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CONSTRUCTION OF AN ORTHONORMAL BASIS IN $C^m(I^d)$

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Summary. With the help of B-splines on $I=\langle 0,1\rangle$ the construction of a basis in the Banach space $C^k(I^d)$ is given for arbitrary $k\geqq 0$ and $d\geqq 1$.

1. Introduction. The sequence $\{x_n, n=1,2,\ldots\}$ of elements of a real Banach space $[X, \|\ \|]$ is called a basis in X whenever each $x\in X$ has unique convergent in the norm $\|\ \|$, expansion

$$x=\sum_{n=1}^{\infty} a_n x_n$$

with real coefficients a_n, $n=1, 2,\ldots$

The space $[C^m(I^d), \|\ \|_m]$ with $m\geqq 0$, $d\geqq 1$, $I=\langle 0,1\rangle$ with the norm

$$\|f\|_m=\sum_{|\alpha|\leqq m}\max_{I^d}|D^\alpha f(t)|$$

is separable Banach space. There is in $C^m(I^d)$ a natural scalar product

$$(f,g)=\int_{I^d} f(t)g(t)dt. \tag{1}$$

Our aim is to describe without proofs a construction of a basis in $[C^m(I^d), \|\ \|_m]$ orthogonal with respect to the scalar product (1).

The question of existence of a basis in $[C^1(I^2), \|\ \|_1]$ was raised already by S. Banach in [12, p. 238]. Only recently S. Schonefeld [6] and independently the author [4] exhibited the same basis in $[C^1(I^2), \|\ \|_1]$. The solution can easily be extended to the case of $C^1(I^d)$ with $d\geqq 1$. However, the problem for higher order derivatives remained open.

The construction presented here was suggested by the author's results on the Franklin orthonormal system ([2] and [3]).

A different construction of a basis in $C^m(T^d)$ with $m\geqq 0$, $d\geqq 1$, where T^d is a d-dimensional torus, was communicated to the author by S. Schonefeld. Without going into details we can say that our approach is of the Franklin type and Schonefeld's of the Schauder type (interpolating). It is remarkable that in both constructions the splines play a central role.

The basis constructed here turns out to be also a basis in the Sobolev spaces $W_p^m(I^d)$ with $m \geqq 0$, $d \geqq 1$ and $p \geqq 1$. This will be discussed in a joint paper of J. Domsta and the author prepared for print in *Studia Mathematica*.

The proof of Theorem 1 will be published in *Studia Mathematica* by J. Domsta.

2. The *B*-splines. Let $\{s_i,\ i=0, \pm 1, \ldots\}$, $s_i < s_{i+1}$, be an arbitrary partition of the real line and let $t_+ = \max(t, 0)$. For $m \geqq 0$ and for each j $(s_j - t)_+^{m+1}$ is a spline of degree $m+1$. Linear combinations of these functions are again splines. Thus, for each $i=0, \pm 1, \ldots$ the function

$$(2) \qquad N_{i,m}(t) = (s_{i+m+2} - s_i)[s_i, \ldots, s_{i+m+2}\,; (s-t)_+^{m+1}]$$

is a spline and it is called *B*-spline (basic spline); $[s_i, \ldots, s_{i+m+2}\,; f(s)]$ is the divided difference of f taken at the points $s_i, s \ldots, s_{i+m+2}$. In the right hand side of (2) t plays a role of a parameter. The *B*-splines were introduced by I. J. Schoenberg [7]. They have the following properties:

1°. $N_{i,m}(t) \geqq 0$ for $t \in (-\infty, \infty)$, $i=0, \pm 1, \pm 2, \ldots$

2°. $\operatorname{supp} N_{i,m} = \langle s_i, s_{i+m+2} \rangle$, $i=0, \pm 1, \pm 2, \ldots$

3°. For each t $\sum_i N_{i,m}(t) = 1$.

4°. For $m \geqq 1$ and for arbitrary real ξ_i

$$\Big(\sum_i \xi_i N_{i,m}(t)\Big)' = (m+1) \sum_i \frac{\xi_i - \xi_{i-1}}{s_{i+m+1} - s_i} N_{i,m-1}(t).$$

5°. Let $\langle a, b \rangle$ be given. The set of all function $N_{i,m}$ which do not vanish identically in $\langle a, b \rangle$ is linearly independent over this interval.

6°. For each i $N_{i,m} \in C^m$.

3. The sequence of special partitions. For $n=1$ we put $s_{i,n} = i$ and for $n = 2^q + \nu > 1$ where $q \geqq 0$, $1 \leq \nu \leqq 2^q$, we define

$$s_{i,n} = \begin{cases} \dfrac{i}{2^{q+1}} & \text{for } i \leqq 2\nu, \\[2mm] \dfrac{i-\nu}{2^q} & \text{for } i > 2\nu. \end{cases}$$

Notice that $s_{0,n} = 0$ and $s_{n,n} = 1$ for all $n \geqq 1$.

