The Majority Voting Parliament Is either Oligarchic or Inconsistent

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1 Introduction

What has been written on Arrow's celebrated *Impossibility Theorem* is immense and there is a big variety of formulations. All of them consider a set of *alternatives* together with a set of *voters* each of them arranging the alternatives according to his or her own *preference ordering*. The central problem is to formulate a general rule (the *social choice function*, it is Arrow's "social welfare function") which would produce a preference relation arranging the alternatives so to reflect "the common will". The main results are negative: it is impossible to find a choice function satisfying *en bloc* certain conditions appearing to be natural. The different formulations propose different ensembles of "natural" conditions (more or less strong) but one of them is permanent: the non-existence of a *dictator*, that is a person whose preference of a certain alternative dictates this alternative to be the social choice regardless of the preferences of the other voters. The advantages and disadvantages of a large number of voting systems (i. e., of social choice rules) are compared in [Str]. Such a comparison is useful for feeling the degrees of "naturalness" of the conditions being imposed on the choice function. Furthermore, many formulations of the Impossibility Theorem together with demonstrations can be found, e. g., in the two books of J. Kelly [Kel 1], [Kel 2] and of course, in the book of K. Arrow himself [Arr]. The most interesting for us are the "logical" proofs using the fact that any ultrafilter on a finite set coincides with the family of all subsets containing a fixed element. In our case, the elements of the ultrafilters are the so-called *decisive sets* of voters, namely, collectives which are able to impose some decision. The conditions of the Theorem together with the properties of the preference orders ensure the family of decisive sets to be an ultrafilter. Hence, there always exists an one-element decisive set — a *dictator*. Independent proofs using ultrafilters were proposed by B. Hansson [Han] and A. Kirman and D. Sondermann [KS]; see also Ch. 8 of [Kel 1].

I will not expose here any demonstration of the Impossibility Theorem but I shall make two remarks. First, all formulations deal with at least *three* alternatives. Something more, Arrow showed [Arr, p. 48] that in the case of *two* alternatives, a social welfare function satisfying all "good" conditions (and avoiding the existence of a dictator) does exist and an example of such a function is the popular majority voting. According to it the winner is the one of the two opponents who has collected more *pros*. Arrow saw in the last result "the logical foundation of the Anglo-American two-party system" [*ibid*.]. Second, the principle difference between choosing a *person* (or date, value, act, etc.) and voting a *sentence* (law, declaration, order, etc.) should be emphasized. Although the electoral body may be one and the same, no decision can lead to a logical contradiction in the first case, while in the second case, the decisions constitute a logical system performing to make implications, conjunctions, etc. Therefore consistency is mandatory for such a system.

It will be shown in this paper that the majority choice between two alternatives is vulnerable as well, because the dictatorship can be recognized in the behavior of a certain group (a *clique*, or an *oligarchy*) or in the behavior of a Speaker of a Parliament. The term 'oligarchy' will be used not in its most popular sense of a *small* group of persons (e. g., financial oligarchy) but according to the Aristotelian division of classes into singular, particular, and universal. The fact that usually it is a minority, as Aristotle notes, is contingent: "... whether in oligarchies or in democracies, the number of the governing body, whether the greater number, as in a democracy, or the smaller number, as in an oligarchy, is an accident" (*The Politics*, Γ 1279b30). The same term is used, e. g., in [Sen].

We have many reasons to examine exactly such a situation, if the analysis is devoted neither to abstract preference relations connected with economics nor to the choice of a person from a list of candidates, but to the standard procedures of approving bills in the parliaments over the world. Indeed, the cases in which the deputies have to choose between more than two alternatives are quite rare. Such cases arise, e. g., when a few values of a certain tax (3, 3.5 or 5 %) had been offered and the parliament must fix one of them, or when a few candidates for a Chair of a certain Commission had been nominated and one of them must be chosen. However, even in these cases the parliamentarian procedure usually requires all alternatives to be arranged and to be voted one after another. In their quotidian work the parliamentarians have in fact *one* offer at each moment and they may either accept (voting *pro*) or reject it (voting *con*); the case of abstaining deputies will not be considered here. In such conditions the majority voting is the only natural one.

