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LAGRANGE INTERPOLATION ON INFINITE INTERVAL

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Summary. In the first part of this paper we present some basic theorems about orthogonal polynomials, which are necessary for the study of interpolation. The second part refers about my latest result obtained together with G. Róna.

§ 1. Let us denote by $P_n(x)$ the orthogonal polynomials associated with the weightfunction $w(x)$ on the interval $(-\infty, \infty)$, where $w(x)$ satisfies the nequality

$$ae^{-x^2} \leq w(x) \leq be^{-x^2} \tag{1}$$

and let the coefficient of x^n in $P_n(x)$ be positive.

Let us consider the following polynomials of degree n:

$$Q_n = P_n(x) + A \,.\, P_{n-1}(x) + B \,.\, P_{n-2}(x), \tag{2}$$

where $B \leq 0$, which are called Quasi-Hermite polynomials. It was proved in the book of G. Freud [1] that the zeros of quasi orthogonal polynomials are real and distinct.

Let

$$\xi_{1,n} < \xi_{2,n} < \dots < \xi_{n,n} \tag{3}$$

be the zeros of $Q_n(x)$ in increasing order.

The following theorem is proved:

Theorem. There exist constants c_1 and c_2 independent of n so, that the inequality

$$\frac{c_1}{\sqrt{n}} \leq \xi_{r+1,n} - \xi_{r,n} \leq \frac{c_2}{\sqrt{n}}, \quad \text{where } |\xi_{r+1,n}| \leq c_3\sqrt{n} \tag{4}$$

holds.

The proof of this theorem uses the following lemmas with the following notations:

$l_{r,n}$ are the fundamental polynomials of Lagrange interpolation;

$\lambda_{r,n}$ are the Christoffel number of mechanical quadrature.

Lemma 1. If $\Pi_{2n-3}(x)$ is an arbitrary polynomial of degree at most $2n-3$, then

(5) $$\int_{-\infty}^{\infty} \Pi_{2n-3}(x)w(x)dx = \sum_{r=1}^{n} \lambda_{r,n}\Pi_{2n-3}(\xi_{r,n})$$

holds.

Lemma 2.

(6) $$0 < \int_{-\infty}^{\infty} l_{r,n}^2(x)w(x)dx \leq \lambda_{r,n}, \quad r=1, 2, \ldots, n.$$

Lemma 3. Let $\xi_{r,n}$ and $\xi_{r+1,n}$ are the two consecutive zeros of $Q_n(x)$. Then

(7) $$\int_{\xi_{r,n}}^{\xi_{r+1,n}} w(x)dx \leq \lambda_{r,n} + \lambda_{r+r,n}$$

holds.

Lemma 4. If $|\xi_{r,n}| \leq \sqrt{n}$, then

(8) $$\lambda_{r,n} \leq \frac{c_1 \exp(-\xi_{r,n}^2)}{\sqrt{n}}$$

holds.

Lemma 5. If $\Pi_{n-1}(x)$ is a polynomial of degree at most $n-1$ and

(9) $$\int_{-\infty}^{\infty} \Pi_{n-1}^2(x)e^{-x^2}dx \leq 1,$$

then

(10) $$\Pi_{n-1}^2(x) \leq \sum_{r=1}^{n} h_r^2(x)$$

holds, where $h_r(x)$ are the orthogonal Hermite polynomials.

Lemma 6. Let $\|f\|$ denote

(11) $$\|f\| = \max_{-\infty<x<\infty} |f(x) . e^{-x^2/2}|$$

and

(12) $$\alpha_n = \sup \frac{\|\pi_n'\|}{\|\pi_n\|},$$

where π_n are polynomials of degree at most n, then there exist constants c_5 and c_6, for which

(13) $$c_5\sqrt{n} \leq \alpha_n \leq c_6\sqrt{n}$$

holds.

Lemma 6 was proved by G. Freud, while the remaining lemmas are the infinite equivalent of same classical finite results.

§ 2. Let $f(x)$ be continuous and bounded function in $[0, \infty)$ and let

(14) $$\|f\| = \sup_{0\leq x<\infty} |f(x)|.$$

We consider the r-th modulus of continuity of

$$w_r(f;\delta)=\sup_{\substack{|h|\leq\delta\\ 0\leq x<\infty}}\left|\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(x+sh)\right|. \tag{15}$$

Denote by $a_n(f;x)$ the Lagrange interpolation polynomials.

The lower estimate for the approximation error of interpolation is as follows:

$$|f(x)-a_n(f;x)|\leq\begin{cases} c_r^{(1)}\cdot n^{\frac{1}{2}}\cdot w_r(f;n^{-2})\cdot w\left(\frac{x}{2}\right), & \frac{1}{n}\geq x\geq 0;\\ c_r^{(2)}\cdot\left(\frac{4n-x}{x}\right)^{\frac{1}{4}}\cdot w_r\left(f;n^{-\frac{1}{4}}\right)\cdot w\left(\frac{x}{2}\right), & \frac{1}{n}<x\leq 4n^{\frac{3}{4}};\\ 2\cdot\|f\|\cdot\left(2n^{\frac{3}{4}}\right)^{-a}\left(2e^{-\frac{3}{2}}\right)^n\cdot w\left(\frac{x}{2}\right), & x\geq 4n^{\frac{3}{4}}, \end{cases} \tag{16}$$

where $w(x)$ is the weightfunction for which the asymptotical Laguerre polynomials are orthogonal, and for which the inequality

$$ax^a e^{-x}\leq w(x)\leq bx^a e^{-x} \tag{17}$$

holds.

REFERENCES

1. G. Freud. Orthogonale Polynome. Berlin, 1969.

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