Труды Международной Конференции по конструктивной теории функций Варна, 19—25 мая 1970

Proceedings of the International Conference on Constructive Function Theory, Varna, May 19—25, 1970

APPROXIMATION THEOREMS FOR SEQUENCES OF COMMUTATIVE OPERATORS IN BANACH SPACES

P. L. Butzer and K. Scherer

Dedicated to Fritz Reutter on the occasion of his 70th birthday on August 26, 1971

Summary. The purpose of this lecture is to give a survey and at the same time an extension of general approximation theorems in [5, 6] for sequences of commutative bounded linear operators satisfying Jackson-and Bernstein-type inequalities. Whereas the theorem of Banach-Steinhaus gives necessary and sufficient conditions for convergence of operator sequences in Banach spaces, the present theorems yield equivalent conditions for the rate of convergence. The results are presented in the setting of the theory of intermediate spaces and include many classical cases such as various summation processes of Fourier series [3] or of the Fourier-inversion integral, and holomorphic one-parameter semigroups of bounded linear operators [2]; they include as well recent important contributions by N. P. Kuptsov [9, Chap. 1]. Although some of the results are provided with proofs, complete proofs of Theorems 1—3 are to be found in Butzer—Scherer [7].

We first recall some basic facts on intermediate spaces and semigroups of operators (see Butzer—Berens [2, Chap. 3]). Given two Banach spaces X, Z with Z continuously embedded in X, thus $Z\subset X$, the K-functional (introduced by J. Peetre [11]) is defined for $f\in Y$ by

$$K(t, f; X, Z)=\inf_{g\in Z}(\| f-g\|_X+t\|g\|_Z),\quad 0<t<\infty.$$

The spaces ($1\leqq q<\infty$)

$$(1)\quad \begin{aligned}(X, Z)_{\sigma,q}&=\left\{f\in X: \|f\|_{\sigma,q}=\left[\int_0^\infty (t^{-\sigma}K(t, f; X, Z))^q\frac{dt}{t}\right]<\infty\right\},\ 0<\sigma<1,\\ (X, Z)_{\sigma,\infty}&=\{f\in X: \|f\|_{\sigma,\infty}=\sup_{0<t<\infty}(t^{-\sigma}K(t, f; X, Z))<\infty\},\quad 0<\sigma\leqq 1\end{aligned}$$

are intermediate spaces of X and Z, i. e., Banach spaces under the norm $\|\cdot\|_{\sigma,q}$ with the prime constituent

$$(2)\quad Z\subset(X, Z)_{\sigma,q}\subset X.$$

Whereas the spaces $(X, Z)_{\sigma,\infty}$ are substitutes for the classical Lipschitz spaces (see (5) below), the spaces $(X, Z)_{\sigma,q}$ form an additional scale of intermediate spaces monotone increasing for $1\leqq q<\infty$ thus

$$(\mathsf{X}, \mathsf{Z})_{\sigma',\infty}\subset(\mathsf{X}, \mathsf{Z})_{\sigma,q}\subset(\mathsf{X}, \mathsf{Z})_{\sigma,\infty}, \quad \sigma'>\sigma.$$

The K-functional itself is a norm on X for each fixed $0<t<\infty$ with $K(t,f;\ \mathsf{X}, \mathsf{Z})=\|f\|_{\mathsf{X}}$ for $t\geq 1$. As a function of t it is continuous and monotone decreasing with

$$(3) \qquad \lim_{t\to 0+} K(t,f;\ \mathsf{X}, \mathsf{Z})=0$$

provided $\overline{\mathsf{Z}}=\mathsf{X}$. In a Banach space setting it can take the place of the modulus of continuity and thus presents a structural property upon f. Indeed, whenever there exists an equi-bounded one-parameter semigroup of operators $\{T(t):0<t<\infty\}$ of class (C_0) on X, the k-th modulus of continuity can be defined by

$$(4) \qquad \omega_k(t,f;T(\cdot)f)=\sup_{0\leq s\leq t}(\|(T(s)-I)^k f\|_{\mathsf{X}}), \quad f\in\mathsf{X};$$

it is related to the K-functional in case $\mathsf{Z}=\mathsf{D}(A^k)$ by the basic inequality (see [2, p. 192])