We will show that any parliament using majority voting is either oligarchic or inconsistent. In the first case, a fixed group is able to dictate all decisions of the parliament. In the second case, any decision will follow. In fact, only three votings will be sufficient to get a contradiction! Our theorem was suggested by a popular article in the Russian school magazine *Kvant* [Sha] (unfortunately, with no reference). The final effect, the article summarizes, is that the parliament is superfluous in both cases: when an oligarchic group (that is a *stable majority*) exists, it will be enough for the Spokesman of the group to come and announce the decision of the governing party. No voting *in* the parliament will be necessary: voting, if any, would have been carried out outside! In the opposite case (that of *versatile majority*) it will be sufficient for the Speaker, who usually rules the agenda, only to include the suitable three bills. Voting will be superfluous again because its results will be known in advance! One may conclude that although majority voting is dictator-proof in the Arrow sense, its fruits do not bring fame to democracy.

2 Basic notions and definitions

The *parliament* Π is a non-empty set of *deputies*. In fact, it may be a certain commission consisting of three members only or even the whole people. The only condition is the number of the elements of the "parliament" to be *odd*. Arbitrary deputy will be denoted by x; subsets of Π (named *parliamentary groups*) will be denoted by $a, b, c, m, n, \ldots$ *Majority* is any group containing more than the half of the deputies. Sentences to be voted are called *bills* and will be denoted by A, B, C, P, Q,...; they will be treated as propositional formulae and hence their set is closed under Boolean connectives. A group a *votes pro* A when any member of a is voting *pro* A (in such a case, some deputy out of a may vote *pro* A, too). A certain bill A is *voted exactly by* a when any member of a is voting *pro* A and any member of $\overline{a}$ is voting *con* (it is not obligatory for a to form a majority). A group a *imposes* a bill A when A has been voted exactly by a and a is a majority, i. e., when a is the exact majority voting pro A. In such a case A will be called a *law*. In other words, A is a law when the parliament passes it by an explicit voting. In some cases however a decision can be impute to the parliament without having been voted and that is when it is a *consequence* of certain laws voted before. Namely, the parliament will be said to *produce* A either when A is a law or when it is a logical consequence of laws already passed. In such a way, the *decisions* of the parliament would be all voted laws together with all their consequences. The parliament can be identified with the set of the decisions which it produces.

Some characteristics of the deputies follow. Deputy's voting *con* A is equivalent to voting *pro* $\neg A$. A deputy is: *honest* when votes *pro* no bill A together its negation $\neg A$; *competent* when never abstains, i. e., he always has opinion and votes either *pro* A or *pro* $\neg A$; *regular* when having voted *pro* A and *pro* $A \Rightarrow B$ will vote *pro* B as well; *logical* when votes *pro* any propositional tautology. A deputy which is honest, competent, consistent and logical will be called *ideal*. Deputies we will deal with are ideal. One can conclude that any ideal deputy produces an ultrafilter over the set of all bills.

The parliament is *inconsistent* when there is a contradiction among its decisions. In the opposite case it is *consistent*.

A majority a is named *stable iff* each time when some law is imposed by a majority b, $a \subseteq b$. Note that the stable majority need not be exact but the requirement to be included in all *exact* majorities is essential. The parliament is said to be *oligarchic* if a stable majority exists.

If one desire to put our terminology in accordance with the most commonly used, each act of voting must be consider separately. The *alternatives* are only two: the bill A and its negation $\neg A$. The *individual preference ordering* $\leq_x$ is reduced to what x will vote: *pro* A or *con* A. In our case, the *social choice function* prescribes to prefer A *iff* the set of persons having preferred A is a majority. If A is a law, the *decisive set* for A is the exact majority that had imposed it (see, e. g., [Bro]). We will modify this definition further.

