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ON A CLASS OF ORTHOGONAL POLYNOMIALS

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Summary. We say $w \in \Phi_r(\xi)$ if $w(x)=q(x)e^{-x}$ is a weight function vanishing for, $x<0$ and for a $\xi \in (0, \infty)$ there exist positive constants A, B and r so that for $0<x_1<\xi<x_2$ we have $0 \leq q(x_1) \leq Aq(\xi) \leq A^2 q(x_2)$ and $x_1^{-r} q(x_1) \geq B^{-1} \xi^{-r} q(\xi) \geq B^{-2} x_2^{-r} q(x_2) \geq 0$. Let further $\varphi_n(x_n)=(x^{1/2}+n^{-1/2})/(|4n-x|^{1/2}+n^{1/6})$ and $\{p_n(w; x)\}$, $n=0, 1, \ldots$, be the sequence of orthonormal polynomials with respect to $w(x)$ and let $\lambda_n(w; \xi)=\left\{\sum_0^{n-1} p_\nu^2(\omega; \xi)\right\}^{-1}$ be the corresponding Christoffel function.

Some theorems about these polynomials are proved, as for instance

Theorem 1. For $w \in \Phi_r(\xi)$ and $0 \leq \xi \leq 4n+\varrho n^{1/3}$, $\varrho<0$, we have $0<c_0 w(\xi)\varphi_n(\xi) \leq \lambda_n(w; \xi) \leq (c_1+c_2\varrho+c_3(\xi_n)^{-r})\, w(\xi)\varphi_n(\xi)$, where c_0, $c_1, \ldots$ depend on A, B and r only.

Applications to different problems of approximation theory are discussed.

1. Introduction. Notations. The main part of this paper (estimates of $\Delta_n(w; \xi)$ and $\lambda_n(w; \xi)$) is a shortened version of a contribution submitted for publication in April 1970 to *Математические заметки*. Details of proof (where they are scatched only here) may be found in that publication.

Using the same notations as in our book [1], let $\{p_n(w; x)\}$ be the sequence of orthonormal polynomials with respect to the weight $w(x) \geq 0$ let $x_{kn}(w)$, $k=1, 2, \ldots, n$, be the zeros of $p_n(w; x)$ in decreasing order and let

$$\psi_n(w;\ x,\ \xi)=p_{n-1}(w; \xi)p_n(w; x)-p_n(w; \xi)p_{n-1}(w; x)$$

be the quasiorthogonal polynomial vanishing at $x=\xi$. (The support of the weights considered will be in $[0, \infty]$.)

In the present paper we denote by ξ^* the greatest positive zero of $\psi_n(w; x, \xi)$ smaller than ξ, provided that such a zero exists, and $\xi^*=0$ else, and let us define

$$\Delta_n(w; \xi)=\xi-\xi^*.$$

We are also considering the Christoffel function

$$\lambda_n(w; \xi)=\left[\sum_{k=0}^{n-1} p_k^2(w; \xi)\right]^{-1}.$$

Let $\Phi_r(\xi)$ $(r>0, \xi>0)$ be the class of non-negative weight functions $w(x)=q(x)e^{-x}$ vanishing for $x<0$, satisfying the following conditions.

There exist two numbers $0<A\leq 1\leq B$ so that for every $0<x_1\leq\xi\leq x_2$ we have

$$0\leq q(x_1)\leq Aq(\xi)\leq A^2q(x_2)$$

and

$$B^2x_1^{-r}q(x_1)\geq B\xi^{-r}q(\xi)\geq x_2^{-r}q(x_2).$$

We say that $w\in\Phi_r$ iff $w\in\Phi_r(\xi)$ for every $\xi>0$.

The Laguerre polynomials $L_n^a(x)$ are apart of a constant factor the orthogonal polynomials $p_n(w_a;x)$ with respect to the weight $w_a(x)=x^a e^{-x}$. We observe that for $a\geq 0$ $w_a\in\Phi_r$ for every $r\geq a$.

In this paper we first give estimates of $\Lambda_n(w;\xi)$ and $\lambda_n(w;\xi)$ under the conditions $w\in\Phi_r(\xi)$, resp. $w\in\Phi_r$ and then apply these estimates to some problems in approximation theory.

