Extended contact algebras and internal connectedness

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Abstract

The notion of contact algebra is one of the main tools in the region based theory of space. It is an extension of Boolean algebra with an additional relation C, called contact. Standard models of contact algebras are topological and are the contact algebras of regular closed sets in a given topological space. In such a contact algebra we add the predicate internal connectedness with the following meaning - a regular closed set x is internally connected if and only if $Int(x)$ is a connected topological space in the subspace topology. We obtain an axiomatization of the theory, consisting of the universal formulas, true in all topological contact algebras with added relation internal connectedness.

Keywords: mereotopology, contact algebras, extended contact algebras, internal connectedness, topological representation.

1 Introduction

In classical Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast the region-based theory of space (RBTS) has as primitives the more realistic notion of region as an abstraction of physical body, together with some basic relations and operations on regions. Some of these relations are mereological - part-of ($x \leq y$), overlap (xOy), its dual underlap ($x\widehat{O}y$). Other relations are topological - contact (xCy), nontangential part-of ($x \ll y$), dual contact ($x\widehat{C}y$) and some others definable by means of the contact and part-of relations. This is one of the reasons that the extension of mereology with these new relations is commonly called *mereotopology*. There is no clear difference in the literature between RBTS and mereotopology, and by some authors RBTS is related rather to the so called *mereogeometry*, while mereotopology is considered only as a kind of point-free topology, considering mainly topological properties of things. The origin of RBTS goes back to Whitehead [36] and de Laguna [21]. According to Whitehead points, as well as the other primitive notions in Euclidean geometry like lines and planes, do not have separate existence in reality and because of this are not appropriate for primitive notions; but points have to be definable by the other primitive notions.

Survey papers about RBTS are [32, 5, 14, 24] (see also the handbook [1] and [4] for some logics of space). Surveys concerning various applications are [6, 7] and the book [15] (see also special issues of Fundamenta Informaticæ[9] and the Journal of Applied Nonclassical Logics [3]). RBTS has applications in computer science because of its more simple way of representing qualitative spatial information and it initiated a special field in Knowledge Representation

(KR) called Qualitative Spatial Representation and Reasoning (QSRR). One of the most popular systems in QSRR is the Region Connection Calculus (RCC) introduced in [25].

The notion of contact algebra is one of the main tools in RBTS. This notion appears in the literature under different names and formulations as an extension of Boolean algebra with some mereotopological relations [35, 27, 33, 34, 5, 11, 8, 10]. The simplest system, called just contact algebra was introduced in [8] as an extension of Boolean algebra $\underline{B} = (B, 0, 1, \cdot, +, *)$ with a binary relation C called contact and satisfying several simple axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,

(C2) If aCb and $a \leq c$ and $b \leq d$, then cCd,

(C3) If $aC(b + c)$, then aCb or aCc,

(C4) If aCb, then bCa,

(C5) If $a \cdot b \neq 0$, then aCb.

The elements of the Boolean algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the empty region. The contact relation is used also to define some other important mereotopological relations like non-tangential inclusion, dual contact and others.

The standard model of Boolean algebra is the algebra of subsets of a given universe. This model cannot express all kinds of contact, for example, the external contact in which the regions share only a boundary point. Because of this standard models of contact algebras are topological and are the contact algebras of regular closed sets in a given topological space.

In [30] is presented a complete quantifier-free axiomatization of several logics on region-based theory of space, based on contact relation and connectedness predicates c and $c^{\leq n}$, and completeness theorems for the logics in question are proved. It was shown in [30] that c and $c^{\leq n}$ are definable in contact algebras by the contact predicate C. The predicates c and $c^{\leq n}$ were studied for the first time in [22, 23] (see also [32]). The expressiveness and complexity of spatial logics containing c and $c^{\leq n}$ has been investigated in [16, 17, 18, 19, 20]. In this paper we consider the predicate c^o - internal connectedness. Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology. We prove that the predicate of internal connectedness cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol $\vdash$ which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$. It turns out that the predicate c^o can be defined in the new language. We define *extended contact algebras* as Boolean algebras with added relations $\vdash$, C and c^o, satisfying some axioms, and prove that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the universal formulas true in all topological contact algebras with added relations $\vdash$ and c^o.

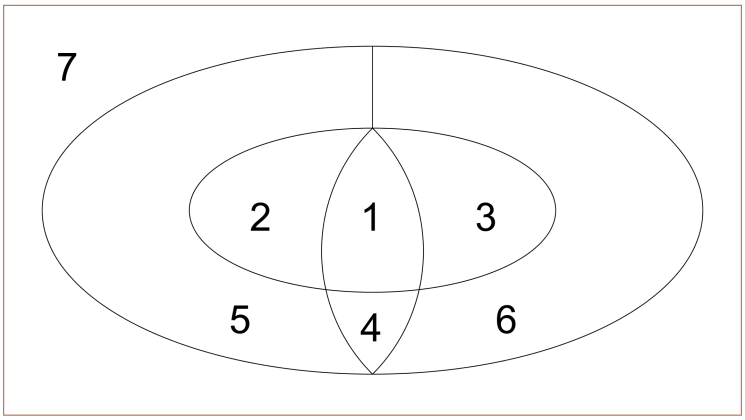

Figure 1: The topological space (X, O)

2 Undefinability of internal connectedness

Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology.

Proposition 2.1 *There does not exist a formula $A(x)$ in the language of contact algebras such that: for arbitrary topological space, for every regular closed subset x of this topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.*

Proof. Suppose for the sake of contradiction that there exists a formula $A(x)$ in the language of contact algebras such that: for any topological space, for every regular closed subset x of this topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.

We consider the topological space (X, O), where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and the topology is defined by an open basis: $\{\{1, 2, 3\}, \{7\}, \{2, 5, 7\}, \{3, 6, 7\}, \{2\}, \{3\}, X, \emptyset\}$ (see Figure 1).

It can be easily verified that the open sets are $\{1, 2, 3\}$, $\{7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{2\}$, $\{3\}$, $\{2, 3, 5, 6, 7\}$, $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 7\}$, $\{2, 7\}$, $\{3, 7\}$, $\{2, 3\}$, $\{2, 3, 7\}$, $\{1, 2, 3, 5, 7\}$, $\{1, 2, 3, 6, 7\}$, $\{2, 3, 5, 7\}$, $\{2, 3, 6, 7\}$, X, $\emptyset$. It can be easily verified that the regular closed sets are $\{4, 5, 6, 7\}$, $\{1, 2, 3, 4, 5, 6\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 2, 4, 5, 6, 7\}$, $\{1, 3, 4, 5, 6, 7\}$, X, $\emptyset$.

We consider the subspace of X, $Y = X \backslash \{1\}$. It can be easily proved that:

$$Int_Y(c \backslash \{1\}) = Int_X c \backslash \{1\} \qquad \text{for every } c \text{ - closed subset of } X \tag{2.1}$$

Using (2.1) and the fact that for every t, $Cl_Y t = Cl_X t \cap Y = Cl_X t \backslash \{1\}$, we prove that $RC(Y) = \{x \backslash \{1\} : x \in RC(X)\}$.

