Logics for extended distributive contact lattices

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Abstract

The notion of contact algebra is one of the main tools in the region based theory of space. It is an extension of Boolean algebra with an additional relation C called contact. There are some problems related to the motivation of the operation of Boolean complementation. Because of this in [18] this operation is dropped and the language of distributive lattices is extended by considering as non-definable primitives the relations of contact, nontangential inclusion and dual contact. It is obtained an axiomatization of the theory consisting of the universal formulas in the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). In this paper we consider several logics, corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for these logics. It turns out that they are decidable.

Keywords: mereotopology, distributive mereotopology, contact algebras, extended distributive contact lattices, spacial logics, completeness theorems.

1 Introduction

In the classical Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast the region-based theory of space (RBTS) has as primitives the more realistic notion of region as an abstraction of physical body, together with some basic relations and operations on regions. Some of these relations are mereological - part-of ($x \leq y$), overlap (xOy), its dual underlap ($x\widehat{O}y$). Other relations are topological - contact (xCy), nontangential part-of ($x \ll y$), dual contact ($x\widehat{C}y$) and some others definable by means of the contact and part-of relations. This is one of the reasons that the extension of mereology with these new relations is commonly called *mereotopology*. There is no clear difference in the literature between RBTS and mereotopology, and by some authors RBTS is related rather to the so called *mereogeometry*, while mereotopology is considered only as a kind of point-free topology, considering mainly topological properties of things. The origin of RBTS goes back to Whitehead [31] and de Laguna [19]. According to Whitehead points, as well as the other primitive notions in Euclidean geometry like lines and planes, do not have separate existence in reality and because of this are not appropriate for primitive notions; but points have to be definable by the other primitive notions.

Survey papers about RBTS are [27, 5, 16, 20] (see also the handbook [1] and [4] for some logics of space). Surveys concerning various applications are

[6, 7] and the book [17] (see also special issues of Fundamenta Informaticæ[9] and the Journal of Applied Nonclassical Logics [3]). RBTS has applications in computer science because of its more simple way of representing qualitative spatial information and it initiated a special field in Knowledge Representation (KR) called Qualitative Spatial Representation and Reasoning (QSRR). One of the most popular systems in QSRR is the Region Connection Calculus (RCC) introduced in [21].

The notion of contact algebra is one of the main tools in RBTS. This notion appears in the literature under different names and formulations as an extension of Boolean algebra with some mereotopological relations [30, 23, 28, 29, 5, 13, 8, 10]. The simplest system, called just contact algebra was introduced in [8] as an extension of Boolean algebra $\underline{B} = (B, 0, 1, \cdot, +, *)$ with a binary relation C called contact and satisfying several simple axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,

(C2) If aCb and $a \leq c$ and $b \leq d$, then cCd,

(C3) If $aC(b + c)$, then aCb or aCc,

(C4) If aCb, then bCa,

(C5) If $a \cdot b \neq 0$, then aCb.

The elements of the Boolean algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the empty region. The contact relation is used also to define some other important mereotopological relations like non-tangential inclusion, dual contact and others.

The standard model of Boolean algebra is the algebra of subsets of a given universe. This model cannot express all kinds of contact, for example, the external contact in which the regions share only a boundary point. Because of this standard models of contact algebras are topological and are the contact algebras of regular closed sets in a given topological space.

Non-tangential inclusion and dual contact are defined by the operation of Boolean complementation. But there are some problems related to the motivation of this operation. A question arises: if the region a in some universe represents a physical body, then what kind of body represents a^*? - it depends on the universe. To avoid this problem, we can drop the operation of complement and replace the Boolean part of a contact algebra with distributive lattice. First steps in this direction were made in [11, 12], introducing the notion of distributive contact lattice. In a distributive contact lattice the only mereotopological relation is the contact relation. In [18] the language of distributive contact lattices is extended by considering as non-definable primitives the relations of contact, nontangential inclusion and dual contact. It is obtained an axiomatization of the theory consisting of the universal formulas in the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). The well known RCC-8 system of mereotopological relations is definable in the language of EDC-lattices. In [18] are considered also some axiomatic extensions of EDC-lattices yielding representations in T_1 and T_2 spaces. In this paper we consider a first-order language without quantifiers corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for several logics for this language. It turns out that all these logics are decidable.

2 Extended distributive contact lattices

Definition 2.1 *[18]* **Extended distributive contact lattice.** *Let $\underline{D} = (D, \leq, 0, 1, \cdot, +, C, \widehat{C}, \ll)$ be a bounded distributive lattice with three additional relations $C, \widehat{C}, \ll$, called respectively* ***contact***, ***dual contact*** *and* ***nontangential part-of***. *The obtained system, denoted shortly by $\underline{D} = (D, C, \widehat{C}, \ll)$, is called* ***extended distributive contact lattice*** *(EDC-lattice, for short) if it satisfies the axioms listed below.*

Notations: if R is one of the relations $\leq, C, \widehat{C}, \ll$, then its complement is denoted by $\overline{R}$. We denote by $\geq$ the converse relation of $\leq$ and similarly $\gg$ denotes the converse relation of $\ll$.

Axioms for C alone: *The axioms (C1)-(C5) mentioned above.*

Axioms for $\widehat{C}$ alone:

($\widehat{C}1$) *If $a\widehat{C}b$, then $a, b \neq 1$,*

($\widehat{C}2$) *If $a\widehat{C}b$ and $a' \leq a$ and $b' \leq b$, then $a'\widehat{C}b'$,*

($\widehat{C}3$) *If $a\widehat{C}(b \cdot c)$, then $a\widehat{C}b$ or $a\widehat{C}c$,*

($\widehat{C}4$) *If $a\widehat{C}b$, then $b\widehat{C}a$,*

($\widehat{C}5$) *If $a + b \neq 1$, then $a\widehat{C}b$.*

Axioms for $\ll$ alone:

($\ll 1$) $0 \ll 0$,

($\ll 2$) $1 \ll 1$,

($\ll 3$) *If $a \ll b$, then $a \leq b$,*

($\ll 4$) *If $a' \leq a \ll b \leq b'$, then $a' \ll b'$,*

($\ll 5$) *If $a \ll c$ and $b \ll c$, then $(a + b) \ll c$,*

($\ll 6$) *If $c \ll a$ and $c \ll b$, then $c \ll (a \cdot b)$,*

($\ll 7$) *If $a \ll b$ and $(b \cdot c) \ll d$ and $c \ll (a + d)$, then $c \ll d$.*

Mixed axioms:

($MC1$) *If aCb and $a \ll c$, then $aC(b \cdot c)$,*

($MC2$) *If $a\overline{C}(b \cdot c)$ and aCb and $(a \cdot d)\overline{C}b$, then $d\widehat{C}c$,*

($M\widehat{C}1$) *If $a\widehat{C}b$ and $c \ll a$, then $a\widehat{C}(b + c)$,*

($M\widehat{C}2$) *If $a\overline{\widehat{C}}(b + c)$ and $a\widehat{C}b$ and $(a + d)\overline{\widehat{C}}b$, then dCc,*

($M \ll 1$) *If $a\overline{\widehat{C}}b$ and $(a \cdot c) \ll b$, then $c \ll b$,*

($M \ll 2$) *If $a\overline{C}b$ and $b \ll (a + c)$, then $b \ll c$.*

For the language of EDCL we can introduce the following principle of duality: dual pairs $(0,1), (\cdot,+), (\leq,\geq), (C,\widehat{C}), (\ll,\gg)$. For each statement (definition) A of the language we can define in an obvious way its dual $\widehat{A}$. For each axiom Ax from the list of axioms of EDCL its dual $\widehat{Ax}$ is also an axiom.

Theorem 2.2 *[18]* **Relational representation Theorem of EDC-latices.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then there is a relational system $\underline{W} = (W, R)$ with reflexive and symmetric R and an embedding h into the EDC-lattice of all subsets of W.*

Corollary 2.3 *[18] Every EDC-lattice can be isomorphically embedded into a contact algebra.*

In [18] are formulated several additional axioms for EDC-lattices which are adaptations for the language of EDC-lattices of some known axioms considered in the context of contact algebras:
(Ext O) $a \not\leq b \rightarrow (\exists c)(a \cdot c \neq 0$ and $b \cdot c = 0)$ - *extensionality of overlap*,
(Ext $\widehat{O}$) $a \not\leq b \rightarrow (\exists c)(a + c = 1$ and $b + c \neq 1)$ - *extensionality of underlap.*
(Ext C) $a \neq 1 \rightarrow (\exists b \neq 0)(a\overline{C}b)$ - *C-extensionality*,
(Ext $\widehat{C}$) $a \neq 0 \rightarrow (\exists b \neq 1)(a\overline{\widehat{C}}b)$ - *$\widehat{C}$-extensionality*,
(Con C) $a \neq 0, b \neq 0$ and $a + b = 1 \rightarrow aCb$ - *C-connectedness axiom* ,
(Con $\widehat{C}$) $a \neq 1, b \neq 1$ and $a \cdot b = 0 \rightarrow a\widehat{C}b$ - *$\widehat{C}$-connectedness axiom* ,
(Nor 1) $a\overline{C}b \rightarrow (\exists c,d)(c + d = 1, a\overline{C}c$ and $b\overline{C}d)$,
(Nor 2) $a\overline{\widehat{C}}b \rightarrow (\exists c,d)(c \cdot b = 0, a\overline{\widehat{C}}c$ and $b\overline{\widehat{C}}d)$,
(Nor 3) $a \ll b \rightarrow (\exists c)(a \ll c \ll b)$.
(U-rich $\ll$) $a \ll b \rightarrow (\exists c)(b + c = 1$ and $a\overline{C}c)$,
(U-rich $\widehat{C}$) $a\overline{\widehat{C}}b \rightarrow (\exists c,d)(a + c = 1, b + d = 1$ and $c\overline{C}d)$,
(O-rich $\ll$) $a \ll b \rightarrow (\exists c)(a \cdot c = 0$ and $c\overline{\widehat{C}}b)$,
(O-rich C) $a\overline{C}b \rightarrow (\exists c,d)(a \cdot c = 0, b \cdot d = 0$ and $c\overline{\widehat{C}}d)$.
The following lemma relates topological properties to the properties of the relations C, $\widehat{C}$ and $\ll$ and shows the importance of the additional axioms for EDC-lattices.

Lemma 2.4 *[18]*
(i) If X is semiregular, then X is weakly regular iff $RC(X)$ satisfies any of the axioms (Ext C), (Ext $\widehat{C}$).
(ii) X is κ-normal iff $RC(X)$ satisfies any of the axioms (Nor 1), (Nor 2) and (Nor 3).
(iii) X is connected iff $RC(X)$ satisfies any of the axioms (Con C), (Con $\widehat{C}$).
(iv) If X is compact and Hausdorff, then $RC(X)$ satisfies (Ext C), (Ext $\widehat{C}$) and (Nor 1), (Nor 2) and (Nor 3) .