$$(5) \qquad \omega_k(t,f;\ T(\cdot)f)\leq M_1K(t^k,f;\ \mathsf{X},\mathsf{D}(A^k))\leq M_2[\omega_k(t,f;\ T(\cdot)f) + \min(1,t^k)\|f\|_{\mathsf{X}}],$$

valid for each $f\in\mathsf{X}$ and positive constants M_1, M_2 depending only upon k. Here $\mathsf{D}(A^k)$ is the domain of the k-th power of the infinitesimal generator A of the semigroup, with norm

$$\|f\|_{\mathsf{D}(A^k)}=\|f\|_{\mathsf{X}}+\|A^kf\|_{\mathsf{X}}.$$

We now turn to the actual problem of this lecture. Let $\mathfrak{V}=\{V_n\}_{n=0}^{\infty}$ be a sequence in $\mathfrak{E}(\mathsf{X})$ with $V_0(f)=0$ for $f\in\mathsf{X}$, $\mathfrak{E}(\mathsf{X})$ being the Banach algebra of (linear) endomorphisms of X. The fundamental convergence theorem of Banach — Steinhaus gives, in particular, necessary and sufficient conditions upon $\mathfrak{V}$ such that

$$(6) \qquad \lim_{n\to\infty}\|V_nf-f\|_{\mathsf{X}}=0 \quad \text{each } f\in\mathsf{X}.$$

These are

$$(7) \qquad \|V_n-I\|_{[\mathsf{X},\mathsf{X}]}\leq M, \quad n\in N,$$

$$(8) \qquad \lim_{n\to\infty}\|V_nf-f\|_{\mathsf{X}}=0, \quad \text{each } f\in\mathsf{A},\ \overline{\mathsf{A}}=\mathsf{X}.$$

We first wish to extend this convergence theorem so as to give results upon the rate of convergence of the sequence $\mathfrak{V}$ to the identity operator I. This can easily be achieved with the aid of the K-functional. Indeed,

Lemma 1. Let $\{V_n\}_0^{\infty}$ be any sequence in $\mathfrak{E}(\mathsf{X})$. For every $f\in\mathsf{X}$

$$(9) \qquad \|V_nf-f\|_{\mathsf{X}}\leq\max(1,\|V_n-I\|_{[\mathsf{X},\mathsf{X}]})K(\|V_n-I\|_{[\mathsf{Z},\mathsf{X}]},f;\mathsf{X},\mathsf{Z}).$$

The proof is rather simple and follows along the standard lines (see [11], [2], [4]). For, let $f=(f-g)+g$ for any $g\in Z$. In view of the linearity* of $\mathfrak{V}$

* One could relax the linearity assumption upon $\mathfrak{V}$ here. Instead, one could assume that the V_n satisfy the condition

$$\|(V_n-I)(f+g)\|\leq\|(V_n-I)f\|+\|(V_n-I)g\|.$$

$$\|(V_n-I)f\|_X \leq \|(V_n-I)(f-g)\|_X + \|(V_n-I)g\|_X$$
$$\leq \|V_n-I\|_{[X,X]}\|f-g\|_X + \|V_n-I\|_{[Z,X]}\|g\|_Z$$
$$\leq \max 1, \|V_n-I\|_{[X,X]})(\|f-g\|_X + \|V_n-I\|_{[Z,X]}\|g\|_Z).$$

Taking the infimum over $g \in Z$ one immediately obtains (9).

We specialize this lemma in the form of several corollaries.

First let us assume that information about the rate of convergence of $\mathfrak{V}$ in the subspace Z is given in the form of a Jackson-type inequality of order $\varrho > 0$

(10) $$\|V_n f - f\|_X \leq C n^{-\varrho} \|f\|_Z, \quad f \in Z;\ n \in N,$$

C being a constant depending only on $\mathfrak{V}$, Z and ϱ. The uniform boundedness principle then implies $\|V_n - I\|_{[Z,X]} \leq C n^{-\varrho}$, and hence by (9) that

(11) $$\|V_n f - f\|_X \leq \max(1, \|V_n - I\|_{[X,X]}) K(C n^{-\varrho}, f; X, Z).$$

(11) yields by (3) the following weaker version of one direction of the Banach-Steinhaus convergence theorem.