We define a function f from $RC(X)$ to $RC(Y)$ in the following way:

$$f(t) = \begin{cases} t & \text{if } 1 \notin t \\ t \backslash \{1\} & \text{if } 1 \in t \end{cases}$$

It can be easily proved that f is an isomorphism from $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ to $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$.

Let $a = \{1, 2, 3, 4, 5, 6\}$. We will prove that a is internally connected. Since $Int_X a = \{1, 2, 3\}$, therefore the closed sets in $Int(a)$ are: $\{1, 2, 3\}$, $\emptyset$, $\{1, 2\}$, $\{1, 3\}$, $\{1\}$. Thus, $Int(a)$ cannot be represented as the union of two non-empty disjoint closed sets and hence $Int(a)$ is connected. Consequently a is internally connected.

Let $b = \{2, 3, 4, 5, 6\}$. $Int_Y b = \{2, 3\}$. We will prove that b is not internally connected. We will prove that $\{2, 3\}$ is not connected. Since $\{2, 3\} = \{2\} \cup \{3\}$, it suffices to prove that $\{2\}$ and $\{3\}$ are closed in $\{2, 3\}$. Remark that $\{2, 4, 5\}$ is closed in Y and hence $\{2\} = \{2, 4, 5\} \cap \{2, 3\}$ is closed in $\{2, 3\}$. Remark that $\{3, 4, 6\}$ is closed in Y and hence $\{3\} = \{3, 4, 6\} \cap \{2, 3\}$ is closed in $\{2, 3\}$. Consequently $\{2, 3\}$ is not connected, i.e. b is not internally connected.

We have $a \in RC(X)$, $c^o(a)$. Consequently $A(a)$. Now consider the topological space Y. Using $b \in RC(Y)$ and $\neg c^o(b)$, we have $\neg A(b)$. We also have $b = f(a)$. $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ and $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$ are isomorphic structures for the language of contact algebras, A is a formula in the same language. Consequently: $A(a)$ is true in $(RC(X) \ldots)$ iff $A(f(a))$ i.e. $A(b)$ is true in $(RC(Y) \ldots)$. We have proven that $A(a)$ is true in $(RC(X) \ldots)$; so $A(b)$ is true in $(RC(Y) \ldots)$ - a contradiction. $\square$

3 Definability of internal connectedness in an extended language

Let X be a topological space. We define the relation $\vdash$ in $RC(X)$ in the following way: $a, b \vdash c$ iff $a \cap b \subseteq c$.

Proposition 3.1 *Let X be a topological space. For every a in $RC(X)$, $c^o(a)$ iff* $\forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

Proof. $\rightarrow$) Let $c^o(a)$. Let $b, c \in RC(X)$, $b \neq 0$, $c \neq 0$, $a = b + c$. We will prove that $b, c \nvdash a^*$. We have $a^* = Cl_X - a = -Int_X a$. Suppose for the sake of contradiction that $b, c \vdash -Int_X a$. It follows that $b \cap c \subseteq -Int_X a$ (1). Suppose for the sake of contradiction that $b \cap Int_X a = \emptyset$. We also have $a = b \cup c$ and consequently $Int_X a \subseteq c$. We will prove that $Int_X b = \emptyset$. Suppose for the sake of contradiction that $Int_X b \neq \emptyset$. Using $Int_X a \subseteq c$ and (1), we have that $Int_X b \cap Int_X a = \emptyset$, but $Int_X b \neq \emptyset$, so $Int_X a \neq Int_X a \cup Int_X b$ (2). We have $a = b \cup c$. Consequently $Int_X a \cup Int_X b \subseteq a$, but $Int_X a \cup Int_X b$ is an open set, so $Int_X a \cup Int_X b \subseteq Int_X a$, i.e. $Int_X a \cup Int_X b = Int_X a$ - a contradiction. Consequently $Int_X b = \emptyset$. We have $b \in RC(X)$, so $b = Cl_X Int_X b = Cl_X \emptyset = \emptyset$ - a contradiction. Consequently $b \cap Int_X a \neq \emptyset$. Similarly $c \cap Int_X a \neq \emptyset$. Let $b_1 = b \cap Int_X a$, $c_1 = c \cap Int_X a$. We have $b_1 \cup c_1 = Int_X a \cap (b \cup c) = Int_X a \cap a = Int_X a$. From $a = b \cup c$ and (1) we get $b_1 \cap c_1 = \emptyset$. We have $Int_X a = b_1 \cup c_1$, $b_1 \neq \emptyset$, $c_1 \neq \emptyset$, $b_1 \cap c_1 = \emptyset$, b_1 and c_1 are closed in $Int_X a$ and therefore $Int_X a$ is not connected, i.e. a is not internally connected - a contradiction.

$\leftarrow$) Let $\forall b, c \in RC(X)(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$. We will prove that $Int_X a$ is connected. Suppose for the sake of contradiction that $Int_X a$ is not connected. Consequently there are b_1, c_1 - closed in $Int_X a$, such that $Int_X a = b_1 \cup c_1$ (1), $b_1 \neq \emptyset$, $c_1 \neq \emptyset$, $b_1 \cap c_1 = \emptyset$. We have $b_1 = b \cap Int_X a$,

$c_1 = c \cap Int_X a$, where b and c are closed in X because b_1 and c_1 are closed in $Int_X a$. Let $b' = Cl_X b_1$, $c' = Cl_X c_1$. a and b are closed sets in X, $b_1 \subseteq b$, $b_1 \subseteq a$ and therefore $b' \subseteq b$, $b' \subseteq a$. Similarly $c' \subseteq c$, $c' \subseteq a$. Suppose for the sake of contradiction that $a \nsubseteq b' \cup c'$. b' and c' are closed in X and consequently $b' \cup c'$ is closed in X. From $b_1 \subseteq b'$, $c_1 \subseteq c'$, (1) we obtain that $Int_X a \subseteq b' \cup c'$, but $b' \cup c'$ is closed in X and consequently $Cl_X Int_X a \subseteq b' \cup c'$. We have $b' \cup c' \subseteq a$, $b' \cup c' \neq a$. Consequently $Cl_X Int_X a \neq a$ - a contradiction with $a \in RC(X)$. Consequently $a \subseteq b' \cup c'$ and thus $a = b' \cup c'$ (3). We have $c_1 = c \cap Int_X a$, $Int_X a = b_1 \cup c_1$, $b_1 \cap c_1 = \emptyset$ and therefore $b_1 = -c \cap Int_X a$. We have that c is closed in X and hence $-c$ is open in X; $Int_X a$ is open in X; so b_1 is open in X, but $b_1 \subseteq b'$, so $b_1 \subseteq Int_X b'$. Suppose for the sake of contradiction that $Int_X b' \neq b_1$. From (3) we get $Int_X b' \subseteq Int_X a$ (4). From $Int_X a = b_1 \cup c_1$, $b_1 \subseteq Int_X b'$, $b_1 \neq Int_X b'$, (4) we obtain $c_1 \cap Int_X b' \neq \emptyset$, but $Int_X b' \subseteq b' \subseteq b$, so $c_1 \cap b \neq \emptyset$. Consequently $b \cap Int_X a \cap c_1 \neq \emptyset$, but $b \cap Int_X a = b_1$, so $b_1 \cap c_1 \neq \emptyset$ - a contradiction. Consequently $Int_X b' = b_1$. $b' = Cl_X b_1 = Cl_X Int_X b'$, so $b' \in RC(X)$. Similarly $c' \in RC(X)$. We also have $b', c' \neq \emptyset$, $a = b' + c'$, so $b' \cap c' \nsubseteq a^* = -Int_X a$. Consequently $b' \cap c' \cap Int_X a \neq \emptyset$, but $b' \subseteq b$, $c' \subseteq c$, so $b \cap c \cap Int_X a \neq \emptyset$, i.e. $(b \cap Int_X a) \cap (c \cap Int_X a) \neq \emptyset$. Consequently $b_1 \cap c_1 \neq \emptyset$ - a contradiction. Consequently $Int_X a$ is connected, i.e. $c^o(a)$. □