2. Investigation of Laguerre-polynomials. Relations between λ_n and Λ_n. Let us denote by x_{kn}^a, $k=1,2,\ldots,n$, the zeros of the Laguerre polynomial $L_n^a(x)=c_np_n(w_a;x)$ in decreasing order, and let $x_{n+1,n}^a=0$.

Applying classical theorems to the Sturm-Liouville type differential equation

$$\frac{d^2v_n}{dt^2}+\left(4n-t^2+2a+2+\frac{1-4a^2}{4x^2}\right)v_n=0,$$

having $v_n(t)=e^{-t^2/2}t^{a+1/2}L_n^a(t^2)$ as one of its solutions, we find after some calculation that

$$b_1\varphi_n(x_{kn}^a)\leq x_{kn}^a-x_{k+1,n}^a\leq b_2\varphi_n(x_{kn}^a),\ k=1,2,\ldots,n, \tag{1}$$

where $b_1,b_2,\ldots$ are positive numbers depending on the occurring parameters a or r only and

$$\varphi_n(x)=\frac{x^{1/2}+n^{-1/2}}{|4n-x|^{1/2}+n^{1/6}}. \tag{2}$$

The following lemmata allow us to derive from (1) estimates of $\lambda_n(w_a;\xi)$

Lemma 1. Let $w(x)=q(x)e^{-x}$ and let $q(x)$ be non-decreasing, then $e^{\xi}\lambda_n(w;\xi)$ is increasing.

P r o o f. Using Theorem I.4.2 of [1], we compare $\lambda_n(w;\xi)$ and $\lambda_n(w_h;\xi)$, where $w_h(x)=e^{-h}w(x-h)\leq w(x)$ for $h>0$.

Lemma 2. For $w\in\Phi_r(\xi)$ we have

$$e^{\xi}\lambda_n(w;\xi)+e^{\xi^*}\lambda_n(w;\xi^*)\geq c_1(\xi-\xi^*)q(\xi).$$

Here $c_1,c_2,\ldots$ denote positive numbers depending on r, A and B.

P r o o f. This is a consequence of the Chebishev-Posse inequality (see formula I. (5.10) in [1]).

Lemma 3. For $w\in\Phi_r(\xi)$ and $\xi\in[x_{n-1,n-1}(w),x_{1,n-1}(w)]$ we have

$$e^{\xi}\lambda_n(w;\xi)\leq c_2(\xi-\xi^*)q(\xi).$$

P r o o f. We apply the Chebishev-Posse inequality

$$e^{(1-\varrho)\xi}\,\lambda_n(w;\xi)\leqq\int\limits_{\xi^*}^{\infty}e^{(1-\varrho)x}\;w(x)dx$$

with $\varrho=(\xi-\xi^*)^{-1}$.

We now apply these lemmata first to the weights $w_a(x)$.

Lemma 4. We have for $0\leqq a\leqq r$ and $\xi\in[0,4n]$

$$b_3(\xi^a+n^{-a})\varphi_n(\xi)\leqq e\xi\lambda_n(w_a;\xi)\leqq b_4(\xi^a+n^{-a})\varphi_n(\xi). \tag{3}$$

(Here b_3 and b_4 depend only on r.)

Proof. For $\xi=x^a_{kn}$, $k=2,3,\ldots,n-1$, (3) is a consequence of Lemma 2 and Lemma 3 and for $\xi=0$ one can verify it directly. The extension to all $\xi\in[0,4n]$ is derived from Lemma 1.

3. Inequalities for $\lambda_n(w;\xi)$ and $\Delta_n(w;\xi)$. From $w\in\Phi_r(\xi)$ and $0<\xi\leqq 2$ we have

$$c_3q(\xi)w_0\left(x-\frac{\xi}{2}\right)\leqq w(x)\leqq c_4q(\xi)\xi^{-r}e^{\xi}w_r(x+\xi) \tag{4}$$

and for $\xi\geqq 2$ we obtain from $w\in\Phi_r(\xi)$

$$c_5q(\xi)\xi^{-r}w_r[(1+\xi^{-1})x]\leqq w(x)\leqq c_6q(\xi)w_0[(1-\xi^{-1})x].$$

By applying Theorem I. 4. 2 of [1] taking into consideration the elementary relation

$$\lambda_n[w(\varrho t);\xi]=\varrho^{-1}\lambda_n[w(t);\varrho\xi]$$

we obtain from Lemma 4 the following estimates on $\lambda_n(w;\xi)$:

Theorem 1. We have for $w\in\Phi_r(\xi)$ and $0\leqq\xi\leqq 4n+\varrho n^{1/3}$ (ϱ fixed)

$$c_5[1+(\xi/n)^+]^{-1}w(\xi)\varphi_n(\xi)\leqq\lambda_n(w;\xi)\leqq c_6[1+\varrho+(n\xi)^{-r}]w(\xi)\varphi_n(\xi), \tag{5}$$

where c_5 and c_6 are depending on r, A and B.