Thus, for each $n \geqq 1$ we have a partition:

$$\ldots < s_{-1,n} < s_{0,n} = 0 < \ldots < s_{n,n} = 1 < s_{n+1,n} < \ldots$$

4. The sequence of finite dimensional subspaces. Let us denote by $[x_1, \ldots, x_p]$ the linear space spanned by the elements $x_1, \ldots, x_p$. For each $n \geqq -m$, $m \geqq 0$, we define a subspace of $C^0(I)$ in the following way:

$$C_n^m(I) = [1, \ldots, t^{m+n}] \quad \text{for} \quad -m \leq n \leq 1,$$

$$C_n^m(I) = [N_{i,m}^n, i = -m-1, \ldots, n-1] \quad \text{for} \quad n > 1,$$

where $N_{i,m}^m$ are the B-splines corresponding to the partition $\{s_{i,m}, i = 0, \pm 1, \pm 2, .\}$.

It should be remembered that $\dim C_n^m(I) = m+n+1$ and that $C_n^m(I) > C_{n+1}^m(I)$. Moreover, the set $\bigcup_n C_n^m(I)$ is dense in $[C^0(I), \|\;\|_0]$.

5. The special orthonormal set in $C^0(I)$. We define for each $m \geq 0$ an orthonormal sequence $\{f_{i,m}, i \geq -m\}$ as follows: $f_{-m,m} = 1$, and for $i > -m$ $f_{i,m}$ is defined as one of the two different from zero elements in $C_i^m(I)$ which are orthonormal (with respect to the scalar product (1)) to the subspace $C_{i-1}^m(I)$. We normalize $\{f_{i,m}, i \geq -m\}$ the sequence so that $(f_{i,m}, f_{i,m}) = \delta_{ij}$.

For $m = 0$ the orthonormal set $\{f_{i,m}, i \geq -m\}$ is known as Franklin system and therefore it is a basis in $[C^0(I), \|\;\|_0]$ (cf. [2]).

6. The main result. In the finite dimensional space $C_n^m(I)$, $n > 1$, we have the basis $\{N_{i,m}^n, -m-1 \leq i \leq n-1\}$. Now, let $g_{i,j}(n, m) = (N_{i,m}^n, N_{j,m}^n)$, then $G(n, m) = (g_{i,j}(n, m))_{i,j=-m-1, \ldots, n-1}$ is the Gramm matrix and it has the inverse $G^{-1}(n, m) = ((a_{i,j}(n, m))_{i,j=-m-1, \ldots, n-1}$.

The Dirichlet's kernel for $\{f_{i,m}, i \geq -m\}$ can be written as follows

$$(3) \qquad K_n^m(t, s) = \sum_{i=-m}^{n} f_{i,m}(t) f_{i,m}(s) = \sum_{i,j=-m-1}^{n-1} a_{i,j}(n, m) N_{i,m}^n(t) N_{j,m}^n(s).$$

The following result is due for $m = 0$ to the author [3] and for $m \geq 1$ to J. Domsta.

Theorem 1. For each $m \geq 0$ there exist constants C_m and q_m, $0 < q_m < 1$, independent of n and such that

$$|a_{i,j}(n, m)| < C_m n q_m^{|i-j|}$$

holds for $n > 1$, $i, j = -m-1, \ldots, n-1$.

7. The special basis in $C^0(I)$. Let us define the integration operator as follows

$$(Hf)(t) = \int_t^1 f(s) ds$$

and let $(Df)(t) = f'(t)$. Using Theorem 1 and formula (3) one can prove the following inequality

$$(4) \qquad |I_s^k D_t^k K_n^m(s, t)| < \widetilde{C}_m n \widetilde{q}_m^{n|t-s|},$$

for $s, t \in I$, $m \geq 0$, $0 \leq k \leq m$ and $n > 1$, where $\widetilde{C}_m$ and $\widetilde{q}_m$, $0 < \widetilde{q}_m < 1$, are constants independent of n.

It follows from the construction of $\{f_{i,m}, i \geq -m\}$ that for $f \in C^k(I)$ we have

$$(5) \qquad (f, f_{i,m}) = (D^k, H^k f_{i,m}), \; i \geq k-m, \; 0 \leq k \leq m.$$

Finally, from (4) and (5) we can deduce the following

Theorem 2. Let $m \geq 0$ and $0 \leq k \leq m$. Then $\{D^k f_{i,m}, i \geq k-m\}$ is a basis in $[C^0(I), \| \|_0]$.

This result for $m=0$ is due to Franklin (cf. [2]) and for $m=1$ to J. Radecki [5].

8. The orthonormal bases in $C^m(I^d)$. Let $m \geq 0$ and let $d \geq 1$. We consider on I^d the following family of functions

$$\{f_{i_1,m}(t_1) \ldots f_{i_d,m}(t_d),\ i_1 \geq -m, \ldots, i_d \geq -m\}. \tag{6}$$

Theorem 3. The family of functions (6) can be arranged into a single sequence $\{F_{i,m}, i=0, 1, \ldots\}$ which is a basis in $[C^m(I^d), \| \|_m]$.

It should be clear that $\{F_{i,m}, i=0, 1, \ldots\}$ is an orthonormal basis and therefore it is a stability point for the Schmidt orthogonalization operator whence by the result of A. Olevskii (see p. 147—150 the paper by A. Olevskii at this Conference) we obtain

Theorem 4. Let $m \geq 0$ and $d \geq 1$. Then there exists in $[C^m(I^d), \| \|_m]$ an orthonormal (with respect to the scalar product (1)) basis of algebraic polynomials.

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