3 The main result

It follows from the definition that the parliament is non-oligarchic *iff* for any majority a there are a bill B and a majority b such that b imposes B but $a \not\subseteq b$. Therefore, a deputy x exists such that $x \in a, x \notin b$. Because b is the *exact* majority voting *pro* B and x is competent, then x is a deputy of a voting *pro* $\neg B$.

Theorem *The consistent parliament is oligarchic. Or, in order the parliament to be consistent, a stable majority must appear.*

The theorem follows from the three lemmas listed below.

Lemma 1 *If a group a votes pro A and a group b votes pro B, then $a \cup b$ votes pro $A \vee B$ and $a \cap b$ votes pro $A \wedge B$; if a imposes A then $\overline{a}$ imposes $\neg A$.*

The demonstration uses the fact that the deputies are logical and regular (in the first two cases), and honest and competent (in the last one). When a and b are disjoint, $a + b$ will be written on the place of $a \cup b$.

Lemma 2 *If there are two bills P and Q as well as three groups a, b, c such that:*
a votes pro $P \wedge \neg Q$,
b votes pro $\neg P \wedge Q$,
c votes pro $P \wedge Q$,
and if $a + b, b + c, a + c$ are majorities, then the parliament is inconsistent.

Proof. Any deputy is honest and hence using $+$ is correct. By Lemma 1 $a+b$ votes pro $(P \wedge \neg Q) \vee (\neg P \wedge Q)$, which is equivalent to $P \equiv \neg Q$, i. e., to $\neg(P \equiv Q)$. It is not obligatory for $a+b$ to be the *exact* majority imposing $\neg(P \equiv Q)$, but it *is* a majority. The majority $b+c$ votes pro $(\neg P \wedge Q) \vee (P \wedge Q)$, which is equivalent to Q. Finally, the majority $a+c$ votes pro $(P \wedge \neg Q) \vee (P \wedge Q)$, equivalent to P. Therefore, the parliament passes P, Q and $\neg(P \equiv Q)$. In this way it produces $\neg Q$ and hence is inconsistent. Note that $\neg Q$ follows from 3 voted laws although there is not a majority explicitly voting *pro* $\neg Q$ (something more, such a majority cannot exist).

Lemma 3 *If in the notations of Lemma 2 $a+c$ and $b+c$ are majorities but $a+b$ and c are not, let d be $\overline{a+b+c}$. Then $a+b+d, c+d$ and $a+b+c$ are three majorities producing a contradiction.*

Proof. All deputies being honest, a, b, c and d make a partition of the parliament into 4 disjoint groups; d votes *pro* $\neg P \wedge \neg Q$. The group $a+b+d$ is a majority being the complement of c which is not a majority. The group $c+d$ is a majority as well, being the complement of $a+b$. Finally $a+b+c$ is a majority because $a+c$ is. Voting then gives:

$a+b+c$ is *pro* $\neg(\neg P \wedge \neg Q)$, which is equivalent to $P \vee Q$;

$c+d$ is *pro* $(P \wedge Q) \vee (\neg P \wedge \neg Q)$, which is equivalent to $P \equiv Q$;

$a+b+d$ is *pro* $(P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$, which is equivalent to $(P \equiv \neg Q) \vee (\neg P \wedge \neg Q)$, i. e., to $\neg((P \equiv Q) \wedge (P \vee Q))$, the negation of the conjunction of both previous laws.

Proof of the Theorem. It will be proved that any non-oligarchic parliament is inconsistent.

Let m_0 be arbitrary majority, e. g., the whole Π. By the assumption of non-oligarchy there exist a majority m_1 ($\neq m_0$) and a law A_1 such that A_1 is imposed by m_1. m_1 being a majority, by the same assumption there are a law B_1 and a majority n_1 such that n_1 imposes B_1 and the relative complement of n_1 to m_1 is non-empty. Denote this complement by $m_1 - n_1$: obviously its members vote *pro* $\neg B$. Denote $m_2 = m_1 \cap n_1$: it is non-empty, being an intersection of two majorities. $A_1 \wedge B_1$ is voted exactly by m_2 because A_1 and B_1 are voted exactly by m_1 and n_1 respectively. m_2 is *smaller* then m_1 being equal to $m_1 - (m_1 - n_1)$.