C o r o l l a r y 1. Let $\{V_n\}_{n=0}^{\infty}$ be any sequence in $\mathfrak{E}(X)$ satisfying (7) and (10) with Z dense in X. Then $\lim_{n\to\infty}\|V_n f - f\|_X = 0$ for each $f \in X$.

By definition (1) relation (11) also yields a direct approximation theorem concerning the rate of convergence, namely

C o r o l l a r y 2. Let $\{V_n\}_{n=0}^{\infty}$ be any sequence in $E(X)$ satisfying (7) and (10). Then

(12) $$f \in (X, Z)_{\theta/\varrho,\infty} \longrightarrow \|V_n f - f\|_X = O(n^{-\theta}), \quad 0 < \theta < \varrho.$$

The proof could also be derived by those interpolation theorems for linear operators which are well-known in the theory of intermediate spaces (compare [2]) and have their model in the famous Riesz-Thorin theorem.

Secondly, instead of estimating $\|V_n - I\|_{[Z,X]}$ as in the above by $Cn^{-\varrho}$ to deduce (11), we take for Z the particular choice $Z = D(A^k)$. In view of (9) and (5) this results in

C o r o l l a r y 3. Let $\{V_n\}_{n=0}^{\infty}$ be any sequence in $\mathfrak{E}(X)$ and $\{T(t): 0 < t < \infty\}$ any equi-bounded one-parameter semigroup of class (C_0) on X. Then

$$\|V_n f - f\|_X \leq M_1^{-1} M_2 \max(1, \|V_n - I\|_{[X,X]})\{\omega_k((\|V_n - I\|_{[Z,X]})^{1/k}, f; T(\cdot)f)$$
$$+ \min(1, \|V_n - I\|_{[Z,X]}\}.$$

In the theory of intermediate spaces results of the type expressed by Corollary 2 are said to be of interpolatory type. Indeed, one can say that (12) interpolates the limiting cases $\theta = 0$ for $f \in X$ in (7) and $\theta = \varrho$ for $f \in Y$ in (10) to $f \in (X, Z)_{\theta/\varrho,\infty}$. In approximation theory this is known as the principle that rapid convergence for a 'small' subspace Z of 'smooth' elements implies less rapid convergence for a 'larger' (or intermediate) subspace $(X, Z)_{\theta/\varrho,\infty}$ of 'less smooth' elements. It was used in the special case that Z consists of 2π-periodic functions which are differentiable in some sense by G. F r e u d [8], S. B. S t e c k i n [12], G. S u n o u c h i [13] and G. L o r e n t z [10]. K u p t s o v [9] generalized it to a form analogous to Corollary 3.

The next problem is to consider a converse approximation theorem for $\{V\}_{n=0}^{\infty}$. This will be accomplished by introducing a Bernstein-type inequality of order ϱ. To this end, let P_n be the linear subspace of X spanned by the ranges of V_k, $0\le k\le n$. Then the P_n are monotone increasing, i. e. $P_0=\{0\}\subset P_1\subset P_2\ldots\subset P_n\ldots\subset X$. A (strong) Bernstein-type inequality for V of order ϱ with respect to $Z\subset X$ is given by

$$(13)\qquad P_n\subset Z,\ \|p_n\|_Z\leq Dn^{\varrho}\|p_n\|_X;\quad p_n\in P_n;\ n\in N.$$

Introducing the best approximation of $f\in X$ by

$$E_n(f;X)=\inf_{d_n\in P_n}\|f-p_n\|_X,$$

we now prove

Lemma 2. Let $\mathfrak{V}$ in $\mathfrak{E}(X)$ satisfy inequality (13) with P_n as above. Then

$$K(t,f;X,Z)\leq M_3[t\|f\|_X+t\sum_{k=1}^{n}(k^{\varrho-1}E_k(f;\ X))+E_n(f;X)].$$