Definition 2.5 *[18]* **U-rich and O-rich EDC-lattices.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:*

(i) $\underline{D}$ is called U-rich EDC-lattice if it satisfies the axioms (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) $\underline{D}$ is called O-rich EDC-lattice if it satisfies the axioms (Ext O), (O-rich $\ll$) and (O-rich $\widehat{C}$).

Theorem 2.6 *[18]* **Topological representation theorem for U-rich EDC-lattices**
Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an U-rich EDC-lattice. Then there exists a compact semiregular T_0-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover:

(i) $\underline{D}$ satisfies (Ext C) iff $RC(X)$ satisfies (Ext C); in this case X is weakly regular.

(ii) $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C); in this case X is connected.

(iii) $\underline{D}$ satisfies (Nor 1) iff $RC(X)$ satisfies (Nor 1); in this case X is κ-normal.

3 Preliminaries

Here we have constructions almost the same as in [4] (pages 57-59).

Let $\mathcal{L}$ be a quantifier-free countable first-order language with equality. Let δ be a formula in $\mathcal{L}$. We define $\bot = \delta \wedge \neg\delta$, $\top = \delta \vee \neg\delta$. Let I be an arbitrary set; for every $i \in I$ β_i be a formula for $\mathcal{L}$ with variables among $p_{i_1}, \ldots, p_{i_{n_i}}, q_{i_1}, \ldots, q_{i_{m_i}}$; for every $i \in I$ γ_i be a formula for $\mathcal{L}$ with variables among $q_{i_1}, \ldots, q_{i_{m_i}}$. ($p_{i_1}, \ldots, p_{i_{n_i}}, q_{i_1}, \ldots,$
$q_{i_{m_i}}$ are different variables.)

Let $\mathbb{L}$ be a logic for $\mathcal{L}$, containing all axioms of the classical propositional logic, whose rules are MP and all rules of the type:

$$\frac{\varphi \to \beta_i\left[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\right] \text{ for all sequences of variables } r_{i_1}\ldots r_{i_{n_i}}}{\varphi \to \gamma_i\left[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\right]}, \tag{3.1}$$

where $i \in I$, φ is a formula for $\mathcal{L}$, $a_{i_1}\ldots a_{i_{m_i}}$ are terms for $\mathcal{L}$. Let also if α is an axiom of $\mathbb{L}$ with variables $p_1, \ldots, p_n$ and $a_1, \ldots, a_n$ are terms in $\mathcal{L}$, then $\alpha\left[\frac{p_1,\ldots,p_n}{a_1,\ldots,a_n}\right]$ is also an axiom of $\mathbb{L}$. (Here $[\ldots]$ means a simultaneous substitution.)

We call the following axiom corresponding to the rule 3.1:
$\neg\gamma_i\left[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\right] \to \exists x_{i_1}\ldots\exists x_{i_{n_i}}\neg\beta_i\left[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{x_{i_1}\ldots x_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\right]$, where $x_{i_1}, \ldots, x_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \ldots, a_{i_{m_i}}$, different from $p_{i_1}, \ldots, p_{i_{n_i}}, q_{i_1} \ldots, q_{i_{m_i}}$.

Remark 3.1 *Another approach is to be considered rules of the kind:*
$\frac{\varphi\to\beta_i\left[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\right]}{\varphi\to\gamma_i\left[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\right]}$*, where $r_{i_1}\ldots r_{i_{n_i}}$ are variables not occuring in $a_{i_1}, \ldots, a_{i_{m_i}}$ and φ (see [4]).*

Definition 3.2 *[4] A set of formulas for $\mathcal{L}$ Γ is a $\mathbb{L}$-theory, if satisfies the following conditions:*
(i) Γ contains all theorems of $\mathbb{L}$;
(ii) If α, $\alpha \to \beta \in \Gamma$, then $\beta \in \Gamma$;
(iii) For every rule of the type above, we have: if $\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right] \in \Gamma$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$, then $\varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right] \in \Gamma$.

A $\mathbb{L}$-theory Γ is consistent, if $\bot \notin \Gamma$.

Γ is a maximal $\mathbb{L}$-theory, if it is a consistent $\mathbb{L}$-theory and for every consistent $\mathbb{L}$-theory Δ, if $\Gamma \subseteq \Delta$, then $\Gamma = \Delta$.

Lemma 3.3 *[4] [Extension lemma] Let Γ be a $\mathbb{L}$-theory and α be a formula. Let $\Delta = \Gamma + \alpha \overset{def}{=} \{\beta:\ \alpha \to \beta \in \Gamma\}$. Then:*
(i) Δ is the smallest $\mathbb{L}$-theory, containing Γ and α;
(ii) Δ is inconsistent $\leftrightarrow \neg\alpha \in \Gamma$;
(iii) For any $i \in I$, φ - a formula for $\mathcal{L}$, $a_{i_1},\dots,a_{i_{m_i}}$ - terms for $\mathcal{L}$, we have: if $\Gamma + \neg\Big(\varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is consistent, then there are variables $r_{i_1},\dots,r_{i_{n_i}}$ such that $\Big(\Gamma + \neg\Big(\varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\Big) + \neg\Big(\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is a consistent $\mathbb{L}$-theory.

Proof. (i) We will prove that $\Gamma \subseteq \Delta$. Let $\gamma \in \Gamma$. We will prove that $\gamma \in \Delta$. It suffices to prove that $\alpha \to \gamma \in \Gamma$. The formula $\gamma \to (\alpha \to \gamma)$ is a theorem of the classical propositional logic, Γ is a $\mathbb{L}$-theory, so $\gamma \to (\alpha \to \gamma) \in \Gamma$. We also have $\gamma \in \Gamma$, Γ is closed under MP, so $\alpha \to \gamma \in \Gamma$.

We will prove that $\alpha \in \Delta$. It suffices to prove that $\alpha \to \alpha \in \Gamma$. But this is true because $\alpha \to \alpha$ is a theorem of $\mathbb{L}$.

We will prove that Δ is a $\mathbb{L}$-theory. Γ contains all theorems of $\mathbb{L}$ and $\Gamma \subseteq \Delta$, consequently Δ contains all theorems of $\mathbb{L}$. Let γ_1, $\gamma_1 \to \gamma_2 \in \Delta$. We will prove that $\gamma_2 \in \Delta$. We have $\alpha \to \gamma_1 \in \Gamma$, $\alpha \to (\gamma_1 \to \gamma_2) \in \Gamma(1)$. The formula $(\alpha \to \gamma_1) \to ((\alpha \to (\gamma_1 \to \gamma_2)) \to (\alpha \to \gamma_2))$ is a theorem of the classical propositional logic and consequently is in Γ. Using this fact, (1) and the closeness of Γ under MP, we get $\alpha \to \gamma_2 \in \Gamma$, so $\gamma_2 \in \Delta$. Let $i \in I$, φ is a formula for $\mathcal{L}$, $a_{i_1},\dots,a_{i_{m_i}}$ are terms for $\mathcal{L}$. Let $\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right] \in \Delta$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$. Let $\gamma_i' = \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]$. We will prove that $\varphi \to \gamma_i' \in \Delta$. We have $\alpha \to \Big(\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]\Big) \in \Gamma$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$, so $(\alpha \wedge \varphi) \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right] \in \Gamma$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$. From here and the fact that Γ is a $\mathbb{L}$-theory, we obtain $(\alpha \wedge \varphi) \to \gamma_i' \in \Gamma$, so $\alpha \to (\varphi \to \gamma_i') \in \Gamma$, so $\varphi \to \gamma_i' \in \Delta$. Consequently Δ is a $\mathbb{L}$-theory.

Let Δ' is a $\mathbb{L}$-theory, containing Γ and α. We will prove that $\Delta \subseteq \Delta'$. Let $\gamma \in \Delta$. We will prove that $\gamma \in \Delta'$. We have $\alpha \to \gamma \in \Gamma$, $\Gamma \subseteq \Delta'$, so $\alpha \to \gamma \in \Delta'$. But $\alpha \in \Delta'$ and Δ' is closed under MP, so $\gamma \in \Delta'$. Consequently Δ is the smallest $\mathbb{L}$-theory, containing Γ and α.
(ii) Let Δ is inconsistent. We will prove that $\neg\alpha \in \Gamma$. $\bot \in \Delta$ and hence $\alpha \to \bot \in \Gamma$. $(\alpha \to \bot) \to \neg\alpha$ is a theorem of the classical propositional logic and therefore is in Γ. Consequently $\neg\alpha \in \Gamma$.

Let $\neg\alpha \in \Gamma$. We will prove that Δ is inconsistent, i.e. that $\bot \in \Delta$. The formula $\neg\alpha \to (\alpha \to \bot) \in \Gamma$, $\neg\alpha \in \Gamma$, so $\alpha \to \bot \in \Gamma$, i.e. $\bot \in \Delta$.

(iii) Let $i \in I$, φ be a formula for $\mathcal{L}$, $a_{i_1}, \ldots, a_{i_{m_i}}$ be terms for $\mathcal{L}$, $\Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ is consistent. Suppose for the sake of contradiction that for all sequences of variables $r_{i_1}, \ldots, r_{i_{n_i}}$ $\Big(\Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)\Big) + \neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ is inconsistent. Consequently $\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big) \in \Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ for all sequences of variables $r_{i_1}, \ldots, r_{i_{n_i}}$. Thus we get $\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big] \in \Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$. We also have $\neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big) \in \Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$, so $\bot \in \Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ - a contradiction. Consequently there is a sequence of variables $r_{i_1}, \ldots, r_{i_{n_i}}$ such that $\Big(\Gamma + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)\Big) + \neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ is consistent.
□

Lemma 3.4 *[4] [Lindenbaum lemma for* $\mathbb{L}$*-theories] Every consistent* $\mathbb{L}$*-theory* Γ *can be extended to a maximal* $\mathbb{L}$*-theory* Δ*.*

Proof. Let Γ be a consistent $\mathbb{L}$-theory and the formulas of $\mathcal{L}$ be $\alpha_1, \ldots, \alpha_n, \ldots$, $n < \omega$. Let an enumeration of the finite sequences of variables be fixed. We define a sequence of consistent $\mathbb{L}$-theories $\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$ by induction in the following way: $\Gamma_1 = \Gamma$ and let $\Gamma_1, \ldots, \Gamma_n$ be defined. We define Γ_{n+1} in the following way:

Case 1: $\Gamma_n + \alpha_n$ is consistent

Case 1.1: α_n is not of the kind $\neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$, where φ is a formula for $\mathcal{L}$, $i \in I$, $a_{i_1}, \ldots, a_{i_{m_i}}$ are terms for $\mathcal{L}$.

