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THE UNIFORM ASYMPTOTIC EXPANSION OF A RATIO OF GAMMA FUNCTIONS

J. L. Fields

Summary. The following formula is established:

$$w^{\beta-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim\sum_{j=0}^{m-1}\frac{\Gamma(\beta-\alpha+2j)}{\Gamma(\beta-\alpha)(2j)!}B_{2j}^{(2p)}(\varrho)w^{-2j}+o\left(\left(\frac{1+|\alpha-\beta|^{2}}{w^{3}}\right)^{m}\right),$$

$$z\to\infty,\ |\arg z|<\pi,$$

where $w=z+\dfrac{\alpha+\beta-1}{2}$, $2\varrho=1+\alpha-\beta$ and $B_k^{(\sigma)}(\varrho)$ is the generalized Bernoulli polynomial.

Introduction. In [1] I established the result, for α, β bounded, m arbitrary,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\frac{\Gamma(w+p)}{\Gamma(w+1-p)}=\sum_{j=0}^{m-1}\frac{\Gamma(1-2\varrho+2j)}{\Gamma(1-2\varrho)(2j)!}B_{2j}^{(2p)}(\varrho)w^{2p-1-2j}+O(w^{2\varrho-1-2m}),$$

$$w\to\infty,\ |\arg(w+\varrho)|<\pi,\ 2w=2z+\alpha+\beta-1,\ 2\varrho=1+\alpha-\beta,$$

which is a computational improvement of the Tricomi-Erdélyi result [2],

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(\beta-\alpha+j)}{\Gamma(\beta-\alpha)\,j!}B_j^{(\alpha-\beta+1)}(\alpha)+O(z^{\alpha-\beta-m}),\tag{2}$$

$$z\to\infty,\ |\arg(z+\alpha)|<\pi,$$

where the $B_j^{(\sigma)}(\alpha)$ are the generalized Bernoulli polynomials defined by

$$\left(\frac{t}{e^t-1}\right)^{\sigma}e^{\alpha t}=\sum_{j=0}^{\infty}\frac{t^j}{j!}B_j^{(\sigma)}(\alpha),\quad B_0^{(\sigma)}(\alpha)=1,\quad |t|<2\pi.$$

Here, I improve (1), (2) by showing that $O(w^{2\varrho-1-2m})$, $O(z^{\alpha-\beta-m})$ can be replaced by $O(w^{2\varrho-1-2m}(1+|\varrho|)^{3m})$, $O(z^{\alpha-\beta-m}(1+|\alpha-\beta|^m)(1+|\alpha|+|\alpha-\beta|)^m)$ respectively, which permits α and β to be large, i. e., $\varrho=o(w^2)$, $(1+|\alpha-\beta|)(1+|\alpha|+|\alpha-\beta|)=o(z)$. The proof essentially consists in considering a situation where Watson's Lemma [3] can be generalized to include secondary parameters which are large. A point of interest is that two integral representations of the error are apparently necessary instead of the usual single integral representation.

I begin by recording some easily proved technical results which will be used later.

Lemma 1. If

$$h(v)=\sqrt{\frac{2(v-1-\log v)}{(v-1)^2}},\quad h(1)=1,$$

$$x(v)=2^{-1/2}e^{-i\pi/2}(v-1)h(v)=\sqrt{1-v+\log v},$$

$$y(v)=2^{-1/2}(v-1)h(v)=\sqrt{v-1-\log v}$$

and the v-contours $C(x,\varphi)$, $C(y,\varphi)$ are defined by

$$C(x,\varphi):\operatorname{Im}\{e^{i\varphi}x(v)\}=0,\quad |\varphi|<\pi/4,$$

$$C(y,\varphi):\operatorname{Im}\{e^{i\varphi}y(v)\}=0,\quad \varphi=0,$$

then for $|\varphi|\leq\pi/4-\varepsilon<\pi/4$ if $z=x$, and $\varphi=0$ if $z=y$,

(i) $z(v)$ maps the v-contour $C(z,\varphi)$ onto the z-contour $C'(z,\varphi):z=Re^{-i\varphi}$, $-\infty<R<\infty$;

(ii) $v=v(z)$ is an analytic function of z for $z\in C'(z,\varphi)$;

(iii) $$\frac{dv(z)}{dz}=\frac{(-1)^{\sigma}2zv(z)}{1-v(z)}=O(1+|z|),\ \sigma=0,\ 1 \text{ if } z=x,\ y,$$

$$v(z)=O(1+|z|^2),$$

uniformly for $z\in C'(z,\varphi)$.

