{c}$ a similar result as in $(4.5)_i$ is true with a correction term. In general, without this correction term the convexity of λ_L fails as one can see in the case of coefficients of the special type, $c^j = a_j^k p_{x_k}$, $\tilde{c}^j = a_j^k\tilde{p}_{x_k}$ where p, $\tilde{p} \in W^{2,\infty}(\Omega)$ are arbitrary functions. For convenience we denote with $\tilde{L}$ the operator

$$\tilde{L} = -(a_j^k u_{x_k} + (d^j - \tilde{c}^j)u)_{x_k} + (d^j + \tilde{c}^j)u_{x_j} + b^0 u.$$

Proposition 4.4. *Let the operators L, $\tilde{L}$ satisfy (1.2), (1.3). Then for every $0 < s < 1$ the inequality $\lambda_S \geq (1-s)\lambda_c + s\lambda_{\overline{c}}$ holds, where*

$$S = (1-s)\tilde{L} + s\tilde{L} + s(1-s)\alpha_j^k(c^j - \tilde{c}^j)(c^k - \tilde{c}^k).$$

Proof. Let z^i, $\tilde{z}^i$ be defined in (3) for the operators L, $\tilde{L}$. Then from (3.4) we have

$$\lambda_L = \lim_{i\to\infty}\lambda_{L_{z^i,0}}, \quad \lambda_{\tilde{L}} = \lim_{i\to\infty}\lambda_{\tilde{L}_{\overline{z}^i,0}} \quad \text{and}$$

$L_{z,0}$, $\tilde{L}_{z,0}$ are given in (3.1). For every $0 < s < 1$ we introduce the notations $C = (1-s)c + s\tilde{c}$, $Z^i = (1-s)z^i + s\tilde{z}^i$. Easy calculations give us the chain of inequalities

$$C^j Z^i_{x_j} - \frac{1}{4}a_j^k Z^i_{x_j}Z^i_{x_k} = (1-s)(c^j z^i_{x_j} - \frac{1}{4}a_j^k z^i_{x_j}z^i_{x_k}) + s(\tilde{c}^j\tilde{z}^i_{x_j} - \frac{1}{4}a_j^k\tilde{z}^i_{x_j}\tilde{z}^i_{x_k})$$

$$-s(1-s)(\tilde{c}^j - c^j)(\tilde{z}^i_{x_j} - z^i_{x_j}) + \frac{1}{4}s(1-s)a_j^k(\tilde{z}^i_{x_j} - z^i_{x_j})(\tilde{z}^i_{x_k} - z^i_{x_k})$$

$$\geq (1-s)(c^j z^i_{x_j} - \frac{1}{4}a_j^k z^i_{x_j}z^i_{x_k}) + s(\tilde{c}^j\tilde{z}^i_{x_j} - \frac{1}{4}a_j^k\tilde{z}^i_{x_j}\tilde{z}^i_{x_k})$$

$$-s(1-s)\alpha_j^k(\tilde{c}^j - c^j)(\tilde{c}^k - c^k).$$

If

$$S_{z,0}u = \frac{1}{2}\left(e^{-z/2}S(ue^{z/2}) + e^{z/2}S^*(ue^{-z/2})\right)$$

$$= L_0 u + \left(C^j z_{x_j} - \frac{1}{4}a_j^k z_{x_j} z_{x_k} + s(1-s)\alpha_j^k(\tilde{c}^j - c^j)(\tilde{c}^k - c^k)\right) u$$

then from theorem 3.1 we get the estimate

$$\lambda_S = \sup_z \lambda_{S_{z,0}} \geq \lambda_{S_{z^i,0}} \geq (1-s)\lambda_{L_{z^i,0}} + s\lambda_{\tilde{L}_{z^i,0}}, \quad z \in C^{0,1}(\overline{\Omega})$$

which after the limit $i \to \infty$ proves proposition 4.4 In the above considerations we used the simple fact that for symmetric operator $Mu = -(a_j^k u_{x_k} + d^j u)_{x_j} + d^j u_{x_j}$ the inequality $\lambda_{M+b^0} \geq (1-s)\lambda_{M+p} + s\lambda_{M+q}$ holds for every $0 < s < 1$, $b^0, p, q \in L^\infty(\Omega)$, $b^0 \geq (1-s)p + sq$. The proof of this statement follows directly from variational formula (2.2). □

We will finish this section with some simple properties of λ_c which illustrate how complicated is the dependence of λ_c from c. For this purpose let us introduce the following set of functions

$$N_c = \{h(x) = (h^1(x), \cdots, h^n(x)),\ h^j(x) \in C^{0,1}(\overline{\Omega}),\ \mathrm{div}(h\phi_c^2) = 0 \text{ for a.e. } \ x \in \Omega\}.$$

It is clear that N_c for every fixed c is a linear subspace of the Lipschitz vectors defined in $\overline{\Omega}$.