Combining Theorem 1 with Lemma 2 resp. with Lemma 3, we obtain

Theorem 2. We have for $w\in\Phi_r(\xi)$ and $0\leqq\xi\leqq 4n+\varrho n^{1/3}$

$$\Delta_n(w;\xi)=\xi-\xi^*\leqq c_7[1+\varrho+(n\xi)^{-r}]\varphi_n(\xi) \tag{6a}$$

and we have for $w\in\Phi_r(\xi)$, $x_{n-1,n-1}(w)\leqq\xi\leqq x_{1,n-1}(w)$

$$\Delta_n(w;\xi)\geqq c_8\varphi_n(\xi). \tag{6b}$$

We mention the following important consequence of Theorem 2.

Theorem 3. Let $w\in\Phi_r$ and let $x^{-r}q(x)=e^xx^{-r}w(x)$ be non-increasing, then we have

$$c_9\varphi_{n+1}[x_{kn}(w)]\leqq x_{kn}(w)-x_{k+1,n}(w)\leqq c_{10}\varphi_{n+1}[x_{kn}(w)],\;k=1,2,\ldots,n. \tag{7}$$

Proof. From the fact that $w(x)/w_r(x)$ is non-increasing it follows by a classical theorem of A. A. Markov* that $x_{1n}(w)\leqq x_{1n}(w_r)\leqq 4n+O(n^{1/3})$, so that we obtain (7) by inserting $\xi=x_{k,n-1}(w)$ in (6a) and (6b).

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* For a simple proof see G. Szegö [3, § 6. 12].

4. Applications to approximation problems. a) As a first application, we investigate the weighted L-approximation of functions by polynomials. Let $f \in L_w$ and let us define

$$\varepsilon_n^{(1)}(w;f) = \min_{\pi_n} \int_0^\infty |f(t) - \pi_n(t)| \, w(t) dt.$$

We assume that $w \in \Phi_r$ and that $f(t)$ is of bounded variation in every finite interval. We put

$$f_n(t) = \begin{cases} f(t) & \text{for } 0 \leqq t \leqq n, \\ f(n+0) & \text{for } t > n \end{cases}$$

then clearly

(8) $$\varepsilon_n^{(1)}(w;f) \leq \varepsilon_n^{(1)}(w;f_n) + \varepsilon_n^{(1)}(w;f-f_n)$$

and (approximating by $\pi_n \equiv 0$)

(9) $$\varepsilon_n^{(1)}(w;f-f_n) \leqq \int_n^\infty |f(t) - f(n+0)| \, w(t) \, dt.$$

To approximate f_n, we make use of a duality principle of S. M. Nikolski [2]:

(10) $$\varepsilon_n^{(1)}(w;f_n) = \max_{g \in \gamma_n} \int_0^\infty f_n(t) g(t) w(t) \, dt,$$

where γ_n is the class of measurable functions satisfying the conditions

(11) $$\operatorname*{vrai\,max}_{0 \leqq x < \infty} |g(x)| \leqq 1$$

and

(12) $$\int_0^\infty g(t) \pi_n(t) w(t) \, dt = 0$$

for every polynomial π_n of degree at most n.