Proof. The lemma follows from the fact that the derivative of $1-v+\log v$ equals zero only if v equals one.

Remarks. The orientation of $C(z,\varphi)$ is determined by $C'(z,\varphi)$, which moves from left to right. Clearly, $C(y,0)$ is just the positive v axis. It is not hard to see that the curves $C(x,\varphi)$ in polar coordinates, $v=re^{i\theta}$, are among the solutions of

$$r\sin(2\theta+\varphi)-(1+\log r)\sin 2\varphi-\varphi\cos 2\varphi=0,$$

and that for $C(x,\varphi)$

$$\theta=-2\varphi\pm\pi+O\left(\frac{\log r}{r}\right),\quad r\to\infty,\ |\varphi|<\pi/4.$$

The reader will find that a rough sketch of $C(x,0)$ and $C(x,\pm\pi/4)=\lim\limits_{\varphi\to\pm\pi/4}C(x,\varphi)$ is very instructive.

Lemma 2. If

$$\theta(t,a,\sigma)=at-\sigma\log\left(\frac{e^t-1}{t}\right),\quad \theta(0,a,\sigma)=0,$$

$$r_m(t,a,\sigma)=t^{-m}\left\{e^{\theta(t,a,\sigma)}-\sum_{j=0}^{m-1}\frac{t^j}{j!}B_j^{(\sigma)}(a)\right\},$$

then

(i) $$r_m(t,\alpha,\sigma)=O((1+|\alpha|+|\sigma|)^m e^{|\theta(t,\alpha,\sigma)|}),$$

(ii) $$r_{2m}(t,\alpha,2\alpha)=O((1+|\alpha|)^m e^{|\theta(t,\alpha,2\alpha)|}),$$

uniformly in t.

Proof. From $\theta(t,\alpha,2\alpha)=\theta(-t,\alpha,2\alpha)$, it follows that $B^{(2\alpha)}_{2j+1}(\alpha)=0$. Let $r=(1+|\alpha|+|\sigma|)^{-1}$ if $\sigma\neq 2\alpha$, and $r=(1+|\alpha|)^{1/2}$ if $\sigma=2\alpha$. Then by Cauchy's estimates on the circle $|t|=2r<2\pi$

$$\left|\frac{B_j^{(\sigma)}(\alpha)}{j!}\right|\leq(2r)^{-j}\max_{|t|=2r}|e^{\theta(t,\alpha,\sigma)}|=O(r^{-j}),$$

$$|r_m(t,\alpha,\sigma)|\leq\frac{(2r)^{1-m}}{2r-|t|}\max_{|\zeta|=2r}|e^{\theta(\zeta,\alpha,\sigma)}|=O(r^{-m}),\quad |t|\leq r,$$

whereas when $|tr^{-1}|\geq 1$,

$$|r_m(t,\alpha,\sigma)|\leq|t|^{-m}\left\{|e^{\theta(t,\alpha,\sigma)}|+O(1)\sum_{j=0}^{m-1}|tr^{-1}|^j\right\}$$

$$=|t|^{-m}\{e^{|\theta(t,\alpha,\sigma)|}+O(|tr^{-1}|^m)\}=O(r^{-m}e^{|\theta(t,\alpha,\sigma)|}).$$

Remark. If t is restricted to a smooth curve C which avoids the points $2\pi ik$, $k=\pm1,\pm2,\ldots$; does not wind around $t=0$ an infinite number of times, and approaches infinity in such a way that $e^t\to 0$, then

$$\theta(t,\alpha,\sigma)=O(\alpha t),\quad \sigma\neq 2\alpha,$$

$$\theta(t,\alpha,2\alpha)=\left(\frac{\alpha t^2}{1+|t|}\right),$$

uniformly for $t\in C$.