In this case $\Gamma_{n+1} \stackrel{def}{=} \Gamma_n + \alpha_n$.

Case 1.2: α_n is of the kind $\neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$, where φ is a formula for $\mathcal{L}$, $i \in I$, $a_{i_1}, \ldots, a_{i_{m_i}}$ are terms for $\mathcal{L}$. By the Extension lemma, we get that there are variables $r_{i_1}, \ldots, r_{i_{n_i}}$ such that $\Big(\Gamma_n + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)\Big) + \neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ is a consistent $\mathbb{L}$-theory and let $r_{i_1}, \ldots, r_{i_{n_i}}$ be the first in the enumeration sequence of variables with this property. In this case $\Gamma_{n+1} \stackrel{def}{=} \Big(\Gamma_n + \neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)\Big) + \neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$.

Case 2: $\Gamma_n + \alpha_n$ is not consistent

In this case $\Gamma_{n+1} \stackrel{def}{=} \Gamma_n$.

Let $\Delta = \bigcup_{n=1}^{\infty} \Gamma_n$. Obviously $\Gamma \subseteq \Delta$. We will prove that Δ is a maximal $\mathbb{L}$-theory. Obviously Δ contains all theorems of $\mathbb{L}$. Let α, $\alpha \to \beta \in \Delta$. We will prove that $\beta \in \Delta$. There is an n such that α, $\alpha \to \beta \in \Gamma_n$; Γ_n is a $\mathbb{L}$-theory; so $\beta \in \Gamma_n$, i.e. $\beta \in \Delta$. Let $i \in I$, φ be a formula for $\mathcal{L}$, $a_{i_1}, \ldots, a_{i_{m_i}}$ be terms for $\mathcal{L}$, $\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r_{i_1}\ldots r_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big] \in \Delta$ for all sequences of variables $r_{i_1}, \ldots, r_{i_{n_i}}$(1). For the sake of contradiction suppose that $\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big] \notin \Delta$(2). $\neg\Big(\varphi \to \gamma_i\Big[\frac{q_{i_1}\ldots q_{i_{m_i}}}{a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ is α_m for some m. By the Extension lemma

(ii) and (2), we obtain that $\Gamma_m + \alpha_m$ is consistent. $\Gamma_{m+1} = (\Gamma_m + \alpha_m) + \neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r'_{i_1}\ldots r'_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big)$ for some sequence of variables $r'_{i_1}, \ldots, r'_{i_{n_i}}$ (3). From (1) we get that $\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r'_{i_1}\ldots r'_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big] \in \Gamma_l$ for some l. From here and (3) we obtain that there is a k such that $\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r'_{i_1}\ldots r'_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]$, $\neg\Big(\varphi \to \beta_i\Big[\frac{p_{i_1}\ldots p_{i_{n_i}}\, q_{i_1}\ldots q_{i_{m_i}}}{r'_{i_1}\ldots r'_{i_{n_i}}\, a_{i_1}\ldots a_{i_{m_i}}}\Big]\Big) \in \Gamma_k$. Consequently $\bot \in \Gamma_k$, i.e. Γ_k is not consistent - a contradiction. Consequently Δ is a $\mathbb{L}$-theory.

For every n, Γ_n is consistent and hence $\bot \notin \Gamma_n$ for every n. Consequently $\bot \notin \Delta$, i.e. Δ is consistent.

Let Δ' be a consistent $\mathbb{L}$-theory and $\Delta \subseteq \Delta'$. We will prove that $\Delta' \subseteq \Delta$. Let $\alpha_n \in \Delta'$. We will prove that $\alpha_n \in \Delta$. For the sake of contradiction suppose that $\neg\alpha_n \in \Gamma_n$. Consequently $\neg\alpha_n \in \Delta$ and $\neg\alpha_n \in \Delta'$. We also have $\alpha_n \in \Delta'$, so $\bot \in \Delta'$ - a contradiction. Consequently $\neg\alpha_n \notin \Gamma_n$. From here and the Extension lemma (ii) we get that $\Gamma_n + \alpha_n$ is consistent. Consequently $\alpha_n \in \Delta$. Consequently Δ is a maximal $\mathbb{L}$-theory. □

Lemma 3.5 *[4] Let S be a maximal $\mathbb{L}$-theory. Then:*
(i) for every formula α, $\alpha \in S$ or $\neg\alpha \in S$;
(ii) for all formulas α and β:
1) $\neg\alpha \in S \leftrightarrow \alpha \notin S$;
2) $\alpha \wedge \beta \in S \leftrightarrow \alpha \in S$ and $\beta \in S$;
3) $\alpha \vee \beta \in S \leftrightarrow \alpha \in S$ or $\beta \in S$.

Proof. (i) Let α be a formula for $\mathcal{L}$. For the sake of contradiction suppose that $S' = S + \neg\alpha$ and $S'' = S + \alpha$ are inconsistent. Consequently $\neg\alpha \to \bot \in S$ and $\alpha \to \bot \in S$. The formula $(\neg\alpha \to \bot) \to ((\alpha \to \bot) \to \bot)$ is a theorem of the classical propositional logic and consequently is in S. Thus using that S is closed under MP, we get that $\bot \in S$ - a contradiction. Consequently S' is consistent or S'' is consistent, so $S' = S$ or $S'' = S$, i.e. $\neg\alpha \in S$ or $\alpha \in S$.
(ii) Let α and β be formulas for $\mathcal{L}$.
1) If $\neg\alpha \in S$, then $\alpha \notin S$ because otherwise S is inconsistent. If $\alpha \notin S$, then $\neg\alpha \in S$ because (i) is true.
2) Let $\alpha \wedge \beta \in S$. The formula $(\alpha \wedge \beta) \to \alpha$ is in S. Consequently $\alpha \in S$. Similarly $\beta \in S$. Let α, $\beta \in S$. The formula $\alpha \to (\beta \to \alpha \wedge \beta)$ is in S. Consequently $\alpha \wedge \beta \in S$.
3) Let $\alpha \vee \beta \in S$. Suppose for the sake of contradiction that $\alpha \notin S$, $\beta \notin S$. From (i) we get $\neg\alpha \in S$ and $\neg\beta \in S$. We have $\neg\alpha \to (\neg\beta \to \neg(\alpha \vee \beta)) \in S$. Thus $\neg(\alpha \vee \beta) \in S$. Consequently S is inconsistent - a contradiction.

Let $\alpha \in S$ or $\beta \in S$. The formulas $\alpha \to (\alpha \vee \beta)$ and $\beta \to (\alpha \vee \beta)$ are in S. Consequently $\alpha \vee \beta \in S$. □

Let S be a maximal $\mathbb{L}$-theory. We define the relation $\equiv$ in the set of all terms of $\mathcal{L}$ in the following way: $a \equiv b \Leftrightarrow a = b \in S$. $\equiv$ is an equivalence relation. Let $B_s = \{|a| :\ a \text{ is a term}\}$. We define the structure $\mathcal{B}_s$ with universe B_s in the following way:
- for every constant c: $c^{\mathcal{B}_s} = |c|$;
- for every n-ary function symbol f: $f^{\mathcal{B}_s}(|a_1|, \ldots, |a_n|) = |f(a_1, \ldots, a_n)|$;
- for every n-ary predicate symbol p: $p^{\mathcal{B}_s}(|a_1|, \ldots, |a_n|) \leftrightarrow p(a_1, \ldots, a_n) \in S$.

We define a valuation in $\mathcal{B}_s$ in the following way: $v_s(p) = |p|$ for every variable p. It can be easily verified that $v_s(a) = |a|$ for every term a. We call $(\mathcal{B}_s, v_s)$ canonical model, corresponding to S.

The semantics of $\mathcal{L}$ is the standard one.

Lemma 3.6 *[4] For every formula α: $(\mathcal{B}_s, v_s) \vDash \alpha \Leftrightarrow \alpha \in S$.*

Proof. Induction on the complexity of α. $\Box$

Proposition 3.7 *[4] All theorems of $\mathbb{L}$ are true in $(\mathcal{B}_s, v_s)$. For every $i \in I$ and for any $a_{i_1}, \ldots, a_{i_{m_i}}$ - terms we have: if $(\mathcal{B}_s, v_s) \nvDash \gamma_i \left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$, then there are terms $p'_{i_1}, \ldots, p'_{i_{n_i}}$ such that $(\mathcal{B}_s, v_s) \nvDash \beta_i \left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{p'_{i_1} \ldots p'_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$.*

Proof. Since S contains all theorem of $\mathbb{L}$ by lemma 3.6, we get that all theorems of $\mathbb{L}$ are true in $(\mathcal{B}_s, v_s)$.

Let $i \in I$, $a_{i_1}, \ldots, a_{i_{m_i}}$ be terms and $(\mathcal{B}_s, v_s) \nvDash \gamma_i \left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$. Consequently $\gamma_i \left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right] \notin S$. For simplicity for any k and any terms $\tau_1, \ldots, \tau_k$ we will denote $\tau_1, \ldots, \tau_k$ by $\overline{\tau}$. Thus $\gamma_i \left[\frac{\overline{q}}{\overline{a}}\right] \notin S$. For the sake of contradiction suppose that for any terms $\overline{p'}$, $\beta_i \left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \in S$. For any terms $\overline{p'}$, $\beta_i \left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \to \left(\top \to \beta_i \left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right]\right)$ is a theorem of $\mathbb{L}$ and hence is in S. Consequently $\top \to \beta_i \left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \in S$ for any terms $\overline{p'}$. By condition (iii) from the definition of $\mathbb{L}$-theory, $\top \to \gamma_i \left[\frac{\overline{q}}{\overline{a}}\right] \in S$. Consequently $\gamma_i \left[\frac{\overline{q}}{\overline{a}}\right] \in S$ - a contradiction. $\Box$

Proposition 3.8 *[4] Let S be a maximal $\mathbb{L}$-theory. Then the canonical structure, corresponding to S, $\mathcal{B}_s$ satisfies all axioms of $\mathbb{L}$ and the axioms, corresponding to the rules of $\mathbb{L}$.*

Proof. Let α be an axiom of $\mathbb{L}$ with variables among $p_1, \ldots, p_n$, where $n \geq 0$. Let v be a valuation in $\mathcal{B}_s$. We will prove that $(\mathcal{B}_s, v) \vDash \alpha$, i.e. $\alpha \left[\frac{p_1, \ldots, p_n}{v(p_1), \ldots, v(p_n)}\right]$ is true. There are terms $a_1, \ldots, a_n$ such that $v(p_1) = |a_1|, \ldots, v(p_n) = |a_n|$. (Here we use the definition of the canonical structure $\mathcal{B}_s$, corresponding to S - $B_s = \{|a| : a \text{ is a term}\}$.) $\alpha \left[\frac{p_1, \ldots, p_n}{a_1, \ldots, a_n}\right]$ is also an axiom of $\mathbb{L}$ and hence by lemma 3.6, $(\mathcal{B}_s, v_s) \vDash \alpha \left[\frac{p_1, \ldots, p_n}{a_1, \ldots, a_n}\right]$. Consequently $\alpha \left[\frac{p_1, \ldots, p_n}{|a_1|, \ldots, |a_n|}\right]$ is true.