4 Extended contact algebras

In this section we give an axiomatization of the relation $a, b \vdash c$ used in the characterization of the predicate $c^0(a)$ of internal connectedness given in Section 3.

Definition 4.1 *Extended contact algebra (ECA, for short) is a system* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$*, where* $(B, \leq, 0, 1, \cdot, +, *)$ *is a nondegenerate Boolean algebra,* $\vdash$ *is a ternary relation in* B *such that the following axioms are true:*
(1) $a, b \vdash c \rightarrow b, a \vdash c$,
(2) $a \leq b \rightarrow a, a \vdash b$,
(3) $a, b \vdash a$,
(4) $a, b \vdash x,\ a, b \vdash y,\ x, y \vdash c \rightarrow a, b \vdash c$,
(5) $a, b \vdash c \rightarrow a \cdot b \leq c$,
(6) $a, b \vdash c \rightarrow a + x, b \vdash c + x$,
C *is a binary relation in* B *such that for all* a, $b \in B$: $aCb \leftrightarrow a, b \nvdash 0$. c^o *is a unary predicate in* B *such that for all* $a \in B$: $c^o(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

Lemma 4.2 *If* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *is an ECA, then* C *is a contact relation in* B *and hence* (B, C) *is a contact algebra.*

Proof. Routine verification that the axioms of contact C_1 - C_5 are true. □

The above lemma shows that the notion of ECA is a generalization of contact algebra.

The next lemma shows the standard topological example of ECA.

Lemma 4.3 *Let* X *be a topological space and* $RC(X)$ *be the Boolean algebra of regular closed subsets of* X*. Let for* $a, b, c \in RC(X)$*:*

aCb *iff* $a \cap b \neq \emptyset$,

$a, b \vdash c$ iff $a \cap b \subseteq c$
$c^0(a)$ iff $Int(a)$ is a connected subspace of X.

Then, together with this relations, the Boolean algebra $RC(X)$ is an ECA, called topological ECA over the space X.

Proof. It can be easily verified that the axioms (1)-(6) of ECA are true and for all a, $b \in RC(X)$: $aCb \leftrightarrow a, b \nvdash 0$. Using proposition 3.1, we get that for every $a \in RC(X)$ we have $c^o(a) \leftrightarrow \forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$. □

Our aim is to prove that every ECA can be isomorphically embedded into a topological ECA over a certain topological space X, which will be done in the next section. This will show that the chosen axioms for ECA are right.

Remark 4.4 *Using axioms (2) and (5), we see that in an ECA $\underline{B}$ $a \leq b \leftrightarrow a, a \vdash b$ for every $a, b \in B$, i.e. the predicate symbol $\leq$ can be removed from the language.*

5 A consequence relation

Definition 5.1 *Let $(B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. We inductively define the binary relations $\vDash_0$, $\vDash_1$, ... between subsets of B and elements of B as follows:*

$S \vDash_0 x \overset{def}{\leftrightarrow} x \in S$
$S \vDash_{n+1} x \overset{def}{\leftrightarrow} \exists x_1, x_2 : \ x_1, x_2 \vdash x, \ S \vDash_{k_1} x_1, \ S \vDash_{k_2} x_2,$ *where* $k_1, k_2 \leq n$
$S \vDash x \overset{def}{\leftrightarrow} \exists n : \ S \vDash_n x$

Before being able to present a proof of a representation theorem of EC-algebras we will need several lemmas.

Lemma 5.2 *If $S \vDash_n y$ and $S \subseteq S'$, then $S' \vDash_n y$.*

Proof. An induction on n.
Case 1: $n = 0$
Let $S \vDash_0 y$ and $S \subseteq S'$. We have $y \in S$ and consequently $y \in S'$, i.e. $S' \vDash_0 y$.
Case 2: $n > 0$
Let $S \vDash_n y$ and $S \subseteq S'$. We will prove that $S' \vDash_n y$. From $S \vDash_n y$, $n > 0$ we get that there are x_1, x_2 such that $x_1, x_2 \vdash y$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. Using $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, $S \subseteq S'$ and the induction hypothesis, we have $S' \vDash_{k_1} x_1$, $S' \vDash_{k_2} x_2$. Consequently $S' \vDash_n y$. □

Lemma 5.3 *If $S \vDash_n y$ and $n \leq n'$, then $S \vDash_{n'} y$.*

Proof. Let $S \vDash_n y$ and $n \leq n'$. We will prove that $S \vDash_{n'} y$.
Case 1: $n = 0$
By induction on n' we will prove that $\forall n' \forall S \forall y(S \vDash_0 y$ and $0 \leq n' \rightarrow S \vDash_{n'} y)$.
Case 1.1: $n' = 0$
Obviously $\forall S \forall y(S \vDash_0 y$ and $0 \leq 0 \rightarrow S \vDash_0 y)$.
Case 1.2: $n' > 0$
Let $S \subseteq B$, $y \in B$, $S \vDash_0 y$ and $0 \leq n'$. We will prove that $S \vDash_{n'} y$. From $n' > 0$ we have $0 \leq n' - 1$. By the induction hypothesis we obtain that $\forall S \forall y(S \vDash_0 y$

and $0 \leq n' - 1 \to S \vDash_{n'-1} y$). Consequently $S \vDash_{n'-1} y$. We also have $y, y \vdash y$ (from axiom (2)). Consequently $S \vDash_{n'} y$.

So we proved that $\forall n' \forall S \forall y (S \vDash_0 y$ and $0 \leq n' \to S \vDash_{n'} y)$. We also have $S \vDash_0 y$ and $0 \leq n'$. Consequently $S \vDash_{n'} y$.