Introducing $G(x) = \int_0^x g(t) w(t) \, dt$, we obtain by partial integration

(13) $$\int_0^\infty f_n(t) g(t) w(t) dt = [f_n G]_0^\infty - \int_0^\infty G(t) df_n(t) = -\int_0^{n+0} G(t) \, df(t).$$

All what remained is to estimate

$$(14)\qquad G(x)=\int_0^x g(t)w(t)dt=\int_0^{+\infty} \Gamma(t, x)g(t)w(t)dt,$$

where

$$\Gamma(t, x)=\begin{cases}1 & \text{for } t\leqq x,\\ 0 & \text{for } t>x.\end{cases}$$

We refer now to the following classical result of A. A. Markov and Th. Stieltjes (see G. Freud [1], § 1.5): There exist two polynomials in t of degree $2\nu-2$ at most $p_\nu(t,x)$ and $P_\nu(t,x)$ so that

$$(15)\qquad p_\nu(t, x)\leqq \Gamma(t; x)\leqq P_\nu(t, x), \quad -\infty<t<+\infty,$$

$$(16)\qquad \int_0^\infty [P_\nu(t, x)-p_\nu(t, x)]w(t)dt\leqq \lambda_\nu(w; x).$$

We obtain using Theorem 1 that

$$(17)\qquad \int_0^\infty |\Gamma(t, x)-p_{[n/2]}(t, x)|\, w(t)\, dt$$

$$=\int_0^\infty [\Gamma(t, x)-p_{[n/2]}(t,x)]w(t)dt\leqq \lambda_{[n/2]}(w; x)$$

$$\leqq c_{11}[1+(nx)^{-r}]\sqrt{\frac{x+n^{-1}}{n}}\,w(x), \quad 0\leqq x\leqq n.$$

From (14), $g\in\gamma_n$ and (11) we get

$$(18)\qquad G(x)=\left|\int_0^\infty [\Gamma(t, x)-p_{[n/2]}(t, x)]\,g(t)w(t)\,dt\right|$$

$$\leqq \int_0^\infty |\Gamma(t, x)-p_{[n/2]}(t, x)|\, w(t)\,dt\leqq c_{11}[1+(nx)^{-r}]\sqrt{\frac{x+n^{-1}}{n}}\,w(x)$$

so that by (13), (18) and (10)

$$(19)\qquad \varepsilon_n^{(1)}(f_n)\leqq c_{11}\int_0^{n+0} [1+(nx)^{-r}]\sqrt{\frac{x+n^{-1}}{n}}\,w(x)\,|df(x)|.$$

From (8), (9) and (19) we have

$$(20)\qquad \varepsilon_n^{(1)}(w; f)\leqq c_{11}\int_0^{n+0} [1+(nx)^{-r}]\sqrt{\frac{x+n^{-1}}{n}}\,w(x)\,|df(x)|$$

$$+\int_0^\infty |f(x)-f(n+0)|\, w(x)\,dx;$$

this is the estimate we wanted to prove.

We emphasize the following special case of (20):

Theorem 4. Let $f \in L_{w_\alpha}$ and let $f(x)$ be of bounded variation in every finite interval, then we have for $\alpha \geqq 0$

$$\varepsilon_n^{(1)}(w_\alpha; f) = \min_{\pi_n} \int_0^\infty |f(t) - \pi_n(t)| t^\alpha e^{-t} dt \leqq c_{12} n^{-\alpha-1} \int^{n^{-1}} |df(x)|$$

$$+ c_{12} n^{-\frac{1}{2}} \int_{n^{-1}}^{n+0} x^{\alpha+1/2} e^{-x} |df(x)| + \int_n^\infty |f(x) - f(n+0)| x^\alpha e^{-x} dx.$$

b) In our second application we consider weighted one-sided approximation by polynomials. At this place we mention only the simplest case of application:

Theorem 5. Let $f(t)$ be of bounded variation in $[0, x_0]$ and let $f(t) = f(x_0)$ for $t > x_0$, let further $\alpha \geqq 0$. Then for every $n > x_0$ there exist two polynomials $p_n(x)$ and $P_n(x)$ of degree at most n so that $p_n(x) \leqq f(x) \leqq P_n(x)$, $0 \leqq x < \infty$, and

$$\int_0^\infty [P_n(x) - p_n(x)] x^\alpha e^{-x} dx$$

$$\leqq c_{13} n^{-\alpha-1} \int_0^{n^{-1}} |df(x)| + c_{13} n^{-\frac{1}{2}} \int_{n^{-1}}^{x_0} x^{\alpha+\frac{1}{2}} e^{-x} |df(x)|.$$

This theorem is proved by the aid of Lemma III. 4. 1 of our book [1].

To further extensions and further applications of our results we hope to return soon elsewhere.

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