If $\mathbb{L}$ includes rules, different from MP, we prove that their corresponding axioms are true in $\mathcal{B}_s$, using proposition 3.7, in the following way: For simplicity for any k and any terms $\tau_1, \ldots, \tau_k$ we denote $\tau_1, \ldots, \tau_k$ by $\overline{\tau}$, $|\tau_1|, \ldots, |\tau_k|$ by $\overline{|\tau|}$ and $v(\tau_1), \ldots, v(\tau_k)$ by $\overline{v(\tau)}$, where v is some valuation. Let $i \in I$ and $\overline{a}$ be terms. Let v be a valuation in $\mathcal{B}_s$ and $(\mathcal{B}_s, v) \vDash \neg\gamma_i \left[\frac{\overline{q}}{\overline{a}}\right]$. We will prove that $(\mathcal{B}_s, v) \vDash \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i \left[\frac{\overline{p}, \overline{q}}{\overline{x}, \overline{a}}\right]$, where $\overline{x}$ are some variables, not occurring in $\overline{a}$, different from $\overline{p}$, $\overline{q}$. Let $v(a_{i_1}) = |b_{i_1}|, \ldots, v(a_{i_{m_i}}) = |b_{i_{m_i}}|$. We have $\neg\gamma_i \left[\frac{\overline{q}}{\overline{v(a)}}\right]$, i.e. $\neg\gamma_i \left[\frac{\overline{q}}{\overline{v_s(b)}}\right]$, i.e. $(\mathcal{B}_s, v_s) \vDash \neg\gamma_i \left[\frac{\overline{q}}{\overline{b}}\right]$. By proposition 3.7, we obtain that there are terms $\overline{p'}$ such that $(\mathcal{B}_s, v_s) \nvDash \beta_i\left[\frac{\overline{p},\overline{q}}{\overline{p'},\overline{b}}\right]$, i.e. $\neg\beta_i\left[\frac{\overline{p},\overline{q}}{\overline{|p'|},\overline{v(a)}}\right]$. Consequently $\left(\mathcal{B}_s, v\left[\frac{\overline{x}}{\overline{|p'|}}\right]\right) \vDash \neg\beta_i\left[\frac{\overline{p},\overline{q}}{\overline{x},\overline{a}}\right]$ and hence $(\mathcal{B}_s, v) \vDash \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{\overline{p},\overline{q}}{\overline{x},\overline{a}}\right]$. □

Theorem 3.9 *[4] [Completeness theorem] The following conditions are equivalent for every formula α:*
(i) α is a theorem of $\mathbb{L}$;
(ii) α is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true.

Proof. (i)→(ii) It suffices to prove that for every $i \in I$, φ - a formula, $a_{i_1}, \ldots, a_{i_{m_i}}$ - terms:
(1) if for arbitrary variables $r_{i_1}, \ldots, r_{i_{n_i}}$ $\varphi \to \beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{r_{i_1} \ldots r_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$ is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true, then $\varphi \to \gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$ is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true.

Let $i \in I$, φ be a formula, $a_{i_1}, \ldots, a_{i_{m_i}}$ be terms and the premise of (1) be true. Let $\mathcal{B}$ be a structure for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true, and v be a valuation in $\mathcal{B}$. We will prove that $(\mathcal{B}, v) \vDash \varphi \to \gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$. Suppose for the sake of contradiction the contrary. Consequently $(\mathcal{B}, v) \vDash \varphi$ and $(\mathcal{B}, v) \nvDash \gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$. But in $\mathcal{B}$ is true the corresponding to the considered rule axiom: $\neg\gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right] \to \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{x_{i_1} \ldots x_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$, where $x_{i_1}, \ldots, x_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \ldots, a_{i_{m_i}}$, different from $p_{i_1}, \ldots, p_{i_{n_i}}, q_{i_1} \ldots, q_{i_{m_i}}$. Consequently $(\mathcal{B}, v) \vDash \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{x_{i_1} \ldots x_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$ and hence there are $r'_{i_1}, \ldots, r'_{i_{n_i}} \in B$ such that $\left(\mathcal{B}, v\left[\frac{r_{i_1} \ldots r_{i_{n_i}}}{r'_{i_1} \ldots r'_{i_{n_i}}}\right]\right) \nvDash \beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{r_{i_1} \ldots r_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$, where $r_{i_1}, \ldots, r_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \ldots, a_{i_{m_i}}$ and φ. We have $\left(\mathcal{B}, v\left[\frac{r_{i_1} \ldots r_{i_{n_i}}}{r'_{i_1} \ldots r'_{i_{n_i}}}\right]\right) \vDash \varphi$ and $\left(\mathcal{B}, v\left[\frac{r_{i_1} \ldots r_{i_{n_i}}}{r'_{i_1} \ldots r'_{i_{n_i}}}\right]\right) \vDash \varphi \to \beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{r_{i_1} \ldots r_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$. Consequently $\left(\mathcal{B}, v\left[\frac{r_{i_1} \ldots r_{i_{n_i}}}{r'_{i_1} \ldots r'_{i_{n_i}}}\right]\right) \vDash \beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{r_{i_1} \ldots r_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$ - a contradiction.
(ii)→(i) Let α be true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true. Suppose for the sake of contradiction that α is not a theorem of $\mathbb{L}$. Let T be the set of all theorems of $\mathbb{L}$. T is a $\mathbb{L}$-theory. Let $T' = T + \{\neg\alpha\}$. We have $\neg\neg\alpha \notin T$, so using the extension lemma (ii), we get that T' is a consistent $\mathbb{L}$-theory. From the Lindenbaum lemma it follows that T' can be extended to a maximal $\mathbb{L}$-theory S. From proposition 3.8 we obtain that the canonical structure $\mathcal{B}_s$ satisfies all axioms of $\mathbb{L}$ and the axioms, corresponding to the rules of $\mathbb{L}$. Consequently $(\mathcal{B}_s, v_s) \vDash \alpha$. By lemma 3.6, we get that $\alpha \in S$. But $\neg\alpha$ is also in S. Consequently S is inconsistent - a contradiction. Consequently α is a theorem of $\mathbb{L}$. □

4 Quantifier-free logics for extended distributive contact lattices

We consider the quantifier-free first-order language with equality $\mathcal{L}$ which includes:
• constants: 0, 1;
• function symbols: $+$, $\cdot$;
• predicate symbols: $\leq$, C, $\widehat{C}$, $\ll$.
Let $\bot \stackrel{def}{=} (0 \leq 0) \wedge \neg(0 \leq 0)$, $\top \stackrel{def}{=} (0 \leq 0) \vee \neg(0 \leq 0)$. Every EDCL is a structure for $\mathcal{L}$.

We consider the logic L with rule MP and the following axioms:
• the axioms of the classical propositional logic;
• the axiom schemes of distributive lattice;
• the axioms for C, $\widehat{C}$, $\ll$ and the mixed axioms of EDCL - considered as axiom schemes.

We consider the following additional rules and an axiom scheme:

(R Ext $\widehat{O}$) $\frac{\alpha\to(a+p\neq 1\vee b+p=1) \text{ for all variables } p}{\alpha\to(a\leq b)}$, where α is a formula, a, b are terms

(R U-rich $\ll$) $\frac{\alpha\to(b+p\neq 1\vee aCp) \text{ for all variables } p}{\alpha\to(a\overline{\ll}b)}$, where α is a formula, a, b are terms

(R U-rich $\widehat{C}$) $\frac{\alpha\to(a+p\neq 1\vee b+q\neq 1\vee pCq) \text{ for all variables } p,\ q}{\alpha\to a\widehat{C}b}$, where α is a formula, a, b are terms

(R Ext C) $\frac{\alpha\to(p\neq 0\to aCp) \text{ for all variables } p}{\alpha\to(a=1)}$, where α is a formula, a is a term

(R Nor1) $\frac{\alpha\to(p+q\neq 1\vee aCp\vee bCq) \text{ for all variables } p,\ q}{\alpha\to aCb}$, where α is a formula, a, b are terms

(Con C) $p \neq 0 \wedge q \neq 0 \wedge p + q = 1 \to pCq$

The corresponding to these rules axioms are equivalent respectively to the axioms (Ext $\widehat{O}$), (U-rich $\ll$), (U-rich $\widehat{C}$), (Ext C), (Nor1).

Let L' be for example the extension of L with the rule (R Ext $\widehat{O}$) and the axiom scheme (Con C). Then we denote L' by $L_{ConC,Ext\widehat{O}}$ and call the axioms (Con C) and (Ext $\widehat{O}$) corresponding to L' additional axioms. In a similar way we denote any extension of L with some of the considered additional rules and axiom scheme and in a similar way we define its corresponding additional axioms.

Using theorem 3.9, we obtain:

Theorem 4.1 (Completeness theorem with respect to algebraic semantics) *Let L' be some extension of L with 0 or more of the considered additional rules and axiom scheme. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all EDCL, satisfying the corresponding to L' additional axioms.

We consider the following logics, corresponding to the EDC-lattices, considered in section 2:
1) L;
2) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$;

3) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$;
4) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$;
5) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$;
6) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$;
7) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1,ConC}$;
8) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$;
9) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$.

To every of these logics we juxtapose a class of topological spaces:
1) the class of all T_0, semiregular, compact topological spaces;
2) the class of all T_0, semiregular, compact topological spaces;
3) the class of all T_0, compact, weakly regular topological spaces;
4) the class of all T_0, semiregular, compact, connected topological spaces;
5) the class of all T_0, semiregular, compact, κ - normal topological spaces;
6) the class of all T_0, compact, weakly regular, connected topological spaces;
7) the class of all T_0, semiregular, compact, κ - normal, connected topological spaces;
8) the class of all T_0, compact, weakly regular, κ - normal topological spaces;
9) the class of all T_0, compact, weakly regular, connected, κ - normal topological spaces.

Later we will prove that some of the rules of these logics can be eliminated and these logics are reducible to other logics. Because of this some other logics also will be considered.