Case 2: $n > 0$

From $S \vDash_n y$, $n > 0$ we get that there are x_1, x_2 such that $x_1, x_2 \vdash y$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. But we have $n \leq n'$, so $k_1, k_2 < n'$. Consequently $S \vDash_{n'} y$. □

Lemma 5.4 *If $S \vDash x$ and $x \leq y$, then $S \vDash y$.*

Proof. Let $S \vDash x$ and $x \leq y$. We will prove that $S \vDash y$. From $x \leq y$ and axiom (2) we have that $x, x \vdash y$ (1). From $S \vDash x$ we obtain that: $S \vDash_n x$ for some n (2). From (1) and (2) we have $S \vDash_{n+1} y$, i.e. $S \vDash y$. □

Lemma 5.5 *If $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash z$, then $\{x\} \cup S \vDash z$.*

Proof. Let $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash z$. We will prove that $\{x\} \cup S \vDash z$. We have $\{y\} \cup S \vDash_{n_0} z$ for some n_0. By induction on n we will prove that $\forall n \forall t (\{x\} \cup S \vDash y, \{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. Let n be a natural number and $\forall n' < n \forall t (\{x\} \cup S \vDash y, \{y\} \cup S \vDash_{n'} t \to \{x\} \cup S \vDash t)$. We will prove that $\forall t (\{x\} \cup S \vDash y, \{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. Let $t \in B$, $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_n t$. We will prove that $\{x\} \cup S \vDash t$.

Case 1: $n = 0$

Case 1.1: $t = y$

Obviously $\{x\} \cup S \vDash t$.

Case 1.2: $t \neq y$

We have $\{y\} \cup S \vDash_0 t$. Consequently $t \in \{y\} \cup S$, but $t \neq y$, so $t \in S$. Consequently $\{x\} \cup S \vDash_0 t$.

Case 2: $n > 0$

We have $\{y\} \cup S \vDash_n t$, $n > 0$. Consequently there are t_1, t_2 such that $t_1, t_2 \vdash t$, $\{y\} \cup S \vDash_{k_1} t_1$, $\{y\} \cup S \vDash_{k_2} t_2$, where $k_1, k_2 < n$. By the induction hypothesis for k_1, k_2, we get $\{x\} \cup S \vDash t_1$, $\{x\} \cup S \vDash t_2$. Consequently $\{x\} \cup S \vDash_{l_1} t_1$, $\{x\} \cup S \vDash_{l_2} t_2$ for some integers l_1, l_2. Let l be the greater among l_1 and l_2. We have $\{x\} \cup S \vDash_l t_1$, $\{x\} \cup S \vDash_l t_2$ by lemma 5.3; $t_1, t_2 \vdash t$; consequently $\{x\} \cup S \vDash_{l+1} t$, i.e. $\{x\} \cup S \vDash t$.

We proved that $\forall n \forall t (\{x\} \cup S \vDash y, \{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. We also have $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_{n_0} z$. Consequently $\{x\} \cup S \vDash z$. □

Lemma 5.6 *If $\{x_1\} \cup S \vDash y$, $\{x_2\} \cup S \vDash y$, then $\{x_1 + x_2\} \cup S \vDash y$.*

Proof. Let $\{x_1\} \cup S \vDash y$, $\{x_2\} \cup S \vDash y$. We will prove that $\{x_1 + x_2\} \cup S \vDash y$. There is a n_0 such that $\{x_1\} \cup S \vDash_{n_0} y$, $\{x_2\} \cup S \vDash_{n_0} y$. We will prove by induction on n that:

$(*)$ $\forall n \forall u \forall v \forall w (\{u\} \cup S \vDash_n v \to \{u + w\} \cup S \vDash v + w)$

Let n be a natural number and $\forall t < n \forall u \forall v \forall w (\{u\} \cup S \vDash_t v \to \{u + w\} \cup S \vDash v + w)$. We will prove that $\forall u \forall v \forall w (\{u\} \cup S \vDash_n v \to \{u + w\} \cup S \vDash v + w)$. Let $u, v, w \in B$ and $\{u\} \cup S \vDash_n v$. We will prove that $\{u + w\} \cup S \vDash v + w$.

Case 1: $n = 0$

Case 1.1: $v \in S$

We have $\{u + w\} \cup S \vDash_0 v$ and by lemma 5.4, we obtain that $\{u + w\} \cup S \vDash v + w$.

Case 1.2: $v \notin S$
We have $\{u\} \cup S \vDash_0 v$, $v \notin S$. Consequently $v = u$. It is sufficient to prove that $\{v + w\} \cup S \vDash v + w$ which obviously is true.
Case 2: $n > 0$
We have $\{u\}\cup S \vDash_n v$, $n > 0$. Consequently there are v_1, v_2 such that $v_1, v_2 \vdash v$, $\{u\}\cup S \vDash_{k_1} v_1$, $\{u\}\cup S \vDash_{k_2} v_2$, where $k_1, k_2 < n$. From the induction hypothesis for k_1 and k_2 we get that $\{u+w\}\cup S \vDash v_1+w$ (1) and $\{u+w\}\cup S \vDash v_2+w$ (2). From $v_1, v_2 \vdash v$ and axiom (6) we obtain $v_1+w, v_2 \vdash v+w$; so $v_2, v_1+w \vdash v+w$ (by axiom (1)); so $v_2 + w, v_1 + w \vdash v + w + w$ (by axiom (6)); consequently $v_1 + w, v_2 + w \vdash v + w$ (3) (by axiom (1)). Using (1),(2) and (3) we have $\{u + w\} \cup S \vDash v + w$.

We proved that $(*)$ is true. From $(*)$ and $\{x_1\} \cup S \vDash_{n_0} y$ we get that $\{x_1 + x_2\} \cup S \vDash y + x_2$ (4). From $(*)$ and $\{x_2\} \cup S \vDash_{n_0} y$, we obtain that $\{x_2 + y\} \cup S \vDash y + y$, i.e. $\{y + x_2\} \cup S \vDash y$ (5). Using (4), (5) and lemma 5.5, we have $\{x_1 + x_2\} \cup S \vDash y$. □

Lemma 5.7 *Let $S \vDash x$. Then there is a finite nonempty subset of S, S_0, such that $S_0 \vDash x$.*

Proof. We will prove by induction on n that $\forall n \forall x(S \vDash_n x \to \exists$ finite nonempty subset S_0 of S such that $S_0 \vDash_n x)$.
Case 1: $n = 0$
Let $S \vDash_0 x$. Consequently $x \in S$. Thus $\{x\}$ is a finite nonempty subset of S and $\{x\} \vDash_0 x$.
Case 2: $n \neq 0$
Let $S \vDash_n x$. Consequently there are x_1, x_2 such that $x_1, x_2 \vdash x$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. Using the induction hypothesis, we have that there exist finite nonempty subsets of S, S_1 and S_2, such that $S_1 \vDash_{k_1} x_1$, $S_2 \vDash_{k_2} x_2$. By lemma 5.2, we get $S_1 \cup S_2 \vDash_{k_1} x_1$, $S_1 \cup S_2 \vDash_{k_2} x_2$. Thus $S_1 \cup S_2 \vDash_n x$, $S_1 \cup S_2 \neq \emptyset$, $S_1 \cup S_2$ is finite, $S_1 \cup S_2 \subseteq S$. □