Proof. Although the result is already contained implicitly in [4], we reestablish it for convenience. Choose $p_n(f)\in P_n$ such that $\|f-p_n(f)\|_X\leq 2E_n(f;X)$ and estimate just as in [5]

$$K(t,f;X,Z)\leq\|f-p_n(f)\|_X+t\|p_n(f)\|_Z.$$

Furthermore there holds for n_0 with $2^{n_0}\leq n<2^{n_0+1}$ (see [5])

$$\|p_n(f)\|_Z\leq 4D[\|f\|_X+\sum_{k=1}^{n_0+1}2^{k\varrho}E_{2^{k-1}}(f;X)].$$

Since $2\sum_{2^{k-2}+1}^{2^{k-1}}(1/j)\geq 1$, the last sum is majorized by

$$\sum_{k=2}^{n_0+1}2^{k\varrho}E_{2^{k-1}}(f;X)\leq 2\sum_{k=2}^{n_0+1}\sum_{j=2^{k-2}+1}^{2^{k-1}}2^{k\varrho}E_{2^{k-1}}(f;X)\frac{1}{j}$$

$$\leq 2\,.\,4^{\varrho}\sum_{k=2}^{n_0+1}\sum_{j=2^{k-2}+1}^{2^{k-1}}j^{\varrho-1}E_j(f;X)\leq 2\,.\,4^{\varrho}\sum_{j=2}^{n}j^{\varrho-1}E_j(f;X).$$

Combining all these estimates we obtain the lemma.

Setting $t=n^{-\varrho}$ in Lemma 2 we have in view of (1)

Corollary 4. If V in $\mathfrak{E}(X)$ satisfies inequality (13) with P_n as above, then

$$(14)\qquad E_n(f;X)=O(n^{-\theta})\Longrightarrow f\in(X,Z)_{\theta/\varrho,\infty};\quad 0<\theta<\varrho.$$

Again specializing Z to $Z=D(A^k)$ we obtain a converse approximation theorem parallel to that of Kuptsov [9]. It is a consequence of Lemma 2 in view of (5) and the fact that $E_n(f;X)\leq\|V_nf-f\|_X$.

Corollary 5. Let $\mathfrak{V}$ and $\{T(t):0\leq t<\infty\}$ be given as in Corollary 3 with $\mathfrak{V}$ satisfying (13). Then for each $0<t<\infty$, $n\in N$,

$$\omega_k(t,f;T(\cdot)f)\leq M_3[\| V_nf-f\|_X+t^k\|f\|_X+t^k\sum_{k=1}^{n}k^{\varrho-1}\| V_nf-f\|_X].$$

This leads us to

Theorem 1. Let $\{V_n\}_{n=0}^{\infty}$ be a sequence in $\mathfrak{C}(X)$ satisfying (7) such that the Jackson- and Bernstein-type inequalities (10) and (13) are valid for a subspace $Z=Y_2$ of X with order $\varrho_2>0$. The first three assertions are equivalent for $f\in X$ and $0<\theta<\varrho_2$:

(a) $$E_n(f;X)=O(n^{-\theta}),$$

(b) $$\| V_nf-f\|_X=O(n^{-\theta}),$$

(c) $$f\in(X,Y_2)_{\theta/\varrho_2,\infty},\ \text{i. e.,}\ K(t^{\varrho_2},f;X,Z)=O(t^{\theta}).$$

If the sequence $\mathfrak{V}$ is in addition commutative, i. e.

(15) $$V_{n_1}(V_{n_2}f)=V_{n_2}(V_{n_1}f),\quad n_1,n_2\in N;\ f\in X,$$

then (a) — (c) are equivalent to

(d) $$\| V_nf\|_{Y_2}=O(n^{-\theta+\varrho_2}).$$

Furthermore, if the sequence $\mathfrak{V}$ is such that inequalities (10) and (13) are also valid with respect to a second subspace $Z=Y_1\in X$ with order ϱ_1, $0<\varrho_1<\varrho_2$, then for $\varrho_1<\theta<\varrho_2$ (a) — (d) are equivalent to

(e) $$f\in Y_1,\quad \| V_nf-f\|_{Y_1}=O(n^{-\theta+\varrho_1}).$$

For this theorem together with its proof the reader is referred to [6]. Nevertheless, the preceding already implies the equivalence of the assertions (a) — (c). Indeed, Corollary 2 gives the Jackson-type implication (c) $\Rightarrow$ (b), the fact that (b) $\Rightarrow$ (a) is trivial, and Corollary 4 the Bernstein-type implication (a) $\Rightarrow$ (c). The further implication (b) $\Rightarrow$ (d) is one of Zamansky-type. The equivalence (b) $\Leftrightarrow$ (e) is a reduction theorem since the order of convergence in (e) is reduced to that with respect to the subspace $Y_1\subset X$; in the concrete situation to be considered in Theorems 4, 5 it gives a result on the simultaneous approximation of a function and its derivatives.