Proposition 4.2 *For every EDCL $\underline{B}$, satisfying the corresponding of some of the considered above logics additional axioms, there exists a topological space X from the corresponding class and an embedding of $\underline{B}$ in $RC(X)$.*

Proof. In [27] (Theorem 2.3.9) it is proved that: if $\underline{B}$ is a contact algebra, then there is a compact, semiregular, T_0 topological space X and an embedding of $\underline{B}$ in $RC(X)$. From here and corollary 2.3 in section 2 it follows that: if $\underline{B}$ is an EDCL (i.e. EDCL, satisfying the corresponding to L zero additional axioms), then there is a compact, semiregular, T_0 topological space X and an embedding of $\underline{B}$ in $RC(X)$.

For the other eight logics the proposition follows from theorem 2.6 in section 2. □

Theorem 4.3 (Completeness theorem with respect to topological semantics) *Let L' be any of the considered logics. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all contact algebras over a topological space from the corresponding to L' class.

Proof. From the previous completeness theorem we have: (i)$\leftrightarrow$

$(ii')\alpha$ is true in all EDCL, satisfying the corresponding to L' additional axioms

We will prove that (ii')$\leftrightarrow$ (ii).
(ii')$\rightarrow$(ii) Let α be true in all EDCL, satisfying the corresponding to L' additional axioms. Let X be a topological space from the corresponding to L' class.

From lemma 2.4 in section 2 it follows that $RC(X)$ satisfies the corresponding to L' additional axioms. Consequently α is true in $RC(X)$.
(ii)→(ii') Let α be true in all contact algebras over a topological space from the corresponding to L' class. Let $\underline{B}$ be an EDCL, satisfying the corresponding to L' additional axioms, and v be a valuation in $\underline{B}$. We will prove that $(\underline{B}, v) \models \alpha$. By proposition 4.2, we get that there is a topological space X from the corresponding to L' class and an isomorphic embedding h of $\underline{B}$ in $RC(X)$. We define a valuation v' in $RC(X)$ in the following way: $v'(p) = h(v(p))$ for every variable p. By $\underline{B'}$ we denote the sublattice of $RC(X)$ to which $\underline{B}$ is isomorphic. We have $(RC(X), v') \models \alpha$, so $(\underline{B'}, v') \models \alpha$, so $(\underline{B}, v) \models \alpha$. □

Proposition 4.4 *L and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ have the same theorems.*

Proof. The proposition follows from the completeness theorem with respect to topological semantics because to L and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ corresponds the same class of topological spaces. □

5 Decidability via finite models and admissibility of some rules of inference

In this section we discuss admissibility of some rules of inference and decidability via finite models of the logics introduced in Section 4. We will not discuss in this dissertation the complexity of the corresponding logics.

5.1 Decidability of the logic L

Proposition 5.1 *The following conditions are equivalent for any formula α:*
(i) α is true in all EDCL;
(ii) α is true in all finite EDCL with a number of the elements less or equal to $2^{2^n-1}+1$, where n is the number of the variables of α.

Proof. Obviously (i) implies (ii). Let (ii) be true. We will prove (i). Let $\underline{B}$ be an EDCL, v be a valuation in $\underline{B}$. We will prove that $(\underline{B}, v) \models \alpha$. Let the variables of α be $p_1, \ldots, p_n$, where $n \geq 0$. It is a well known fact that $v(p_1), \ldots, v(p_n)$ generate a distributive sublattice $\underline{B'}$ of $\underline{B}$ with a number of the elements less or equal to $2^{2^n-1}+1$. $\underline{B'}$ is an EDCL. We define a valuation v' in $\underline{B'}$ in the following way:

$$v'(p) = \begin{cases} v(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \ldots \text{ or } p = p_n \\ 0 & \text{otherwise} \end{cases}$$

It suffices to prove that $(\underline{B'}, v') \models \alpha$. But this is true because (ii) is fulfilled. □

Corollary 5.2 *L is decidable.*

5.2 Admissibility of the rule (R Ext C)

As in [4] we define a p-morphism and prove a lemma for it. Let (W, R) and (W', R') be relational structures and f be a surjection from W to W'. We call f p-morphism from (W, R) to (W', R'), if the following conditions are fulfilled

for any x, $y \in W$ and any x', $y' \in W'$:
(p1) If xRy, then $f(x)R'f(y)$;
(p2) If $x'R'y'$, then $(\exists x, y \in W)(x' = f(x), y' = f(y), xRy)$.

Let $\underline{B}$ be the contact algebra over (W, R), $\underline{B'}$ be the contact algebra over (W', R'), v and v' be valuations respectively in $\underline{B}$ and $\underline{B'}$. We say that f is a p-morphism from $(\underline{B}, v)$ to $(\underline{B'}, v')$, if for every variable p and every $x \in W$: $x \in v(p) \leftrightarrow f(x) \in v'(p)$. It can be easily proved that for every term a and every $x \in W$: $x \in v(a) \leftrightarrow f(x) \in v'(a)$.

Lemma 5.3 *[4] Let f be a p-morphism from $(\underline{B}, v)$ to $(\underline{B'}, v')$. Then for any formula for $\mathcal{L}$, φ we have: $(\underline{B}, v) \models \varphi \leftrightarrow (\underline{B'}, v') \models \varphi$.*

Proof. Induction on the complexity of φ. □

Proposition 5.4 *The rule (R Ext C) is admissible in $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$.*

Proof. The construction is almost the same as in [4] (Lemma 6.1). Let L' be any of these logics. Let α be a formula, a be a term. Let $\alpha \rightarrow (p \neq 0 \rightarrow aCp)$ be a theorem of L' for every variable p. We will prove that $\alpha \rightarrow (a = 1)$ is a theorem of L'. Suppose for the sake of contradiction the contrary. There is an EDCL $\underline{B}$, satisfying the corresponding to L' additional axioms, and a valuation in it v such that $(\underline{B}, v) \not\models \alpha \rightarrow (a = 1)$. Consequently $(\underline{B}, v) \models \alpha$ and $(\underline{B}, v) \not\models a = 1$. $\underline{B}$ is an U-rich EDCL and by theorem 2.6 in section 2, we get that there is a topological space X and an embedding h of $\underline{B}$ in $RC(X)$. Moreover if $\underline{B}$ satisfies (Con C), then $RC(X)$ also satisfies (Con C). We define a valuation v' in $RC(X)$ in the following way: $v'(p) = h(v(p))$ for every variable p. We have $(RC(X), v') \models \alpha$ and $(RC(X), v') \not\models a = 1$.

Let Q be the set of all variables, occurring in α and a. $v'(Q)$ is a finite subset of $RC(X)$. The subalgebra $\underline{B_1}$ of $RC(X)$, generated by $v'(Q)$, is a finite Boolean contact algebra. If $RC(X)$ satisfies (Con C), then $\underline{B_1}$ also satisfies the axiom (Con C). We define a valuation v_1 in $\underline{B_1}$ in the following way:

$$v_1(p) = \begin{cases} v'(p) & \text{if } p \in Q \\ 0 & \text{otherwise} \end{cases}$$

We have $(\underline{B_1}, v_1) \models \alpha$ and $(\underline{B_1}, v_1) \not\models a = 1$. There is a relational structure (W_2, R_2) and an isomorphism h_1 from $\underline{B_1}$ to the contact algebra $\underline{B_2}$ over (W_2, R_2). We define a valuation v_2 in $\underline{B_2}$ in the following way $v_2(p) = h_1(v_1(p))$ for every variable p. $(\underline{B_2}, v_2) \models \alpha$ and $(\underline{B_2}, v_2) \not\models a = 1$. Consequently $v_2(a) \neq W_2$. Let $w_1 \in W_2 - v_2(a)$, $w_0 \notin W_2$. We define $W_3 = W_2 \cup \{w_0\}$, $R_3 = R_2 \cup \{(w_0, w_0), (w_0, w_1), (w_1, w_0)\}$. We define $f : W_3 \rightarrow W_2$ in the following way:

$$f(w) = \begin{cases} w & \text{if } w \neq w_0 \\ w_1 & \text{if } w = w_0 \end{cases}$$

Let $\underline{B_3}$ be the contact algebra over (W_3, R_3). We define a valuation v_3 in $\underline{B_3}$ in the following way: $v_3(p) = f^{-1}(v_2(p))$ for every variable p. It can be easily verified that f is a p-morphism from $(\underline{B_3}, v_3)$ to $(\underline{B_2}, v_2)$. Consequently $(\underline{B_3}, v_3) \models \alpha$ and $(\underline{B_3}, v_3) \not\models a = 1$. If $\underline{B}$ satisfies the axiom (Con C), then $\underline{B_1}$ also satisfies (Con C) and since $\underline{B_1}$ is isomorphic to $\underline{B_2}$, we have that $\underline{B_2}$ also satisfies (Con C). From here and the definition of R_3 we get that if $\underline{B}$ satisfies (Con C), then $\underline{B_3}$ also satisfies the axiom (Con C)(1). Since $\underline{B_3}$ is a contact algebra, we have that $\underline{B_3}$ satisfies (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$)(2). Let p be a variable, not occurring in a and α. We have $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \models \alpha$ and $v_3\left[\frac{p}{\{w_0\}}\right](a) = v_3(a) = f^{-1}(v_2(a)) = v_2(a)$; $v_3\left[\frac{p}{\{w_0\}}\right](p) \neq \emptyset$; $v_3\left[\frac{p}{\{w_0\}}\right](a)\overline{C_{R_3}}v_3\left[\frac{p}{\{w_0\}}\right](p)$. Consequently $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \not\models \alpha \to (p \neq 0 \to aCp)$. Also from (1) and (2) it follows that $\underline{B_3}$ satisfies the corresponding to L' additional axioms. But $\alpha \to (p \neq 0 \to aCp)$ is a theorem of L', so $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \models \alpha \to (p \neq 0 \to aCp)$ - a contradiction. □

5.3 Admissibility of the rule (R Nor1)

Proposition 5.5 *The rule (R Nor1) is admissible in the logics* $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ *and* $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$.