Lemma 5.8 *Let $S = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_k\}$ for some $n, k > 0$ and $S \vDash x$. Let $a = a_1 \cdot \ldots \cdot a_n$, $b = b_1 \cdot \ldots \cdot b_k$. Then $a, b \vdash x$.*

Proof. By induction on n we will prove that $\forall n \forall x(S \vDash_n x \to a, b \vdash x)$.
Case 1: $n = 0$
Let $x \in B$, $S \vDash_0 x$. We will prove that $a, b \vdash x$. We have $x \in S$. Without loss of generality $x = a_1$. From $a \leq a_1$ by axiom (2), we obtain that $a, a \vdash a_1$. From axiom (3) we get $a, b \vdash a$. From here and $a, a \vdash a_1$ by axiom (4), we get that $a, b \vdash a_1$.
Case 2: $n \neq 0$
Let $x \in B$ and $S \vDash_n x$. We will prove that $a, b \vdash x$. There are x_1, x_2 such that $x_1, x_2 \vdash x$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. Using the induction hypothesis, we get $a, b \vdash x_1$, $a, b \vdash x_2$. But $x_1, x_2 \vdash x$, so by axiom (4), we obtain $a, b \vdash x$. □

6 Topological representation theory of ECA

Definition 6.1 *Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. A subset of B, Γ, is an abstract point if the following conditions are satisfied:*

1) $1 \in \Gamma$
2) $0 \notin \Gamma$
3) $a + b \in \Gamma \to a \in \Gamma$ *or* $b \in \Gamma$
4) $a, b \in \Gamma,\ a, b \vdash c \to c \in \Gamma$

Note that ultrafilters are abstract points.

Lemma 6.2 *Let X be a topological space. For every n and for any $b_1, \ldots, b_n \in RC(X)$, we have $Cl_X Int_X(b_1 \cap \ldots \cap b_n) = b_1 \cdot \ldots \cdot b_n$.*

Proof. An induction on n.
• $n = 1$
$Cl_X Int_X b_1 = b_1$ because $b_1 \in RC(X)$.
• $n \to n + 1$
We will prove that $Cl_X Int_X(b_1 \cap \ldots \cap b_{n+1}) = b_1 \cdot \ldots \cdot b_{n+1}$. Let $b = b_2 \cap \ldots \cap b_{n+1}$. We will prove that $Int_X(b_1 \cap b) = Int_X(b_1 \cap Cl_X Int_X b)$. We have $Int_X(b_1 \cap b) \subseteq Int_X b$ and hence $Int_X(b_1 \cap b) \subseteq Cl_X Int_X b$. We also have $Int_X(b_1 \cap b) \subseteq b_1 \cap b \subseteq b_1$. Consequently $Int_X(b_1 \cap b) \subseteq b_1 \cap Cl_X Int_X b$. Consequently $Int_X(b_1 \cap b) \subseteq Int_X(b_1 \cap Cl_X Int_X b)$. Since $b_2, \ldots, b_{n+1} \in RC(X)$ and $b = b_2 \cap \ldots \cap b_{n+1}$, we have that b is closed. We also have $Int_X b \subseteq b$, so $Cl_X Int_X b \subseteq b$. Consequently $b_1 \cap Cl_X Int_X b \subseteq b_1 \cap b$ and hence $Int_X(b_1 \cap Cl_X Int_X b) \subseteq Int_X(b_1 \cap b)$. Thus $Int_X(b_1 \cap b) = Int_X(b_1 \cap Cl_X Int_X b)$. We have $Cl_X Int_X(b_1 \cap b) = Cl_X Int_X(b_1 \cap Cl_X Int_X(b_2 \cap \ldots \cap b_{n+1})) = Cl_X Int_X(b_1 \cap (b_2 \cdot \ldots \cdot b_{n+1})) = b_1 \cdot (b_2 \cdot \ldots \cdot b_{n+1})$.
□

Lemma 6.3 *Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. Let $A \neq \emptyset$, $A \subseteq B$, $a \in B$ be such that $A \nvDash a$. Then there exists an abstract point Γ such that $A \subseteq \Gamma$ and $a \notin \Gamma$.*

Proof. We consider the set $(M, \subseteq)$, where $M = \{P \subseteq B :\ A \subseteq P;\ a \notin P;\ x, y \in P,\ x, y \vdash z \to z \in P\}$. We will prove that $(M, \subseteq)$ has a maximal element Γ and Γ is an abstract point. Let $P_0 = \{t :\ A \vDash t\}$. We will prove that $P_0 \in M$. Obviously $A \subseteq P_0$ and $a \notin P_0$. Let $x, y \in P_0$, $x, y \vdash z$. We will prove that $z \in P_0$. We have $A \vDash_n x$ and $A \vDash_n y$ for some n. Consequently $A \vDash_{n+1} z$. Consequently $z \in P_0$. Thus $P_0 \in M$. We will prove that $(M, \subseteq)$ has a maximal element. Let L be a chain in M.
Case 1: $L = \emptyset$
P_0 is an upper bound of L.
Case 2: $L \neq \emptyset$
We will prove that $\bigcup L \in M$. Obviously $\bigcup L \subseteq B$, $A \subseteq \bigcup L$, $a \notin \bigcup L$. Let $x, y \in \bigcup L$, $x, y \vdash z$. We will prove that $z \in \bigcup L$. We have $x \in P_1$, $y \in P_2$, where $P_1, P_2 \in L$. Without loss of generality $P_1 \subseteq P_2$. Thus $x, y \in P_2$, $x, y \vdash z$, $P_2 \in M$. Consequently $z \in P_2$ and hence $z \in \bigcup L$. Consequently $\bigcup L \in M$. Obviously $\bigcup L$ is an upper bound of L.

Thus $(M, \subseteq)$ satisfies the Zorn condition. Consequently $(M, \subseteq)$ has a maximal element Γ. We will prove that Γ is an abstract point. We have $A \neq \emptyset$ and hence $a_1 \in A$ for some a_1. We have $\Gamma \in M$ and therefore $A \subseteq \Gamma$, so $a_1 \in \Gamma$. From $a_1 \leq 1$ by axiom (2), we get that $a_1, a_1 \vdash 1$. We also have $\Gamma \in M$, $a_1 \in \Gamma$, so $1 \in \Gamma$.

Suppose for the sake of contradiction that $0 \in \Gamma$. From $0 \leq a$ by axiom (2), we obtain $0, 0 \vdash a$. Consequently $a \in \Gamma$ - a contradiction with $\Gamma \in M$. Consequently $0 \notin \Gamma$.