I now establish my main result.

Theorem.

(i) $$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(\beta-\alpha+j)}{\Gamma(\beta-\alpha)j!}B_j^{(\alpha-\beta+1)}(\alpha)z^{\alpha-\beta-j}$$
$$+O(z^{\alpha-\beta-m}(1+|\alpha-\beta|)^m(1+|\alpha|+|\alpha-\beta|)^m)$$
$$z\to\infty,\quad |\arg(z+\alpha)|<\pi,\quad (1+|\alpha-\beta|)(1+|\alpha|+|\alpha-\beta|)=o(z);$$

(ii) $$\frac{\Gamma(z+\alpha)}{\Gamma(z+1-\alpha)}=\sum_{j=0}^{m-1}\frac{\Gamma(1-2\alpha+2j)}{\Gamma(1-2\alpha)(2j)!}B_{2j}^{(2\alpha)}(\alpha)z^{2\alpha-1-2j}+O(z^{2\alpha-1-2m}(1+|\alpha|)^{3m}),$$
$$z\to\infty,\quad |\arg(z+\alpha)|<\pi,\quad (1+|\alpha|)^3=o(z^2);$$

(iii) $$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\sum_{j=0}^{m-1}\frac{\Gamma(\beta-\alpha+2j)}{\Gamma(\beta-\alpha)(2j)!}B_{2j}^{(\alpha-\beta+1)}\left(\frac{\alpha-\beta+1}{2}\right)\left(\frac{2z+\alpha+\beta-1}{2}\right)^{\alpha-\beta-2j}$$
$$+O((2z+\alpha+\beta-1)^{\alpha-\beta-2m}(1+|\alpha-\beta|^{3m}),$$
$$z\to\infty,\quad |\arg(z+\alpha)|<\pi,\quad (1+|\alpha-\beta|)^3=o((2z+\alpha+\beta-1)^2).$$

Proof. After proving (i), I will indicate what minor modifications must be made to establish (ii). Using the w, ϱ notation of (1), it is not hard to see that (ii) and (iii) are equivalent. For convenience, let $\sigma=1+\alpha-\beta$.

First assume that $|\arg\sigma|\leq\pi-\varepsilon<\pi$. It follows from the Beta integral and Lemma 2, that

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\frac{\Gamma(\sigma)}{2\pi i}\int_{-\infty e^{i\delta}}^{0+}e^{(z+\alpha)t}(e^t-1)^{-\sigma}dt$$

$$=\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(\beta-\beta+j)}{\Gamma(\beta-\alpha)\,j!}B_j^{(\sigma)}(\alpha)z^{\alpha-\beta-j}+R_m(z,\alpha,\sigma),$$

(3)
$$R_m(z,\alpha,\sigma)=\frac{\Gamma(\sigma)}{4\pi i}\int_{-\infty e^{i\delta}}^{0+}e^{zt}t^{m-\sigma}r_m(t,\alpha,\sigma)dt,$$

$$|\arg[(z+\alpha)e^{i\delta}]|<\pi/2,\quad |\arg[(e^t-1)e^{-i\delta}]|\leq\pi,\quad |\delta|<\pi/2.$$

Note that

$$|\arg(ze^{i\delta})|=\left|\arg[(z+\alpha)e^{i\delta}]-\arg\left(1+\frac{\alpha}{z}\right)\right|\leq|\arg[(z+\alpha)e^{i\delta}]|+O\left(\frac{\alpha}{z}\right)<\frac{\pi}{2},$$

allows the use of Hankel's integral,

$$\frac{1}{2\pi i}\int_{-\infty e^{i\delta}}^{0+}e^{zt}t^{-\lambda}dt=\frac{z^{\lambda-1}}{\Gamma(\lambda)},\quad |\arg(ze^{i\delta})|<\frac{\pi}{2},\quad |\delta|<\frac{\pi}{2},$$

in the above.