* *Case 1*: m_2 is not a majority. Then $n_1 - m_1$ is non-empty (in the opposite case n_1 would be a part of m_1 and their intersection, i. e., m_2 would coincide with n_1, which is a majority). In such a way, the following three groups appears:

$a = m_1 - n_1$ voting *pro* $A_1 \wedge \neg B_1$ because n_1 is the exact group voting *pro* B_1;

$b = n_1 - m_1$ voting *pro* $\neg A_1 \wedge B_1$ because m_1 is the exact group voting *pro* A_1;

$c = m_1 \cap n_1$ voting *pro* $A_1 \wedge B_1$.

Denote $P = A_1, Q = B_1$. Now $a+c = m_1$ and $b+c = n_1$ are majorities. If $a+b$ is a majority either, the Theorem follows from Lemma 2; if $a+b$ is not a majority, the Theorem follows from Lemma 3 because c ($= m_2$) is not a majority by the assumption.

* *Case 2*: m_2 is a majority. Then it is the group *imposing* $A_1 \wedge B_1$. The situation is the same as that of m_1 with A_1: there are a law B_2 and a majority n_2 imposing it. Denote $m_2 \cap n_2$ by m_3; it imposes $A_1 \wedge B_1 \wedge B_2$ and is smaller than m_2. If m_3 is not a majority, the Theorem is proved. If it is still a majority, the process will continue until a group $m_{i+1} = m_i \cap n_i$ will be obtained, which will not be a majority; however, it will be the exact group voting *pro* $A_1 \wedge B_1 \wedge ... \wedge B_i$. Denote:

$a = m_i - n_i$: the group voting *pro* $(A_1 \wedge B_1 \wedge ... \wedge B_{i-1}) \wedge \neg B_i$;

$b = n_i - m_i$: the group voting *pro* $\neg(A_1 \wedge B_1 \wedge ... \wedge B_{i-1}) \wedge B_i$;

$c = m_i \cap n_i$: the group voting *pro* $(A_1 \wedge B_1 \wedge ... \wedge B_{i-1}) \wedge B_i$;

$d = \overline{m}_i \cap \overline{n}_i$: the group voting *pro* $\neg(A_1 \wedge B_1 \wedge ... \wedge B_{i-1}) \wedge \neg B_i$.

Taking $P = A_1 \wedge B_1 \wedge ... \wedge B_{i-1}, Q = B_i$, the Theorem follows from Lemma 2 or Lemma 3.

4 More proofs and algebraic analysis

We will now consider the algebraic structure of "working" majorities, i. e., majorities which pass some law. It is convenient to use here a slight modification of the notion of *decisive set*: it will be a group involving a majority sufficient to impose some law. Denote the family of all decisive sets by F:

$$F = \{m|(\exists m_0, P)(m_0 \subseteq m \text{ and } m_0 \text{ imposes } P)\}$$

It is obvious that $\Pi \in F$; if $m \in F$ and $m \subseteq n$ then $n \in F$. If $m \in F$ and $n \in F$ provided $m \cap n \in F$, F should be a *filter*. Up till now, it is possible to affirm only that there are some bills P and Q together with some majorities $m_0 \subseteq m$ and $n_0 \subseteq n$ imposing them. Of course, the group $m_0 \cap n_0$ is voting *pro* $P \wedge Q$; maybe it is a majority (imposing $P \wedge Q$), maybe not. In the first case there exists a stable majority but in the second case the majority is versatile.