In order to discuss the applicability of Theorem 1 one must check whether for a sequence $\{V_n\}_{n=0}^{\infty}$ of bounded linear operators converging to the identity operator the two inequalities (10) and (13) are satisfied for $Z=Y_1, Y_2$ with orders ϱ_1,ϱ_2, respectively, i. e. for $i=1, 2$, and $n\in N$

(10a, b) $$\| V_nf-f\|_X\leq C_1n^{-\varrho_i}\|f\|_{Y_i},\quad f\in Y_i;$$

(13a, b) $$P_n\subset Y_i,\ \|p_n\|_{Y_i}\leq D_in^{\varrho_i}\|p_n\|_X,\quad p_n\in P_n$$

(as a matter of fact, in general it suffices to check whether the higher order inequalties are satisfied, see the criterium in [7]). Furthermore, assumption (15) must be satisfied.

The first assumption, namely Jackson's inequality (10), is usually well-known in most of the common applications. (see inequalities (23), (24) needed for Theorem 4). It may be noted that this assumption is always satisfied if a saturation theorem is available for the process $\mathfrak{V}$. Such a theorem asserts in particular that

$$(16)\qquad \exists (B)\text{-space } Z\subset X : f\in Z \Longrightarrow \| V_n f - f \|_X = O(n^{-\varrho})$$

(and conversely). According to the uniform boundedness principle (16) is equivalent to (10). In general, assertions of type (16) lie rather deep, nevertheless one has at one's disposal the well-developed saturation theory; see Butzer-Berens [2], Berens [1], Butzer—Nessel [3] and the literature cited there.

The commutativity assumption (15), needed in the equivalence of the assertions (a)—(c) with (d) and (e), is generally not a very restrictive assumption. It is always satisfied by summation processes of Fourier series, singular integrals of convolution type and holomorphic semigroup operators. As a matter of fact, the commutativity assumption replaces the semigroup property; the proof of (b) $\Longleftrightarrow$ (d) in [6] is modelled upon that of Berens (see [2, Sec. 2.3]).

Concerning the Bernstein-type inequalities (13 a, b), needed in the proofs of (b) $\Longrightarrow$ (d) and (a), (b) $\Longrightarrow$ (c) (here the commutativity does not enter), the situation is not as straightforward. Indeed, the order $\varrho_i>0$ (and subspace $Y_i\subset X$) must be the same in the Bernstein inequality (13) as in the corresponding Jackson inequality (10). [This is associated with the fact that a process V satisfying a Jackson and Bernstein-type inequality for the same $\varrho>0$ and $Z\subset X$ is intimately connected with best approximation $E_n(f;X)$ through the equivalence (a) $\Longleftrightarrow$ (b). In this sense such V are 'optimal' approximation processes, they approximate a given f with the same order as that of best 'polynomial' approximation.] As a (counter-)example, consider the classical de La Vallée Poussin means of the Fourier series of $f\in C_{2\pi}$ defined for $n\in N$ by

$$[U_{n+1}f](x)=\sum_{\nu=-n}^{n}{}' p_{\nu,n} f^{\frown}(\nu) e^{ix\nu},\quad f^{\frown}(\nu)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u) e^{-i\nu u}\,du,$$

$$p_{\nu,n}=\begin{cases}\dfrac{(n!)^2}{(n-|\nu|)!\,(n+|\nu|)!}, & |\nu|\le n;\\ 0, & |\nu|>n.\end{cases}$$

Here it is known that (compare [4, p. 137 f])

$$(18)\qquad \| U_{n+1}f-f\|_{C_{2\pi}}\le n^{-1}\| f''\|_{C_{2\pi}},$$

$$(19)\qquad \| U''_{n+1}f\|_{C_{2\pi}}\le n^2\| U_{n+1}f\|_{C_{2\pi}},$$

$$(20)\qquad \| U''_{n+1}f\|_{C_{2\pi}}\le 2n\| f\|_{C_{2\pi}}.$$

In other words the order of the Jackson-type inequality here is $\varrho=1$ (just half as good as the order of best approximation for functions $f\in C_{2\pi}^{(2)}$), while the order of the (classical) Bernstein-type (19) is $\varrho=2$. To overcome this difficulty, in analogy to the inequality (20), a weak Bernstein-type inequality of order ϱ with respect $Z\subset X$ was introduced in [5, 6], namely,

$$(21)\qquad V_n f\in Z, \quad \|V_n f\|_Z\leq D' n^{\varrho}\|f\|_X, \quad f\in X;\ n\in N.$$