If σ is bounded, the change of integration variable $zt=u$ leads directly to the desired order estimate for R_m. If σ is unbounded, make the change of variable $zt=\sigma v$, and choose Φ such that $|\omega|<\pi/4$, $2\omega=\arg\sigma-2\Phi$. Then it is easy to see that

$$R_m(z,\alpha,\sigma)=\Gamma(\sigma)\left(\frac{\sigma}{z}\right)^{m-\sigma+1}\frac{e^\sigma}{2\pi i}\int_{C(x,\Phi)}e^{\sigma(v-1-\log v)}v^m r_m\left(\frac{\sigma v}{z},\alpha,\sigma\right)dv$$

$$=\Gamma(\sigma)\left(\frac{\sigma}{z}\right)^{m-\sigma+1}\frac{e^\sigma}{2\pi i}\int_{-\infty e^{-i\Phi}}^{\infty e^{-i\Phi}}e^{-\sigma x^2}[v(x)]^m r_m\left(\frac{\sigma v(x)}{z},\alpha,\sigma\right)\frac{dv(x)}{dx}dx$$

$$=O\left(z^{\sigma-1-m}\sigma^{\frac{1}{2}+m}(1+|\alpha|+|\sigma|)^m\int_{-\infty e^{-i\Phi}}^{\infty e^{-i\Phi}}(1+|x|)^{4m+1}\left|\exp\left(-\sigma x^2+O\left(\frac{\alpha\sigma v(x)}{z}\right)\right)dx\right|\right)$$

$$=O\left(z^{\sigma-1-m}\sigma^m(1+|\alpha+|\sigma|)^m\int_{-\infty e^{-i\omega}}^{\infty e^{i\omega}}(1+|u|)^{4m+1}/\exp\left(-u^2\left[1+O\left(\frac{\alpha}{z}\right)\right]+O\left(\frac{\alpha\sigma}{z}\right)\right)du\right)$$

$$=O(z^{\sigma-1-m}(1+|\sigma|)^m(1+|a|+|\sigma|)^m),$$

since a, $a\sigma$ are $o(z)$. Here we have made use of Stirling's formula

$$\Gamma(\sigma)=\sqrt{2\pi}\sigma^{\sigma-\frac{1}{2}}e^{-\sigma}\{1+O(\sigma^{-1})\},\quad \sigma\to\infty,\quad |\arg\sigma|\leq\pi-\varepsilon<\pi.$$

For remaining sector $|\arg(-\sigma)|<\pi/2$, an alternate representation for R_m is needed. If $\sigma<1$ and $z>0$, the loop contour in (3) can be shrunk into the straight lines $zt=ue^{\pm i\pi}$, which with an analytic continuation argument leads us to

$$R_m(z,a,\sigma)=(-1)^m\frac{z^{\sigma-1-m}}{\Gamma(1-\sigma)}\int_0^{\infty e^{i\delta}}e^{-u}u^{m-\sigma}r_m\left(-\frac{u}{z},a,\sigma\right)du,$$

$$\delta<\pi/2,\quad Re(-\sigma)>-1.$$

From the change of variable $u=(-\sigma)v(y)$, we obtain

$$R_m(z,a,\sigma)=(-1)^m\left(-\frac{\sigma}{z}\right)^{m+1-\sigma}\frac{e\sigma}{\Gamma(1-\sigma)}\int_0^{\infty}e^{\sigma(v-1-\log v)}v^m r_m\left(\frac{\sigma v}{z},a,\sigma\right)dv$$

$$=(-1)^m\left(\frac{-\sigma}{z}\right)^{m+1-\sigma}\frac{e^{\sigma}}{\Gamma(1-\sigma)}\int_{-\infty}^{\infty}e^{\sigma y^2}[v(y)]^m r_m\left(\frac{\sigma v(y)}{z},a,\sigma\right)\frac{dv(y)}{dy}dy$$