Second proof of the Theorem. It will be shown that if the intersection of any two decisive sets is a decisive set, then the parliament is oligarchic; if the intersection of some two decisive sets is not a decisive set, then the parliament is inconsistent. This means to consider both possibilities:

* *Case 1.* F is a filter, i. e., the intersection of any two decisive sets is a decisive set, too. Something more, the intersection of *all* decisive sets is a decisive set because the number of all such sets is finite. Hence, a constant majority — an *oligarchy* — able to impose every law voted by the parliament is obtained. This result reproduces the well-known fact that any filter over a finite set consists of all its subsets involving a certain fixed subset.

** *Case 2.* F is not a filter. We saw above that there are two bills, P and Q, and two sets: m_0 imposing P, and n_0 imposing Q, such that $m_0 \cap n_0$ is not a majority. Hence, $\overline{m_0 \cap n_0}$ is a majority voting *pro* $\neg(P \wedge Q)$. In such a way, a contradiction arises between the last law, explicitly voted, and the conjunction of both laws previously voted.

Third proof of the Theorem [Sha]. Suppose the sequence of the laws passed by the parliament is $A_1, A_2, A_3, ...$ Form the new sequence $M_1 = A_1, ..., M_n = A_1 \wedge ... \wedge A_n, ...$ If A_i had been imposed by the majority a_i, the members of the group $m_i = a_1 \cap ... \cap a_n$ (it may be empty) would vote *pro* M_n. Obviously $m_n \subseteq m_{n+1}$, so that the cardinality of m_i is decreasing. Because Π is finite, the sequence becomes stable, i. e., there is k such that $m_i = m_k$ for any $i \geq k$. Two cases are possible: in the first case, m_k is a majority (it *is* an oligarchy) and then it would impose all laws in the sequence. In the second case, m_k is not a majority and hence $\overline{m}_k$ is a majority voting *pro* $\neg(A_1 \wedge ... \wedge A_k)$. However, $A_1, ..., A_k$ had been voted before and the contradiction in the decisions of the parliament is on hand.

The inconvenience of the last proof is that the sequence of all parliamentarian acts is presupposed and their inconsistency comes after an indeterminate number of steps. From our first proof it is clear that it is enough to use only the knowledge about the non-existence of a stable majority and the number of uses of this knowledge is not greater than the half of the Parliament.

5 Some practical conclusions

How much is the situation described above real? Yes, it is quite real! On the one hand, it is possible to provide an infinite non-contradictory life to the parliament putting to the vote only logical tautologies, a rise in the deputies' wages, etc. Then the stable majority will coincide with the whole parliament. On the other hand, when the parliament is lacking a stable majority, three votes are sufficient for achieving a contradiction. The Speaker must only wait for two laws to pass thanks to two majorities which intersection would not be a majority. Then he will have to put to vote the disjunction of the negations of that two laws. If such occasion does not appear, the Speaker could provoke it on the base of the own information about the parliamentarian groups — that is his or her job. As we see, the Speaker of a non-oligarchic Parliament turns into a new kind of a dictator being capable to lead or not lead it to a contradiction.

References

[Arr] Arrow K., *Social Choice and Individual Values*, Yale Univ. Press, New Haven, 2nd ed. 1963.

[Bro] Brown D., "An approximative solution to Arrow's problem", *J. Econ. Theory*, **9** (1974), pp. 375–383.

[Han] Hansson B., "The existence of group preferences", *Public Choice*, **28** (1976), pp. 89–98.

[Kel 1] Kelly J., *Arrow Impossibility Theorem*, Acad. Press, New York, 1978.

[Kel 2] Kelly J., *Social Choice Theory. An Introduction*, Springer, Berlin, 1987.

[K&S] Kirman A., D. Sondermann, "Arrow's theorem, many agents, and invisible dictators", *J. Econ. Theory*, **5** (1972), pp. 167–177.

[Sen] Sen A., *Choice, Welfare and Measurment*, Basil Blackwell, Oxford, 1982.

[Sha] Shapiro A., "Logic and parliament", *Kvant*, 1995, n. 3, pp. 6–11 (*in Russian*).

[Str] Straffin Ph., Jr., *Topics in the Theory of Voting*, Birkhäuser, Boston, 1980.

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