Now, if a strong Bernstein-type inequality (13) of order ϱ is satisfied by a process $\mathfrak{V}$, having property (7), so also the weak-type (21). Indeed, by (13) and (7), for $f\in X$

$$\|V_n f\|_Z\leq Dn^{\varrho}\|V_n f\|_X\leq D(M+1)n^{\varrho}\|f\|_X=D'n^{\varrho}\|f\|_X.$$

Although this weak-type inequality does not enable one to connect $\mathfrak{V}$ with best approximation as in Lemma 2, it is satisfied by a wider class of operator sequences and is even powerful enough to enable us to establish the equivalence of the assertions (b) — (e) of Theorem 1 provided commutativity (15) is assumed. For this purpose we need an auxiliary

Theorem 2. Let $\mathfrak{V}$ be a sequence in $\mathfrak{E}(X)$ satisfying (7) such that the inequalities (10) and (21) are valid for a subspace $Z=Y_2$ of X with order $\varrho_2>0$ as well as for a second subspace $Z=Y_1$ of X with order $\varrho_1>0$. If $0<\varrho_1<\varrho_2$, then

(i) $Y_2\subset Y_1$;

(ii) $\mathfrak{V}$ satisfies a Jackson-type inequality of order $(\varrho_2-\varrho_1)$ with respect to $Y_2\subset Y_1$, i. e., $\|V_n f-f\|_{Y_1}\leq C^* n^{-(\varrho_2-\varrho_1)}\|f\|_{Y_2}$ $(f\in Y_2;\ n\in N)$;

(iii) $\mathfrak{V}$ satisfies a Bernstein-type inequality of order $(\varrho_2-\varrho_1)$ with respect to $Y_2\subset Y_1$, i. e., $\|V_n f\|_{Y_2}\leq D^* n^{(\varrho_2-\varrho_1)}\|f\|_{Y_1}$ $(f\in Y_1;\ n\in N)$.

Here C^*, D^* are positive constants depending on $\mathfrak{V}$, Y_1 and Y_2. We now come to the primary result of this lecture.

Theorem 3. Let $\mathfrak{V}$ be a sequence in $\mathfrak{E}(X)$ satisfying (7) and commutativity relation (15) such that the inequalities (10) and (21) are valid for a subspace $Z=Y_2$ of X with order $\varrho_2>0$. The first three assertions are equivalent for $f\in X$ and $0<\theta<\varrho_2$, $1\leq q\leq\infty$,

$$\text{(b)}\qquad \sum_{n=1}^{\infty}(n^{\theta}\|V_n f-f\|_X)^q\frac{1}{n}<\infty,$$

$$\text{(c)}\qquad f\in(X, Y_2)_{\theta/\varrho_2, q},\ \text{ i. e., } \int_0^{\infty}(t^{-\theta}K(t^{\varrho_2}, f; X, Y_2))^q\frac{dt}{t}<\infty,$$

$$\text{(d)}\qquad \sum_{n=1}^{\infty}(n^{\theta-\varrho_2}\|V_n f\|_{Y_2})^q\frac{1}{n}<\infty.$$

Furthermore, if the inequalities (10) and (21) are also valid for $\mathfrak{V}$ with respect to a second subspace $Z=Y_1$ of X with order ϱ_1, $0<\varrho_1<\theta<\varrho_2$, then (b) — (d) are equivalent to

$$\text{(e)} \qquad f\in Y_1, \sum_{n=1}^{\infty}(n^{\theta-\varrho_1}\|V_nf-f\|_{Y_1})^q\frac{1}{n}<\infty.$$

Note that the case $q=\infty$ of this theorem includes the corresponding equivalences of Theorem 1. The proof of (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) is already given in [5, 6], for the proof of (d)$\Longleftrightarrow$(e) see [7].