Condition 4) from the definition of abstract point is satisfied for Γ because $\Gamma \in M$.

Let $x + y \in \Gamma$. We will prove that $x \in \Gamma$ or $y \in \Gamma$. For the sake of contradiction suppose that $\{x\} \cup \Gamma \vDash a$, $\{y\} \cup \Gamma \vDash a$. From lemma 5.6 we have $\{x+y\} \cup \Gamma \vDash a$, but $\{x+y\} \cup \Gamma = \Gamma$, so $\Gamma \vDash a$. Consequently there is a n_0 such that $\Gamma \vDash_{n_0} a$. By induction on n we will prove that
$\forall n \forall x(\Gamma \vDash_n x \to x \in \Gamma)$ (1)
Case 1: $n = 0$
Let $x \in B$ and $\Gamma \vDash_0 x$. Obviously $x \in \Gamma$.
Case 2: $n > 0$
Let $x \in B$ and $\Gamma \vDash_n x$. We will prove that $x \in \Gamma$. There are x_1, x_2 such that $x_1, x_2 \vdash x$, $\Gamma \vDash_{k_1} x_1$, $\Gamma \vDash_{k_2} x_2$, where k_1, $k_2 < n$. By the induction hypothesis and $\Gamma \vDash_{k_1} x_1$, $\Gamma \vDash_{k_2} x_2$, we get that x_1, $x_2 \in \Gamma$. We also have $\Gamma \in M$, $x_1, x_2 \vdash x$, so $x \in \Gamma$.
Consequently (1) is true. We also have $\Gamma \vDash_{n_0} a$. Consequently $a \in \Gamma$ - a contradiction with $\Gamma \in M$. Consequently $\{x\} \cup \Gamma \nvDash a$ or $\{y\} \cup \Gamma \nvDash a$. Without loss of generality, suppose $\{x\} \cup \Gamma \nvDash a$. Let $\Gamma' = \{z : \{x\} \cup \Gamma \vDash z\}$. We will prove that $\Gamma' \in M$. Obviously $\Gamma' \subseteq B$, $A \subseteq \Gamma \subseteq \Gamma'$. Since $\{x\} \cup \Gamma \nvDash a$, $a \notin \Gamma'$. Let x_1, $x_2 \in \Gamma'$, $x_1, x_2 \vdash x_3$. We will prove that $x_3 \in \Gamma'$. We have $\{x\} \cup \Gamma \vDash_n x_1$, $\{x\} \cup \Gamma \vDash_n x_2$ for some n. Consequently $\{x\} \cup \Gamma \vDash_{n+1} x_3$. Consequently $x_3 \in \Gamma'$. Thus $\Gamma' \in M$. We have $\Gamma \subseteq \Gamma'$, Γ is a maximal element of $(M, \subseteq)$, $\Gamma' \in M$, so $\Gamma = \Gamma'$ and hence $x \in \Gamma$. Consequently Γ is an abstract point. $\square$

Theorem 6.4 *(Representation theorem) Let* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *be an ECA. Then there is a compact, semiregular,* T_0 *topological space* X *and an embedding* h *of* $\underline{B}$ *into* $RC(X)$.

Proof. Let X be the set of all abstract points of $\underline{B}$ and for $a \in B$, let $h(a) = \{\Gamma \in X : a \in \Gamma\}$. The set $\{h(a) : a \in B\}$ can be taken as a closed basis for a topology of X.

We will prove that h is an injection. Let a, b are different elements of B. Suppose for the sake of contradiction that $a, a \vdash b$, $b, b \vdash a$. Using axiom (5) of ECA, we get $a \leq b$, $b \leq a$ and hence $a = b$ - a contradiction. Without loss of generality suppose $a, a \nvdash b$. Thus using lemma 5.8, we obtain $\{a\} \nvDash b$. By lemma 6.3, there is an abstract point Γ such that $\{a\} \subseteq \Gamma$ and $b \notin \Gamma$. Consequently $\Gamma \in h(a)$, $\Gamma \notin h(b)$, i.e. $h(a) \neq h(b)$.

From the definition of abstract point we obtain $h(0) = \emptyset$, $h(1) = X$.

Let a, $b \in B$. We will prove that $h(a+b) = h(a) + h(b)$. We have $h(a+b) = \{\Gamma \in X : a + b \in \Gamma\}$, $h(a) + h(b) = \{\Gamma \in X : a \in \Gamma\} \cup \{\Gamma \in X : b \in \Gamma\}$. Obviously $h(a+b) \subseteq h(a)+h(b)$. Let $\Gamma \in h(a) \cup h(b)$. Without loss of generality suppose $\Gamma \in h(a)$, i.e. $\Gamma \in X$ and $a \in \Gamma$. From $a \leq a + b$ and axiom (2) we get $a, a \vdash a + b$. But $a \in \Gamma$ and Γ is an abstract point, so $a + b \in \Gamma$, so $\Gamma \in h(a+b)$.

Let a, b, $c \in B$. Obviously $a, b \vdash c$ implies $h(a), h(b) \vdash h(c)$. Suppose that $h(a), h(b) \vdash h(c)$. We will prove that $a, b \vdash c$. Suppose for the sake of contradiction that $\{a, b\} \nvDash c$. By lemma 6.3, we get that there is an abstract point Γ such that $a, b \in \Gamma$, $c \notin \Gamma$. We have $\Gamma \in h(a) \cap h(b)$; $h(a) \cap h(b) \subseteq h(c)$ (since $h(a), h(b) \vdash h(c)$); so $\Gamma \in h(c)$; so $c \in \Gamma$ - a contradiction. Consequently $\{a, b\} \vDash c$. By lemma 5.8, $a, b \vdash c$.

Let a, $b \in B$. We have $a \leq b \leftrightarrow a, a \vdash b \leftrightarrow h(a), h(a) \vdash h(b) \leftrightarrow h(a) \cap h(a) \subseteq h(b) \leftrightarrow h(a) \subseteq h(b)$.

In a similar way as in [32] (Proposition 2.3.4 (1),(2)) we prove that $h(a^*) = Cl_X(-h(a))$, $h(a)$ is a regular closed set for every $a \in B$. Consequently X is semiregular.

We have $h(a.b) = h((a^* + b^*)^*) = (h(a)^* + h(b)^*)^* = h(a).h(b)$ for all a, $b \in B$.

Let a, $b \in B$. Obviously aCb iff $h(a)Ch(b)$.

Let $a \in B$. We will prove that $c^o(h(a))$ implies $c^o(a)$. Let $c^o(h(a))$. Let $b, c \in B$, $b \neq 0$, $c \neq 0$, $a = b + c$. We will prove that $b, c \nvdash a^*$. We have $h(b) \neq \emptyset$, $h(c) \neq \emptyset$, $h(a) = h(b) + h(c)$, $c^o(h(a))$. Consequently $h(b), h(c) \nvdash (h(a))^*$, so $h(b), h(c) \nvdash h(a^*)$. Consequently $b, c \nvdash a^*$.