Proof. The construction is almost the same as in [4] (Lemma 6.2). Let L' be any of these logics. Let α be a formula, a and b be terms. Let $\alpha \to (p + q \neq 1 \vee aCp \vee bCq)$ be a theorem of L' for all variables p and q. We will prove that $\alpha \to aCb$ is a theorem of L'. Suppose for the sake of contradiction the contrary. The same way as in the proof of the previous proposition we obtain that there is a contact algebra $\underline{B}$ over some relational structure (W, R) and a valuation in it v such that $(\underline{B}, v) \models \alpha$ and $(\underline{B}, v) \models a\overline{C}b$. Moreover if L' is the second logic, then $\underline{B}$ satisfies (Con C). Let $A \subseteq W$. We define $\langle R\rangle A = \{x \in W :\ (\exists y \in A)(yRx)\}$. Let $W_{defects} = \langle R\rangle(v(a)) \cap \langle R\rangle(v(b))$. We define
$W_1 = W \times \{1, 2\}$,
$(x, i)R_1(y, j) \leftrightarrow xRy$ and $((j = 1 \wedge x \in v(a) \wedge y \in W_{defects})$
or $(i = 1 \wedge y \in v(a) \wedge x \in W_{defects})$
or $(j = 2 \wedge x \in v(b) \wedge y \in W_{defects})$
or $(i = 2 \wedge y \in v(b) \wedge x \in W_{defects})$
or $(x \notin v(a) \cup v(b) \cup W_{defects} \wedge y \in W_{defects})$
or $(y \notin v(a) \cup v(b) \cup W_{defects} \wedge x \in W_{defects})$
or $(x \in W_{defects} \wedge y \in W_{defects})$
or $(x \notin W_{defects} \wedge y \notin W_{defects}))$,
$v_1(q) = v(q) \times \{1, 2\}$,
$f((x, i)) = x$.
Let $\underline{B_1}$ be the contact algebra over (W_1, R_1). It can be easily verified that f is a p-morphism from $(\underline{B_1}, v_1)$ to $(\underline{B}, v)$. Consequently $(\underline{B_1}, v_1) \models \alpha$ and $(\underline{B_1}, v_1) \models a\overline{C}b$. It can be easily verified that if L' is the second logic, then $\underline{B_1}$ satisfies (Con C). Let p, q be variables which do not occur in a, b and φ. We define a valuation v_1' in $\underline{B_1}$ eventually different from v_1 only in p and q: $v_1'(p) = \langle R_1\rangle(v_1(b))$, $v_1'(q) = \overline{\langle R_1\rangle(v_1(b))}$. Obviously $v_1'(p) + v_1'(q) = 1$. Suppose for the sake of contradiction that $v_1'(a)Cv_1'(p)$. Consequently $v_1(a)C_{R_1}\langle R_1\rangle v_1(b)$. From here we obtain that there are $(x, i) \in v_1(a)$, $(y, j) \in \langle R_1\rangle v_1(b)$ such that $(x, i)R_1(y, j)$. From $(y, j) \in \langle R_1\rangle(v_1(b))$ we obtain that there is $(z, k) \in v_1(b)$ such that $(z, k)R_1(y, j)$. Consequently $z \in v(b)$ and yRz and hence $y \in \langle R\rangle(v(b))$ (1). From $(x, i) \in v_1(a)$ we obtain $x \in v(a)$ (2). From $(x, i)R_1(y, j)$ we get xRy (3).

Using (2) and (3), we get $y \in \langle R\rangle(v(a))$ (4). From (4) and (1) we get $y \in W_{defects}$. From $(x,i)R_1(y,j)$, $y \in W_{defects}$, $x \in v(a)$, $(\underline{B}, v) \models a\overline{C}b$ and the definition of R_1 we get $j = 1$. Using $(z,k)R_1(y,j)$, $y \in W_{defects}$, $z \in v(b)$, $(\underline{B}, v) \models a\overline{C}b$ and the definition of R_1, we get $j = 2$ - a contradiction. Consequently $v'_1(a)\overline{C}v'_1(p)$. From the definition of $v'_1(q)$ we obtain that $v'_1(b)\overline{C}v'_1(q)$. Thus $(\underline{B_1}, v'_1) \not\models p + q \neq 1 \vee aCp \vee bCq$ and $(\underline{B_1}, v'_1) \models \alpha$; $\underline{B_1}$ satisfies the corresponding to L' additional axioms - a contradiction. $\square$

5.4 The rule (R U-rich $\ll$) is not admissible in L_{ConC}

Lemma 5.6 *Let $\underline{B} = (B, \ldots)$ be an EDCL, satisfying (U-rich $\ll$) and (Con C). Then for every $a \in B$, different from 0 and 1, we have $a \overline{\ll} a$.*

Proof. Let $a \in B$, $a \neq 0$, $a \neq 1$. Suppose for the sake of contradiction that $a \ll a$. Since $\underline{B}$ satisfies (U-rich $\ll$), there is a $c \in B$ such that $c + a = 1$ and $a\overline{C}c$. We have that $\underline{B}$ satisfies (Con C) and $c + a = 1$, $a \neq 0$, $c \neq 0$ (because $a \neq 1$), so aCc - a contradiction. Consequently $a\overline{\ll}a$. $\square$

Proposition 5.7 *The rule (R U-rich $\ll$) is not admissible in L_{ConC}.*

Proof. We will prove that there is a theorem of $L_{ConC,U-rich\ll}$ which is not a theorem of L_{ConC}. We consider the formula $\alpha: \ p \neq 0 \wedge p \neq 1 \rightarrow p\overline{\ll}p$. Using lemma 5.6, we obtain that α is true in every EDCL, satisfying (Con C) and (U-rich $\ll$). Consequently α is a theorem of $L_{ConC,U-rich\ll}$.

We consider the relational structure (W, R), where $W = \{x, y\}$, $R = \{(x, x), (y, y)\}$. Let $\underline{B}$ be the contact algebra over (W, R). Let $B' = \{\emptyset, W, \{x\}\}$. It can be easily verified that B' is closed under $\cup$ and $\cap$. Consequently $\underline{B'} = (B', \subseteq, \emptyset, W, \cap, \cup)$ is a distributive lattice. We can consider $\underline{B'}$ as a substructure $(B', \subseteq, \emptyset, W, \cap, \cup, C_R, \widehat{C_R}, \ll_R)$ of $\underline{B}$. $\underline{B}$ is an EDCL and the axioms of EDCL are quantifier-free and therefore $\underline{B'}$ is an EDCL. We have $\{x\} \neq \emptyset$, $\{x\} \neq W$ and $\{x\} \ll \{x\}$, so α is not true in $\underline{B'}$. It can be easily verified that $\underline{B'}$ satisfies (Con C). Consequently α is not a theorem of L_{ConC}. $\square$

5.5 A technical lemma with applications to admissibility of some rules of inference and decidability of some logics

Lemma 5.8 *Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$) and v be a valuation in it. Let α be a formula in $\mathcal{L}$. Then there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that: $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. The number of the elements of $\underline{B^*}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α.*

Proof. Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$), and v be a valuation in it. Let α be a formula in $\mathcal{L}$. From the relational representation theorem of EDC-lattices (Theorem 2.2 in section 2) it follows that there is a relational structure (W', R') with R' reflexive and symmetric and an isomorphic embedding h of $\underline{B}$ in the contact algebra $\underline{B'}$ over (W', R'). $\underline{B}$ is isomorphic of some substructure of $\underline{B'}$, $\underline{B_1}$, which is an EDCL, satisfying (Con C) and (U-rich $\ll$). We define a valuation v_1 in $\underline{B_1}$ in the following way: for every variable

p $v_1(p) \overset{def}{=} h(v(p))$. It can be easily proved that $(\underline{B}, v) \vDash \alpha$ iff $(\underline{B_1}, v_1) \vDash \alpha$. Let the variables of α be $p_1, \ldots, p_n$, where $n \geq 0$. $v_1(p_1), \ldots, v_1(p_n)$ generate a finite sublattice $\underline{B_2} = (B_2, \subseteq, \emptyset, W', \cap, \cup, C_{R'}, \widehat{C_{R'}}, \ll_{R'})$ of $\underline{B_1}$ which is an EDCL, satisfying (Con C), and has number of the elements $\leq 2^{2^n - 1} + 1$. We define a valuation v_2 in $\underline{B_2}$ in the following way:

$$v_2(p) = \begin{cases} v_1(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \ldots \text{ or } p = p_n \\ \emptyset & \text{otherwise} \end{cases}$$

We have $(\underline{B_1}, v_1) \vDash \alpha$ iff $(\underline{B_1}, v_2) \vDash \alpha$ iff $(\underline{B_2}, v_2) \vDash \alpha$.

For every $A \in W'$ we define, using that B_2 is finite, $s_A \overset{def}{=} \bigcap\{a \in B_2 : A \in a\}$, i.e. s_A is the smallest element of B_2 which contains A.

We will define *special sets* and with their help we will obtain a Boolean algebra $\underline{B_3}$. Let $A \in W'$ and $b \in B_2$, $b \subseteq s_A$, $A \notin b$, $\forall c(c \neq \emptyset, c \in B_2, c \subseteq s_A, A \notin c \rightarrow b \cap c \neq \emptyset)$. Then $s_A - b$ we call a special set, determined by the ordered pair (s_A, b).

Let (a, b) be an ordered pair of elements of B_2. We have:
1) if $b \subseteq a$, $a \neq b$, then (a, b) determines at most one special set;
2) if b is not a proper subset of a, then (a, b) does not determine a special set;

Using this fact, we get that the number of the special sets is $\leq$ half of the ordered pairs of different elements of B_2. Let C be the set of all special sets, N be the number of the elements of B_2. We have $|C| \leq \frac{N(N-1)}{2} \leq \frac{(2^{2^n-1}+1)2^{2^n-1}}{2}$. Let D be the set of all finite unions of special sets. We have that $|D| \leq$ the number of the nonempty subsets of C, i.e. $|D| \leq 2^{|C|} - 1 \leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}} - 1$. Let $B_3 \overset{def}{=} D \cup \{\emptyset\}$. We have $|B_3| \leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$; $B_3 \subseteq B'$.