$$=O(z^{\sigma-1-m}(-\sigma)^{\frac{1}{2}+m}(1+|a|+|\sigma|)^m\int_{-\infty}^{\infty}(1+|y|)^{4m+1}\left|\exp\left(\sigma y^2+O\left(\frac{a\sigma v(y)}{z}\right)\right)\right|dy,$$

which reduces as before to the correct order estimate. The analysis o $R_{2m}(z,a,2a)$ is almost identical, but it must be kept in mind that when $zt=2av(x)$, $v(x)=O(1+|x|^2)$,

$$\frac{at^2}{1+|t|}=O\left(\frac{a^2t(1+|x^2|)}{z(1+|t|)}\right)=O\left(\frac{a^2t}{z}\right)+O\left(\frac{a^2x^2}{z}\right)=o(1)+ax^2o(1).$$

A similar remark holds when $v=\tilde{v}(y)$.

Corollary 1. Under the changes of variable

$$2z=v(\lambda+\mu)+a+b-1,$$

$$2\alpha=v(\lambda+\mu)+a-b+1,$$

$$2\beta=v(\mu-\lambda)-a+b+1,$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=\frac{\Gamma(\lambda v+a)}{\Gamma(\mu v+b)}$$

and (iii) of the theorem yelds an asymptotic expansion for $\frac{\Gamma(\lambda v+a)}{\Gamma(\mu v+b)}$ in terms of the variable $[v(\lambda+\mu)+a+b-1]^2$ as $(\lambda+\mu)v\to\infty$, when $|\arg[(\lambda+\mu)v]|<\pi$, and $[v(\lambda-\mu)+a-b]^3=o([v(\lambda+\mu)+a+b-1]^2$, i. e. $\lambda-\mu=o(v^{-\frac{1}{3}})$.

Remark. This form of (iii) was first mentioned by P. C. Consul in the January 1969 *Notices of the Americal Mathematicai Society.* In private correspondence, he informed me that he had not established this result analytically, but only investigated its first few terms by direct computation.

Corollary 2. If $o\leq k\leq n$, k an integer, then

$$\binom{n}{k}=\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n+1-k)}=\frac{n^k}{k!}\left\{1+\frac{k(k-1)}{2n}+O\left(\frac{(k+1)^4}{n^2}\right)\right\}$$

$$=\frac{(n+(1-k)/2)^k}{k!}\left\{1+\frac{k-k^3}{6(2n+1-k)^2}+O\left(\frac{(k+1)^6}{(2n+1-k)^4}\right)\right\},$$

and for λ a parameter,

$$\frac{\Gamma(n+1)\Gamma(n+\lambda+k)}{\Gamma(n+1-k)\Gamma(n+\lambda)}=n^{2k}\left\{1+\frac{k\lambda}{n}+O\left(\frac{(2k+|\lambda|+1)^2+(|\lambda|+1)^2}{n^2}\right)\right\}$$

$$=\left(n+\frac{\lambda}{2}\right)^{2k}\left\{1-\frac{k[4k^2+6k(\lambda-1)+3\lambda^2-6\lambda+2]}{3(2n+\lambda)^2}+O\left(\frac{(2k+|\lambda|+1)^6}{(2n+\lambda)^4}\right)\right\}.$$

Remark. This last expansion is derived by multiplying the expansions for $\Gamma(n+1)/\Gamma(n+\lambda)$ and $\Gamma(n+\lambda+k)/\Gamma(n+1-k)$ together. For some purposes, it is more convenient to use the large variable $4n(n+\lambda)=(2n+\lambda)^2-\lambda^2$. The last expression then becomes

$$\frac{\Gamma(n+1)\Gamma(n+\lambda+k)}{\Gamma(n+1-k)\Gamma(m+\lambda)}=[n\,(n+\lambda)]^k\left\{1-\frac{k(k-1)(2k+3\lambda-1)}{6n(n+\lambda)}+O\left(\frac{(2k+|\lambda|+1)^6}{[n(n+\lambda)]^2}\right)\right\}.$$

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University of Alberta Department of Mathematics
Edmonton Canada

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