Let $a \in B$. Let $c^o(a)$. Suppose for the sake of contradiction that $\neg c^o(h(a))$. Consequently there are b, $c \in RC(X)$ such that $b \neq \emptyset$, $c \neq \emptyset$, $h(a) = b \cup c$ and $b \cap c \subseteq h(a)^*$ (proposition 3.1). b and c are closed, so $b = \bigcap_{i \in I} h(b_i)$, $c = \bigcap_{j \in J} h(c_j)$ for some sets I and J. Let $A = \{b_i : i \in I\} \cup \{c_j : j \in J\}$. Suppose for the sake of contradiction that $A \vDash a^*$. Thus by lemma 5.7, we get that there is a finite nonempty subset of A, A', such that $A' \vDash a^*$. Let $b_{i_1} \in \{b_i : i \in I\}$, $c_{j_1} \in \{c_j : j \in J\}$. Let $A' = \{b_{i_2}, b_{i_3}, \ldots, b_{i_k}\} \cup \{c_{j_2}, c_{j_3}, \ldots, c_{j_l}\}$ for some k, $l \geq 1$. Let $b' = b_{i_1} \cdot b_{i_2} \cdot \ldots \cdot b_{i_k}$, $c' = c_{j_1} \cdot c_{j_2} \cdot \ldots \cdot c_{j_l}$. From $A' \vDash a^*$ and lemma 5.2 we get that $\{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \cup \{c_{j_1}, c_{j_2}, \ldots, c_{j_l}\} \vDash a^*$. Using this fact, the definitions of b' and c' and lemma 5.8, we obtain $b', c' \vdash a^*$. Suppose for the sake of contradiction that $b' \cdot a = 0$. Consequently $h(b_{i_1}) \cdot h(b_{i_2}) \cdot \ldots \cdot h(b_{i_k}) \cdot h(a) = h(0) = \emptyset$. Thus by lemma 6.2, we have $Cl_X Int_X(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$, so $Int_X(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$. We have $h(a) = b \cup c$ and therefore $b = b \cap h(a) \subseteq h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)$. Consequently $Int_X b \subseteq Int_X(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$, i.e. $Int_X b = \emptyset$. We have $b \in RC(X)$ and from here $b = Cl_X Int_X b = Cl_X \emptyset = \emptyset$ - a contradiction. Consequently $b' \cdot a \neq 0$ (1). Similarly $c' \cdot a \neq 0$ (2). We have $h(a) = b \cup c \subseteq h(b_{i_m}) \cup h(c_{j_n})$ for all $m = 1, \ldots, k$, $n = 1, \ldots, l$. Consequently $a \leq b_{i_m} + c_{j_n}$ for all $m = 1, \ldots, k$, $n = 1, \ldots, l$. We also have $b' + c' = (b_{i_1} \cdot \ldots \cdot b_{i_k}) + (c_{j_1} \cdot \ldots \cdot c_{j_l}) = (b_{i_1} + c_{j_1}) \cdot \ldots \cdot (b_{i_k} + c_{j_1}) \cdot \ldots \cdot (b_{i_1} + c_{j_l}) \cdot \ldots \cdot (b_{i_k} + c_{j_l})$. Consequently $a \leq b' + c'$. Thus $a = (b' + c') \cdot a = b' \cdot a + c' \cdot a$ (3). From $b' \cdot a \leq b'$ by axiom (2), we have $b' \cdot a, b' \cdot a \vdash b'$ (4). From axiom (3) we get $b' \cdot a, c' \cdot a \vdash b' \cdot a$ (5). From (5) and (4) by axiom (4), we obtain $b' \cdot a, c' \cdot a \vdash b'$ (6). Similarly $c' \cdot a, b' \cdot a \vdash c'$ and from here by axiom (1), we have $b' \cdot a, c' \cdot a \vdash c'$ (7). From (6), (7) and $b', c' \vdash a^*$ we get, by axiom (4), that $b' \cdot a, c' \cdot a \vdash a^*$ (8). From $c^o(a)$, (1), (2) and (3) we obtain $b' \cdot a, c' \cdot a \nvdash a^*$ - a contradiction with (8). Consequently $A \nvDash a^*$. Thus by lemma 6.3, we get that there is an abstract point Γ_1 such that $A \subseteq \Gamma_1$, $a^* \notin \Gamma_1$. Since $A \subseteq \Gamma_1$, we have $b_i \in \Gamma_1$ for every $i \in I$ and $c_j \in \Gamma_1$ for every $j \in J$. We also have that Γ_1 is an abstract point, so $\Gamma_1 \in h(b_i)$ for every $i \in I$ and $\Gamma_1 \in h(c_j)$ for every $j \in J$. Consequently $\Gamma_1 \in b$, $\Gamma_1 \in c$. We have $a^* \notin \Gamma_1$, so $\Gamma_1 \notin h(a^*)$. Thus $b \cap c \nsubseteq h(a^*)$, i.e. $b \cap c \nsubseteq h(a)^*$ - a contradiction. Consequently $c^o(h(a))$. Consequently h is an embedding.

As in [32] (Lemma 2.3.6), replacing the notion of clan with abstract point, we prove that X is a compact, T_0 space. $\square$

7 Concluding remarks

One of the motivations to introduce ECA is that its language is richer and makes possible to express the predicate of internal connectedness of a region. Here we mention without proof some other things which can be expressed in its language and also some things which are not expressible and need further extension.

It is known that the intersection of regular closed sets is not in general a regular closed set. Let X be a topological space and for the elements of $RC(X)$ consider the relation: $RC_{\cap}(a,b) \leftrightarrow a \cap b$ is a regular closed set. Very probably this relation is not expressible in contact algebras, but it is expressible in ECA as follows: $RC_{\cap}(a,b) \leftrightarrow a, b \vdash a \cdot b$.

Another interesting property which is expressible in ECA is related to the existence or non-existence of cavities in a region, presenting a physical body. Then the complement $-a$ is an open set which is not connected. So connectedness of $-a$ expresses that a has no cavities. This is expressible in ECA by $c^o(a^*)$.

If the internal part of a region is not connected then we cannot express the number of its components. For that purpose we need a more general relation between finite number of regions, which topological meaning is expressible in $RC(X)$ by the relation: $a_1, \ldots, a_n \vdash b$ iff $a_1 \cap \ldots \cap a_n \subseteq b$. Such relations for all n are studied in the paper [31].

By this relation one can express also n-ary contact by $C_n(a_1, \ldots, a_n)$ iff $a_1, \ldots, a_n \nvdash 0$, which is not expressible neither in contact algebras nor in ECA.

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References

[1] M. Aiello, I. Pratt-Hartmann and J. van Benthem (eds.), *Handbook of spatial logics*, Springer, 2007.