Finally, let us illustrate Theorems 1 and 3 in two concrete situations, namely the typical means and de la Vallée Poussin means of the Fourier series. Given $f\in C_{2\pi}$, the former are defined by

$$[R_{n+1,l}f](x)=\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n+1}\right)^l f^{\wedge}(\nu)e^{i\nu x}, \quad l>0, \ n\in N. \tag{22}$$

Now $R_{n+1,l}f$ is an integral of convolution type and thus defines a sequence of commutative bounded linear operators mapping $C_{2\pi}$ into itself. So to apply Theorem 1 [3] we only need to check whether $\{R_{n+1,l}\}_{n=0}^{\infty}$ satisfies the conditions (7), (10 a, b), (13 a, b) (21 a, b) with respect to $X=C_{2\pi}$ and a suitable subspaces Y_i of $C_{2\pi}$. We choos $Y_2=C_{2\pi}^{\{l\}}$ and $Y_1=C_{2\pi}^{\{k\}}$, where $C_{2\pi}^{\{k\}}=\{f\in C_{2\pi}: f^{\{k\}}\in C_{2\pi}\}$ with $\|f\|_{C_{2\pi}^{\{k\}}}=\|f\|_{C_{2\pi}}+\|f^{\{k\}}\|_{C_{2\pi}}$. $f^{\{k\}}$ stands for the k-th order Riesz derivative of f which may be defined through

$$f^{\{k\}}(x)\sim\sum_{\nu=-\infty}^{\infty}|\nu|^k f^{\wedge}(\nu)e^{i\nu x}.$$

Then it is known (compare Butzer—Scherer [4, p. 149 ff] and the references cited there) that

$$\|R_{n+1,l}f-f\|_{C_{2\pi}}\leq Cn^{-l}\|f\|_{C_{2\pi}^{(l)}}, \quad f\in C_{2\pi}^{(l)}\ ;\ n\in N; \tag{23}$$

$$\|R_{n+1,l}f\|_{C_{2\pi}^{\{k\}}}\leq Dn^k\|R_{n+1,l}f\|_{C_{2\pi}}, \quad k, n\in N; \tag{24}$$

$$\|R_{n+1,l}f\|_{C_{2\pi}}\leq M_l\|f\|_{C_{2\pi}}.$$

Inequality (23) is of Jackson type with order $\varrho=l$; by results in [4] it also implies the validity of one of lower order $0\leq k\leq l$, namely

$$\|R_{n+1,l}f-f\|_{C_{2\pi}}\leq C_kn^{-k}\|f\|_{C_{2\pi}^{\{k\}}}, \quad f\in C_{2k}^{\{k\}}.$$

(24) is a strong Bernstein-type inequality in the sense of (13); it is the counterpart of the classical Bernstein inequality for trigonometric polynomials to derivatives in the sense of Riesz.

These facts enable one to state as a consequence of Theorem 1 (which may also be stated in the framework of the theory of intermediate spaces.

Theorem 4. Let the process $\{R_{n+1,l}\}_{n=0}^{\infty}$, $l>0$, be defined by (22). The following assertions are equivalent for $0<k<r+\alpha<l$, $0<\alpha<2$, $1\leq q\leq\infty$ and for $f\in C_{2\pi}$:

$$\text{(a)} \qquad \sum_{n=1}^{\infty}(n^{(r+\alpha)}E_n(f;C_{2\pi}))^q\frac{1}{n}<\infty,$$

(b) $$\sum_{n=1}^{\infty}(n^{(r+\alpha)}\|R_{n+1,l}f-f\|_{C_{2\pi}})^q\frac{1}{n}<\infty,$$

(c) $$\int_0^{\infty}(t^{-\alpha}\|\Delta_t^2 f^{(r)}(\cdot)\|_{C_{2\pi}})^q\frac{dt}{t}<\infty,$$

(d) $$\sum_{n=1}^{\infty}(n^{(r+\alpha-l}\|R_{n+1,l}^{\{l\}}f\|_{C_{2\pi}})^q\frac{1}{n}<\infty,$$

(e) $$\sum_{n=1}^{\infty}(n^{(r+\alpha-k)}\|R_{n+1,l}^{k)}f-f^{(k)}\|_{C_{2\pi}})^q\frac{1}{n}<\infty,\ f^{(k)}\in C_{2\pi}.$$