We will prove:

Claim 5.9 *If $a, b \in B_2$, then $a - b \in B_3$.*

Proof. Case 1: $a - b = \emptyset$
We have $a - b \in B_3$.
Case 2: $a - b \neq \emptyset$
Let $A \in a - b$. By t_A we denote the largest element of B_2 which is a subset of s_A and does not contain A ($t_A = \cup\{e \in B_2 : e \subseteq s_A, A \notin e\}$). s_A is the smallest element of B_2 which contains A; a is an element of B_2 which contains A; so $s_A \subseteq a(2)$. We have $s_A \cap b \in B_2$, $s_A \cap b \subseteq s_A$, $A \notin s_A \cap b$; so $s_A \cap b \subseteq t_A(3)$. From (2) and (3) we get that $s_A - t_A \subseteq a - b$. Thus we juxtapose to every point A of $a - b$ an ordered pair elements of B_2 (s_A, t_A) such that $A \in s_A - t_A \subseteq a - b$. Let the obtained this way ordered pairs be $(s_1, t_1), \ldots, (s_k, t_k)$, where $k > 0$. Obviously $a - b \subseteq (s_1 - t_1) \cup \ldots \cup (s_k - t_k) \subseteq a - b$, i.e. $a - b = (s_1 - t_1) \cup \ldots \cup (s_k - t_k)$. Let $i \in \{1, \ldots, k\}$. Using the definition of t_i, we get that $s_i - t_i$ is a special set, determined by (s_i, t_i). Consequently $a - b$ is a finite union of special sets. Consequently $a - b \in B_3$. □

Claim 5.10 $\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ *is a Boolean algebra and $B_2 \subseteq B_3$. (We do not use $*$ in the notation of $\underline{B_3}$ because we do not want to change the language.)*

Proof. Let $a \in B_2$. We have $a = a - \emptyset$. Using claim 5.9, we obtain that $a \in B_3$. Consequently $B_2 \subseteq B_3$. Consequently $W' \in B_3$. We will prove that B_3 is closed under $\cup$ and $\cap$. Obviously B_3 is closed under $\cup$. Let a_1, $a_2 \in B_3$. We will prove that $a_1 \cap a_2 \in B_3$. If $a_1 = \emptyset$ or $a_2 = \emptyset$, then obviously $a_1 \cap a_2 \in B_3$. Let a_1, $a_2 \neq \emptyset$. We have $a_1 \cap a_2 = (a_{11} \cup \ldots \cup a_{1l}) \cap (a_{21} \cup \ldots \cup a_{2m})$, where l, $m > 0$, $a_{11}, \ldots, a_{2m}$ are special sets. $a_1 \cap a_2 = (a_{11} \cap a_{21}) \cup \ldots \cup (a_{11} \cap a_{2m}) \ldots (a_{1l} \cap a_{21}) \cup \ldots \cup (a_{1l} \cap a_{2m})$. It is sufficient to prove that the intersection of two special sets is $\emptyset$ or a finite union of special sets. Let $s_{A_1} - b_1$ and $s_{A_2} - b_2$ be special sets. It can be easily verified that $(s_{A_1} - b_1) \cap (s_{A_2} - b_2) = (s_{A_1} \cap s_{A_2}) - ((b_1 \cap s_{A_2}) \cup (b_2 \cap s_{A_1}))$. Using this fact and claim 5.9, we obtain that $(s_{A_1} - b_1) \cap (s_{A_2} - b_2) \in B_3$. Consequently $a_1 \cap a_2 \in B_3$. Thus $\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ is a distributive lattice of sets.

We will prove that for every $a \in B_3$, we have $\overline{a} \in B_3$. Let $a \in B_3$. If $a = \emptyset$, then $\overline{a} \in B_3$. Let $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_l} - b_l)$, where $l > 0$, $s_{A_1} - b_1, \ldots, s_{A_l} - b_l$ are special sets, determined respectively by $(s_{A_1}, b_1), \ldots, (s_{A_l}, b_l)$. $\overline{a} = \overline{s_{A_1} - b_1} \cap \ldots \cap \overline{s_{A_l} - b_l}$. Let $i \in \{1, \ldots, l\}$. We will prove that $\overline{s_{A_i} - b_i} \in B_3$. $\overline{s_{A_i} - b_i} = \overline{s_{A_i}} \cup b_i = (W' - s_{A_i}) \cup b_i$. Using $W' \in B_2$, $s_{A_i} \in B_2$ and claim 5.9, we get that $W' - s_{A_i} \in B_3 (4)$. (s_{A_i}, b_i) determines a special set and therefore $b_i \in B_2$; but $B_2 \subseteq B_3$, so $b_i \in B_3 (5)$. Using (4), (5) and the fact that B_3 is closed under $\cup$, we get that $\overline{s_{A_i} - b_i} \in B_3$ for every $i \in \{1, \ldots, l\}$. But B_3 is closed under $\cap$, so $\overline{a} \in B_3$. Consequently $\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ is a Boolean algebra. $\square$

We will call the elements of W' points. Let T, $U \in W'$ and suppose there is an $a \in B_2$ such that $T \in a$, $U \notin a$. We define $S_{T,U} = \bigcup \{a \in B_2 : T \in a, U \notin a\}$, i.e. $S_{T,U}$ is the largest element of B_2, containing T and not-containing U.

Let T, $U \in W'$ and suppose there is an $a \in B_2$ such that $T \in a$, $U \notin a$. We call U corresponding to T, if $(\forall a \in B_2)(U \in a \rightarrow T \in a)$ and $s_T \overline{\ll_{R'}} S_{T,U}$.

We define a binary relation R in W' in the following way: TRU iff $TR'U$ or U is corresponding to T or T is corresponding to U. Obviously R is reflexive and symmetric. We consider the Boolean contact algebra $\underline{B_4} = (B_3, \subseteq, \emptyset, W', \cup, \cap, C_R, \widehat{C_R}, \ll_R)$. (Here $C_R, \widehat{C_R}, \ll_R$ are defined in the following way: $aC_Rb \leftrightarrow$ there are $F_1 \in a$, $F_2 \in b$ such that F_1RF_2, $a\widehat{C_R}b \leftrightarrow$ there are $F_1 \in \overline{a}$, $F_2 \in \overline{b}$ such that F_1RF_2, $a \ll_R b \leftrightarrow (\forall F_1 \in a)(\forall F_2 \in \overline{b})(F_1\overline{R}F_2)$). We consider the following substructure of $\underline{B_4}$: $\underline{B_5} = (B_2, \subseteq, \emptyset, W', \cup, \cap, C_R, \widehat{C_R}, \ll_R)$. We will prove:

Claim 5.11 *$\underline{B_5}$ is isomorphic to* $\underline{B_2} = (B_2, \subseteq, \emptyset, W', \cup, \cap, C_{R'}, \widehat{C_{R'}}, \ll_{R'})$.

Proof. The isomorphism will be the mapping $id: B_2 \rightarrow B_2$ ($id(a) \stackrel{def}{=} a$ for every $a \in B_2$).

•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1C_{R'}a_2$ iff $a_1C_Ra_2$. Obviously $a_1C_{R'}a_2$ implies $a_1C_Ra_2$. Let $a_1C_Ra_2$. Consequently there are $F_1 \in a_1$, $F_2 \in a_2$ such that F_1RF_2.

Case 1: $F_1R'F_2$

Obviously $a_1C_{R'}a_2$.

Case 2: $F_1\overline{R'}F_2$

F_2 is corresponding to F_1 or F_1 is corresponding to F_2. Without loss of generality F_2 is corresponding to F_1. Consequently every element of B_2 which contains F_2, also contains F_1; $F_2 \in a_2$; $a_2 \in B_2$; so $F_1 \in a_2$. We also have $F_1 \in a_1$, so $a_1C_{R'}a_2$.

•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1\widehat{C_{R'}}a_2$ iff $a_1\widehat{C_R}a_2$. Obviously $a_1\widehat{C_{R'}}a_2$ implies $a_1\widehat{C_R}a_2$. Let $a_1\widehat{C_R}a_2 (6)$. Suppose for the sake of contradiction

that $a_1\overline{\widehat{C_{R'}}}a_2$(7). Consequently $\overline{a_1} \cap \overline{a_2} = \emptyset$(8). From here and (6) we get that there are $F_1 \in a_1$, $F_2 \in a_2$ such that $F_1 R F_2$. There is an element of B_2 (a_2) which contains F_2 but does not contain F_1; there is an element of B_2 (a_1) which contains F_1 but does not contain F_2; so F_2 is not corresponding to F_1 and F_1 is not corresponding to F_2. From (7) we get that $F_1\overline{R'}F_2$. Consequently $F_1\overline{R}F_2$ - a contradiction. Consequently $a_1\widehat{C_{R'}}a_2$.
•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1 \ll_{R'} a_2$ iff $a_1 \ll_R a_2$. Obviously $a_1 \ll_R a_2$ implies $a_1 \ll_{R'} a_2$. Let $a_1 \ll_{R'} a_2$(9). Suppose for the sake of contradiction that $a_1\overline{\ll_R}a_2$. Consequently there are $F_1 \in a_1$, $F_2 \notin a_2$ such that $F_1 R F_2$. From (9) we obtain that $F_1\overline{R'}F_2$. We have $F_1 \in a_2$, $a_2 \in B_2$, $F_2 \notin a_2$, so F_1 is not corresponding to F_2. Consequently F_2 is corresponding to F_1. Consequently $s_{F_1}\overline{\ll_{R'}}S_{F_1,F_2}$(10). We have $a_2 \in B_2$, $F_1 \in a_2$, $F_2 \notin a_2$, so $a_2 \subseteq S_{F_1,F_2}$(11). We have $F_1 \in a_1$, $a_1 \in B_2$, so $s_{F_1} \subseteq a_1$(12). From (9) we get that $a_1 \subseteq a_2$(13). From (10), (12), (13) and (11) we obtain $a_1\overline{\ll_{R'}}a_2$ - a contradiction with (9). Consequently $a_1 \ll_R a_2$.

Consequently $\underline{B_5}$ is isomorphic to $\underline{B_2}$. □

From this claim we get $(\underline{B_2}, v_2) \models \alpha$ iff $(\underline{B_5}, v_2) \models \alpha$. $\underline{B_5}$ is a substructure of $\underline{B_4}$, α is quantifier-free, so $(\underline{B_5}, v_2) \models \alpha$ iff $(\underline{B_4}, v_2) \models \alpha$.