[2] R. Balbes and P. Dwinger. (1974). *Distributive Lattices.* University of Missouri Press, Columbia.

[3] P. Balbiani (Ed.), Special Issue on Spatial Reasoning, *J. Appl. Non-Classical Logics* **12** (2002), No. 3-4.

[4] P. Balbiani, T. Tinchev and D. Vakarelov. Modal Logics for Region-based Theory of Space. *Fundamenta Informaticae*, Special Issue: Topics in Logic, Philosophy and Foundation of Mathematics and Computer Science in Recognition of Professor Andrzej Grzegorczyk, vol. (81), (1-3), (2007), 29-82.

[5] B. Bennett and I. Düntsch. Axioms, Algebras and Topology. In:*Handbook of Spatial Logics*, M. Aiello, I. Pratt, and J. van Benthem (Eds.), Springer, 2007, 99-160.

[6] A. Cohn and S. Hazarika. Qualitative spatial representation and reasoning: An overview. *Fuandamenta informaticae* 46 (2001), 1-20.

[7] A. Cohn and J. Renz. Qualitative spatial representation and reasoning. In: F. van Hermelen, V. Lifschitz and B. Porter (Eds.) *Handbook of Knowledge Representation*, Elsevier, 2008,551-596.

[8] G. Dimov and D. Vakarelov. Contact algebras and region–based theory of space: A proximity approach I. *Fundamenta Informaticae*, vol. 74, No 2-3,(2006) 209-249

[9] I. Düntsch (Ed.), Special issue on Qualitative Spatial Reasoning, *Fundam. Inform.*, 46 (2001).

[10] I. Düntsch and D. Vakarelov. Region-based theory of discrete spaces: A proximity approach. In: Nadif, M., Napoli, A., SanJuan, E., and Sigayret, A. EDS, *Proceedings of Fourth International Conference Journées de l'informatique Messine*, 123-129, Metz, France, 2003. Journal version in: *Annals of Mathematics and Artificial Intelligence*, **49**(1-4):5-14, 2007.

[11] I. Düntsch and M. Winter. A representation theorem for Boolean contact algebras. *Theoretical Computer Science (B)*, 347 (2005), 498-512.

[12] M. Egenhofer and R. Franzosa, Point-set topological spatial relations. *Int. J. Geogr. Inform. Systems*, 5:161-174, 1991.

[13] R. Engelking, *General topology*, Polish Scientiphic Publishers, 1977.

[14] T. Hahmann and M. Gruninger. Region-based Theories of Space: Mereotopology and Beyond. S. Hazarika (ed.): *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions.*, 2012. pp. 1-62. IGI Publishing.

[15] *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions* S. M. Hazarika (Ed.), IGI Global; 1st ed. (May 31, 2012)

[16] R. Kontchakov, I. Pratt-Hartmann, F. Wolter and M. Zakharyaschev. Topology, connectedness, and modal logic, *In C. Areces and R. Goldblatt, editors, Advances in Modal Logic*, volume 7, College Publications 2008, London, 151-176

[17] R. Kontchakov, I. Pratt-Hartmann and M. Zakharyaschev. Interpreting Topological Logics over Euclidean Spaces, *F. Lin, U. Sattler and M. Truszczynski, editors, Proceedings of KR*, Toronto, Canada, May, 9-13, AAAI Press, 2010, 534-544

[18] R. Kontchakov, I. Pratt-Hartmann and M. Zakharyaschev. Topological logics over Euclidean spaces, *Proceedings of Topology, Algebra and Categories in Logic, TACL 2009*, Amsterdam, July, 7-11, 2009

[19] R. Kontchakov, I. Pratt-Hartmann, F.Wolter and M. Zakharyaschev. On the computational complexity of spatial logics with connectedness constraints, *I. Cervesato, H. Veith and A. Voronkov, editors, Proceedings of LPAR*, 2008, Doha, Qatar, November, 22-27, 2008, 574-589, LNAI, vol. 5330, Springer 2008

[20] R. Kontchakov, I. Pratt-Hartmann, F. Wolter and M. Zakharyaschev. Spatial logics with connectedness predicates, *Logical Methods in Computer Science*, 6 (3:7), 2010

[21] T. de Laguna Point, line and surface as sets of solids, *J. Philos*, 19 (1922), 449-461.

[22] I. Pratt-Hartmann Empiricism and Racionalizm in Region-based Theories of Space., *Fundamenta Informaticae*, 45, 2001, 159-186

[23] I. Pratt-Hartmann A topological constraint language with component counting., *J. of Applied Nonclassical Logics*, 12, 2002, 441-467

[24] I. Pratt-Hartmann First-order region-based theories of space, *In: Logic of Space, M. Aiello, I. Pratt and J. van Benthem (Eds.)*, Springer, 2007.

[25] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In: B. Nebel, W. Swartout, C. Rich (Eds.) *Proceedings of the 3rd International Conference Knowledge Representation and Reasoning*, Morgan Kaufmann, Los Allos, CA, pp. 165–176, 1992.

[26] E. Shchepin. Real-valued functions and spaces close to normal. *Siberian mathematical journal*, **13** (1972) 820–830

[27] J. Stell. Boolean connection algebras: A new approach to the Region Connection Calculus, *Artif. Intell.* **122** (2000), 111–136.

[28] M. Stone. Topological representations of distributive lattices and Brouwerian logics. *Časopis Pěst. Mat.*, 67, (1937), 1-25.

[29] W. J. Thron. Proximity structures and grills, *Math. Ann.*, **206** (1973), 35–62.

[30] T. Tinchev and D. Vakarelov. Logics of Space with Connectedness Predicates: Complete axiomatizations. *Proceedings of the conference 8th Advances in Modal Logic. L. Beklemishev, V. Goranko, V. Shehtman (eds.), Advances in Modal Logic, 8*, College Publications, London, (2010), 409-427.

[31] D.Vakarelov. Extended mereotopology based on sequent algebras. *Proceedings of Workshop dedicated to the 65 anniversary of Luis Farinas del Cerro*, 3-4 March, 2016, Toulouse, France.

[32] D. Vakarelov. Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics. In: Dov Gabbay et al. (Eds.) *Mathematical Problems from Applied Logics. New Logics for the XXIst Century.* II. Springer, 2007, 267-348.

[33] D. Vakarelov, I. Düntsch and B. Bennett. A note on proximity spaces and connection based mereology. In: Welty, C. and Smith, B., editors, *Proceedings of the 2nd International Conference on Formal Ontology in Information Systems (FOIS'01)*, ACM, (2001), 139-150.

[34] D. Vakarelov, G. Dimov, I. Düntsch, and B. Bennett. A proximity approach to some region based theory of space. *Journal of applied non-classical logics*, vol. 12, No3-4 (2002), 527-559

[35] H. de Vries. *Compact spaces and compactifications*, Van Gorcum, 1962

[36] A. N. Whitehead. *Process and Reality*, New York, MacMillan, 1929.