Indeed, it only remains to show that the statement (c) of Theorem 3 implies (c) above. This is a consequence of the fact that

$$(C_{2\pi}, C_{2\pi}^{\{l\}})_{(r+\alpha)/l,q}=\left\{f\in C_{2\pi}:\sum_{n=1}^{\infty}(n^{(r+\alpha)}E_n(f;C_{2\pi}))^q\frac{1}{n}<\infty\right\}$$

$$=(C_{2\pi}, C_{2\pi}^{\{l\}})_{(\pi+\alpha/l,q}$$

(25) $$=\left\{f\in C_{2\pi}:\int_0^{\infty}(t^{-(r+\alpha)}\|\Delta_t^1 f(\cdot)\|_{C_{2\pi}})^q\frac{dt}{t}<\infty\right\},$$

(26) $$=\left\{f\in C_{2\pi}:\int_0^{\infty}(t^{-\alpha}\|\Delta_t^2 f(\cdot)\|_{C_{2\pi}})^q\frac{dt}{t}<\infty\right\}.$$

Concerning these facts on intermediate space theory, the first and second equality follow by [4], the third and fourth (a reduction theorem) by [2].

Note that since $R_{n+1,l}^{(k)}(f)=R_{n+1,l}(f^{(k)})$, the fact that (b) and (c) are equivalent already implies (replacing r by $r-k$) that (e) is equivalent to (c).

This is not surprising since in establishing (b)$\Longleftrightarrow$(c) we have used the deep reduction theorem (26) of intermediate space theory, and the equivalence (b)$\Longleftrightarrow$(e) is also of reduction type. Let us finally add that all of the above considerations can also be carried out for non-integral positive k, r, l with $0<k<r+\alpha<l$, $0<\alpha<2$.

As already mentioned, an example of an operator sequence satisfying the hypotheses of Theorem 3 but not of Theorem 1 is given by the de La Vallee Poussin means defined by (17). The inequalities (18), (20) show that we must choose $X=C_{2\pi}$, $Y_2=C_{2\pi}^{(2)}$ with $\varrho_2=1$ for the Jackson and (weak) Bernstein-type inequalities (10) and (21) to be satisfied for $Z=Y_2$. By a general result in [7] this also implies such inequalities for the space $Y_1=C_{2\pi}^{(1)}$ with order $\varrho_1=1/2$.

The application of Theorem 3 now leads to

Theorem 5. Let $\{U_n\}_{n=0}^{\infty}$ be the approximation process defined by (17). The following assertions are equivalent for $f\in C_{2\pi}$ and $1/2<\alpha<1$, $1\leq q\leq\infty$:

(b) $$\sum_{n=1}^{\infty}(n^{\alpha} \| U_{n+1}f-f \|_{C_{2\pi}})^q \frac{1}{n}<\infty,$$

(c) $$\int_0^{\infty}(t^{-2\alpha} \| \Delta_t^2 f(\cdot) \|_{C_{2\pi}})^q \frac{dt}{t}<\infty,$$

(d) $$\sum_{n=1}^{\infty}(n^{\alpha-1} \| U''_{n+1}f \|_{C_{2\pi}})^q \frac{1}{n}<\infty,$$

(e) $$\sum_{n=1}^{\infty}(n^{(\alpha-1/2)} \| U'_{n+1}f-f' \|_{C_{2\pi}})^q \frac{1}{n}<\infty,\ f'\in C_{2\pi}.$$

Note that the assertions (b), (c) and (d) are also equivalent to another for $0<\alpha\leq 1/2$. Assertion (d) above follows by (d) of Theorem 3 (stating that $f\in(C_{2\pi}, C_{2\pi}^{(2)})_{\alpha,q}$) in view of (25) by writing $\alpha=2\alpha/2$.

Finally we remark that our Theorem 3 could also be established for approximation processes generated by a family of operators $\{V_t, 0\leq t<\infty\}$ depending upon a continuous parameter t (instead of the discrete n). In this form (for the precise formulation and proof see Butzer — Scherer [7]) it would subsume results of Butzer-Berens [2] on holomorphic one-parameter semigroups of bounded linear operators in $\mathfrak{E}(X)$.

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Lehrstuhl A für Mathematik,
Technische Hochschule
Aachen BRD

Received on September 20, 1970