Claim 5.12 $\underline{B_4}$ *satisfies (Con C).*

Proof. It suffices to prove that for every non-empty and different from W' $a \in B_3$, there are $F_1 \in a$ and $F_2 \notin a$ such that $F_1 R F_2$. Let $a \in B_3$, $a \neq \emptyset$ and $a \neq W'$. We have $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_k} - b_k)$, where $k > 0$; $s_{A_1} - b_1, \ldots, s_{A_k} - b_k$ are special sets, determined respectively by $(s_{A_1}, b_1), \ldots, (s_{A_k}, b_k)$.
Case 1: $(\exists i \in \{1, \ldots, k\})(\exists T \in b_i - a)(\exists U \in s_{A_i} - b_i)$($U$ is corresponding to T)
We have $U \in a$, $T \notin a$ and URT.
Case 2: $(\forall i \in \{1, \ldots, k\})(\forall T \in b_i - a)(\forall U \in s_{A_i} - b_i)$($U$ is not corresponding to T)
We will prove that for every $i \in \{1, \ldots, k\}$ there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$. Let $i \in \{1, \ldots, k\}$.
Case 2.1: $b_i \subseteq a$
$s_{A_i} - b_i \subseteq s_{A_i} \subseteq a$. We have $s_{A_i} \in B_2 \subseteq B_1$, i.e. $s_{A_i} \in B_1$.
Case 2.2: $b_i \nsubseteq a$
The idea of finding c_i is shortly the following: Let $T \in b_i - a$. For T we divide the points from $s_{A_i} - b_i$ into two kinds:
1 kind) all U such that $(\forall b \in B_2)(U \in b \rightarrow T \in b)$
2 kind) all U such that $(\exists b \in B_2)(U \in b, T \notin b)$
We will prove that there is an element of B_2 t_T such that $s_T \ll_{R'} t_T$ and every point of the first kind is not in t_T. Since B_2 is finite, we can obtain finitely many such pairs (s_T, t_T). For every pair (s_T, t_T), using $s_T \ll_{R'} t_T$, we get that there is a q_r such that q_r does not intersect s_T and q_r contains all points of the first kind. Thus every point T from $b_i - a$ which determines the pair in question (s_T, t_T), is not in q_r, the points for T of the first kind are in q_r. We will find a set q'_r such that $s_{A_i} - b_i \subseteq q_r \cup q'_r$, every point T which determines the pair (s_T, t_T), is not in q'_r. Thus for every pair (s_T, t_T) we get a set $q_r \cup q'_r$ which includes $s_{A_i} - b_i$ and does not contain any point T, determining the pair (s_T, t_T). We consider the intersection q of all sets of the kind $q_r \cup q'_r$. We have

$s_{A_i} - b_i \subseteq q$. Every point T from $b_i - a$ is not in some $q_r \cup q'_r$ and therefore is not in q. As a c_i we can take $q \cap s_{A_i}$.

Now we will give the proof in details. Let $T \in b_i - a$. We consider arbitrary $U \in s_{A_i} - b_i$ such that $(\forall b \in B_2)(U \in b \to T \in b)$. U is not corresponding to T. Consequently $s_T \ll_{R'} S_{T,U}$. We have $b_i \subseteq S_{T,U}$. Let $P_T \stackrel{def}{=} \{S_{T,U} : U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)\}$. B_2 is finite and therefore P_T is finite and let $P_T = \{t_1, \dots, t_l\}$, where $l > 0$. Let $t_T \stackrel{def}{=} t_1 \cap \dots \cap t_l$. We have $t_T \in B_2$. We have $\forall U$(If $U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)$, then $U \notin t_T$)(14); $b_i \subseteq t_T$(15). For every $j \in \{1, \dots, l\}$ $s_T \ll_{R'} t_j$, so $s_T \ll_{R'} t_T$(16).

Let $Q \stackrel{def}{=} \{(s_T, t_T) : T \in b_i - a\}$. Since B_2 is finite, we have that Q is finite and let $Q = \{(p_{11}, p_{12}), \dots, (p_{m1}, p_{m2})\}$, where $m > 0$.

Let $r \in \{1, \dots, m\}$. We consider (p_{r1}, p_{r2}). Using (16), we get $p_{r1} \ll_{R'} p_{r2}$. We also have $p_{r1}, p_{r2} \in B_2 \subseteq B_1$; $\underline{B_1}$ satisfies (U-rich $\ll$); so there is a $q_r \in B_1$ such that $p_{r2} \cup q_r = W'$, $q_r \overline{C_{R'}} p_{r1}$. Consequently $q_r \cap p_{r1} = \emptyset$. Let $V_r = \{T \in b_i - a : (s_T, t_T) = (p_{r1}, p_{r2})\}$. We have:
(17) If $T \in V_r$, then $T \in p_{r1}$ and $T \notin q_r$.
Using (14) and $p_{r2} \cup q_r = W'$, we obtain that:
(18) If $T \in V_r$, then
$\forall U$(If $U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)$, then $U \notin p_{r2}$ and $U \in q_r$).
Let $q'_r \stackrel{def}{=} \bigcup\{s_U : U \in s_{A_i} - b_i, (\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)\}$. We will prove that $s_{A_i} - b_i \subseteq q_r \cup q'_r$. Let $U \in s_{A_i} - b_i$.
Case 1: $(\exists T \in V_r)(\forall b \in B_2)(U \in b \to T \in b)$
Using (18), we get $U \in q_r$.
Case 2: $(\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)$. From the definition of q'_r we obtain that $s_U \subseteq q'_r$. $U \in s_U$, so $U \in q'_r$.
We proved that:
(19) $s_{A_i} - b_i \subseteq q_r \cup q'_r$.

We will prove that: if $T \in V_r$, then $T \notin q'_r$. Let $T \in V_r$. Suppose for the sake of contradiction that $T \in q'_r$. Consequently $T \in s_U$ for some U such that $U \in s_{A_i} - b_i$, $(\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)$. Consequently $(\exists b \in B_2)(U \in b$ and $T \notin b)$ and hence $T \notin s_U$ (we have $s_U \subseteq b$) - a contradiction with $T \in s_U$. Consequently $T \notin q'_r$. We proved that:
(20) if $T \in V_r$, then $T \notin q'_r$.
From (17) and (20) we get that:
(21) if $T \in V_r$, then $T \notin q_r \cup q'_r$.

Let $q \stackrel{def}{=} (q_1 \cup q'_1) \cap \dots \cap (q_m \cup q'_m)$. For every point T of $b_i - a$, there is a $r \in \{1, \dots, m\}$ such that $(s_T, t_T) = (p_{r1}, p_{r2})$. We have $T \in V_r$ and by (21), we obtain $T \notin q_r \cup q'_r$. We proved that for every point T of $b_i - a$, there is a $r \in \{1, \dots, m\}$ such that $T \notin q_r \cup q'_r$. Consequently
(22) $(\forall T \in b_i - a)(T \notin q)$
We have proved ((19)) that
(23) $(\forall r \in \{1, \dots, m\})(s_{A_i} - b_i \subseteq q_r \cup q'_r)$
Consequently $s_{A_i} - b_i \subseteq q$ (24)
We have that for every $r \in \{1, \dots, m\}$: $q_r \in B_1$, $q'_r \in B_2 \subseteq B_1$, so $q \in B_1$ (25).
Let $c_i \stackrel{def}{=} q \cap s_{A_i}$. From here and (25) we obtain $c_i \in B_1$ (26). From (22) we get:
(27) $(\forall T \in b_i - a)(T \notin c_i)$

From (24) we get:
(28) $s_{A_i} - b_i \subseteq c_i$
We will prove that $c_i \subseteq a$. Let $F \in c_i$. Consequently $F \in s_{A_i}$.
Case 1: $F \in s_{A_i} - b_i$
We have $F \in a$.
Case 2: $F \in b_i$
Suppose for the sake of contradiction that $F \notin a$. Consequently $F \in b_i - a$. From (27) we obtain $F \notin c_i$ - a contradiction. Consequently $F \in a$.
We proved that $c_i \subseteq a$ (29)
From (26), (28) and (29) we get that there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$.

We proved that for every $i \in \{1, \ldots, k\}$, there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$. Consequently $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_k} - b_k) \subseteq c_1 \cup \ldots \cup c_k \subseteq a$. Consequently $a = c_1 \cup \ldots \cup c_k$. Consequently $a \in B_1$. We have $a \neq \emptyset$, $a \neq W'$, $\underline{B_1}$ is an EDCL, satisfying (Con C) and (U-rich $\ll$), so by lemma 5.6, we get that $\overline{a \ll_{R'}} a$. Consequently there are $F_1 \in a$, $F_2 \notin a$ such that $F_1 R' F_2$. Consequently $F_1 R F_2$.

We proved that for every non-empty and different from W' $a \in B_3$, there are $F_1 \in a$ and $F_2 \notin a$ such that $F_1 R F_2$. Thus we proved that $\underline{B_4}$ satisfies (Con C). $\square$

Thus $\underline{B_4}$ is a finite connected Boolean contact algebra and v_2 is a valuation in it; $(\underline{B_4}, v_2) \models \alpha$ iff $(\underline{B}, v) \models \alpha$; the number of the elements of $\underline{B_4}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α. $\square$

Proposition 5.13 *The rule (R U-rich $\widehat{C}$) is admissible in $L_{ConC,U-rich\ll}$.*

Proof. It suffices to show that every theorem of $L_{ConC,U-rich\ll,U-rich\widehat{C}}$ is a theorem of $L_{ConC,U-rich\ll}$. Let α be a theorem of $L_{ConC,U-rich\ll,U-rich\widehat{C}}$ (1). We will prove that α is a theorem of $L_{ConC,U-rich\ll}$. It suffices to prove that α is true in all EDCL, satisfying (Con C) and (U-rich $\ll$). Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$) and v be a valuation in it. We will prove that $(\underline{B}, v) \models \alpha$. By lemma 5.8, we get that there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. $\underline{B^*}$ is a Boolean contact algebra and therefore satisfies (U-rich $\ll$) and (U-rich $\widehat{C}$). Using this fact, the connectedness of $\underline{B^*}$ and (1), we have $(\underline{B^*}, v^*) \models \alpha$. Consequently $(\underline{B}, v) \models \alpha$. $\square$

Proposition 5.14 *The rule (R Ext $\widehat{O}$) is admissible in the logic $L_{ConC,U-rich\ll,U-rich\widehat{C}}$.*

Proof. The proof is similar to the proof of proposition 5.13. Here we use that in all Boolean contact algebras are true (U-rich $\ll$), (U-rich $\widehat{C}$) and (Ext $\widehat{O}$). $\square$

Proposition 5.15 *$L_{ConC,U-rich\ll}$ is decidable.*

Proof. It suffices to prove that the following are equivalent for every formula α in $\mathcal{L}$:
(i) α is a theorem of $L_{ConC,U-rich\ll}$;
(ii) α is true in all finite EDCL, satisfying (Con C) and (U-rich $\ll$) with number of the elements $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α.

Let α be a formula in $\mathcal{L}$. Obviously (i) implies (ii). Let (ii) be true. We will prove (i). Let $\underline{B}$ be an EDCL, satisfying (Con C), (U-rich $\ll$) and v be a valuation in it. It suffices to prove that $(\underline{B}, v) \models \alpha$. By lemma 5.8, we get that there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. The number of the elements of $\underline{B^*}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α. We have $(\underline{B^*}, v^*) \models \alpha$. Consequently $(\underline{B}, v) \models \alpha$. □

5.6 The main theorem

Corollary 5.16 *(i) The logics* L, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$ *have the same theorems and are decidable;*
(ii) The logics $L_{ConC,U-rich\ll}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC,Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$ *have the same theorems and are decidable.*

Proof. (i) follows from proposition 5.5, proposition 5.4, proposition 4.4 and corollary 5.2.
(ii) follows from proposition 5.5, proposition 5.4, proposition 5.14, proposition 5.13, proposition 5.15. □

In a dual way we can obtain logics for O-rich EDC-lattices.

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