Proposition 4.5. *Let the operator L satisfy (1.2), (1.3) and $b^j \in W^{1,\infty}(\Omega)$. Then the following statements are true:*

i) for every $h \in N_c$ the equalities $\lambda_{c+h} = \lambda_c$, $\phi_{c+h} = \phi_c$ hold;

ii) $\phi_c = \phi_g \iff \lambda_c = \lambda_g$ and $c - g \in N_c \cap N_g$, as a consequence the identity $N_c = N_g$ holds.

iii) $c \in N_c \iff \lambda_c = \lambda_0$;

iv) for every $h \in N_c$ the estimate $\lambda_{th} \leq \lambda_c$ holds for every $t \in R$.

Proof. i) Since

$$L_0\phi_c + \frac{1}{\phi_c}\mathrm{div}((c+h)\phi_c^2) = L_0\phi_c + \frac{1}{\phi_c}\mathrm{div}(c\phi_c^2) = \lambda_c\phi_c$$

and $\phi_c > 0$ in Ω, $\phi_c = 0$ on $\partial\Omega$ it follows from corollary 2.1 in [4] that $\lambda_{c+h} = \lambda_c$ and $\phi_{c+h} = \phi_c$.

ii) Suppose that $\phi_c = \phi_g$. Then from the identities

$$\left(\lambda_c - \frac{1}{\phi_c^2}\mathrm{div}(c\phi_c^2)\right)\phi_c = L_0\phi_c = L_0\phi_g = \left(\lambda_g - \frac{1}{\phi_g^2}\mathrm{div}(g\phi_g^2)\right)\phi_g$$

$$= \left(\lambda_g - \frac{1}{\phi_c^2}\mathrm{div}(g\phi_c^2)\right)\phi_c$$

we obtain that $\mathrm{div}((c-g)\phi_c^2) = \lambda_c - \lambda_g$. Integrating the above equality in Ω we get immediately that $\lambda_c = \lambda_g$ and $\mathrm{div}((c-g)\phi_c^2) = \mathrm{div}((c-g)\phi_g^2) = 0$ in Ω i.e. from the definition of N_c,N_g we have $c - g \in N_c \cap N_g$.

Suppose that $\lambda_c = \lambda_g$ and $c - g \in N_c \cap N_g$. Since

$$\lambda_c\phi_g = \lambda_g\phi_g = L_0\phi_g + \frac{1}{\phi_g}\text{div}(c\phi_g^2) + \frac{1}{\phi_g}\text{div}(g-c)\phi_g^2$$

$$= L_0\phi_g + \frac{1}{\phi_g}\,\text{div}(c\phi_g^2)$$

and $\phi_g > 0$ in Ω, $\phi_g = 0$ on $\partial\Omega$ it follows that $\phi_g = \phi_c$.

iii) If $\lambda_c = \lambda_0$ then from $(4.2)_i$ we have $\text{div}(c\phi_0^2) = 0$ in Ω, i.e. $c \in N_c$.

Suppose that $c \in N_c$. Then $\text{div}(c\phi_c^2) = 0$ in Ω and from the equality $\lambda_c\phi_c = L_0\phi_c + \frac{1}{\phi_c}\text{div}(c\phi_c^2) = L_0\phi_c$ and the positiveness of ϕ_c in Ω, $\phi_c = 0$ on $\partial\Omega$ it follows that $\phi_c = \phi_0$, $\lambda_c = \lambda_0$.

iv) If $c \in N_c$ we have from iii) that $\lambda_c = \lambda_0$ so that $\lambda_{th} \equiv \lambda_0 = \lambda_c$ for every $t \in R$ and iv) is proved. Suppose now that $c \notin N_c$. From i) for a fixed $h \in N_c$ the identities $\lambda_{c+th} = \lambda_c$, $\phi_{c+th} = \phi_c$ hold for every $t \in R$. If $\lambda_{c+sh} > \lambda_c$ for some $s \in R$ then from the continuous dependence of the first eigenvalue with respect to the coefficients (see proposition 5.1 in [4]) we have the inequality $\lambda_{c+sh-\epsilon c} > \lambda_c$ for a sufficient small positive constant ϵ. However, from $(4.5)_{ii}$ it follows that

$$\lambda_c = \lambda_{c+hs/(1-\epsilon)} = \lambda_{(c+hs-\epsilon c)/(1-\epsilon)} \geq \lambda_{c+hs-\epsilon c} > \lambda_c$$

which is impossible. □

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