*Труды Международной Конференции по конструктивной теории функции Варна, 19—25 мая 1970*

*Proceedings of the International Conference on Constructive Function Theory, Varna, May 19—25, 1970*

# UNIFORM APPROXIMATION TO CONTINUOUS FUNCTIONS WITH VALUES IN LOCALLY CONVEX SPACE

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***Summary.*** Let $X$ and $Y$ be real locally convex spaces and $K$ a compact subset of $X$. Let $C(K, Y)$ denote the family of continuous functions from $K$ to $Y$ and let $P(K, Y)$ denote) the family of continuous polynomial operators from $K$ to $Y$. $P(K, Y)$ is contained in $C(K, Y$ and $C(K, Y)$ carry the uniform topology given by $\|f-g\|=\sup_{x\in K}\|f(x)-g(x)\|$. The purpose of this communication is to give a proof of the following T h e o r e m: $P(K, Y)$ is dense in $C(K, Y)$. This is an improvement of the main theorem of [1]. The proof depends only on standard theorems.

Let $K$ be a compact topological space, and let $E$ be a locally convex topological vector space on $\mathbb{R}$. Let $C(K\ E)$ denote the vector space on $\mathbb{R}$ of all continuous functions $F:K\to E$ from $K$ to $E$ with the topology of uniform convergence on $K$. $C(K, E)$ is a locally convex space. If $A$ is a subalgebra of the algebra $C(K, \mathbb{R})$ let $FA(K, E)$ denote the following vector subspace of $C(K, E)$,

$$FA(K, E)=\{a_1(x)e_1+\cdots+a_m(x)e_m \mid a_i\in A,\ e_i\in E,\ 1\leqq i\leqq m,\ m\geqq 1\}.$$

(Functions with coefficients in $A$).

The main result of this note is the following

*Theorem.* Let the algebra $A$ separate the points of $K$ and is not vanishing in any point of $K$. Then $FA(K, E)$ is dense in $C(K, E)$.

The proof is elementary.

A consequence of this theorem is an improvement of the main theorem of [1].

A family $P$ of continuous seminorms on a locally convex space $E$ will be called a basis of continuous seminorms on $E$ if to any continuous seminorm $p$ on $E$ there is a seminorm $p'$ belonging to $P$ and a constant $C>0$ such that, for all $e\in E$,

$$p(e)\leqq Cp'(e).$$

If $P$ is a basis of continuous seminorms on $E$ then the family $Q$ of the following seminorms

$$q(F)=\sup_{x\in K} p(F(x)),\quad F\in C(K, E),\quad p\in P$$

is a basis of continuous seminorms on $C(K, E)$. In this case $\Phi\in C(K, E)$ is close to $F$ in the topology of uniform convergence on $K$, i. e. $F(x)-\Phi(x)\in V$, for all $x\in K$ where $V$ is a given neighbourhood of $0\in E$ if there is a seminorm $q\in Q$ and a constant $\eta>0$ such that

$$q(F-\Phi)<\eta.$$

The last condition means that $p(F(x)-\Phi(x))<\eta$ for all $x\in K$ ($p$ and $q$ are as above).

P r o o f. The proof consists of two parts.

1°. In the first part we prove that $FC(K, E)$ is dense in $C(K, E)$. Let $F\in C(K, E)$, $p\in P$, $\eta>0$. We will show that there exist realvalued continuous functions $\varphi_1, \dots, \varphi_m$ on $K$ and vectors $e_1, \dots, e_m\in E$ such that if $\Phi(x)=\varphi_1(x)e_1+\dots+\varphi_m(x)e_m$ then $p(F(x)-\Phi(x))<\eta$ for all $x\in K$. Since $F$ is continuous on $K$ and $K$ is compact, then $F$ is uniformly continuous on $K$. Hence there is a finite open cover $U_1, \dots, U_m$ of $K$ such that if $x', x''\in U_i$ then $p(F(x')-F(x''))<\eta$, $1\leq i\leq m$. Since $K$ is compact there is a subordinate partition of unity on $K$ $\varphi_1, \dots, \varphi_m$, i. e. for $i=1, \dots, m$ $\varphi_i\in C(K, [0, 1])\subset C(K, \mathbb{R})$, if $\varphi_i(x)\neq 0$ then $x\in U_i$ and $\sum_{i=1}^{m}\varphi_i(x)=1$. Let $z_i\in U_i$; without loss of generality we may assume that $z_i\notin U_j$ for all $j\neq i$. Now we denote with $\Phi$ the function $\Phi(x)=\varphi_1(x)F(z_1)+\dots+\varphi_m(x)F(z_m)$. $\Phi\in FC(K, E)$, $\Phi(z_i)=F(z_i)$ for all $i=1, \dots, m$ and we have

$$F(x)-\Phi(x)=F(x)-\sum_{i=1}^{m}\varphi_i(x)F(z_i)=\sum_{i=1}^{m}\varphi_i(x)[F(x)-F(z_i)].$$

If $\varphi_i(x)\neq 0$, then $p(F(x)-F(z_i))<\eta$ for all $x\in K$ (since $x, z_i\in U_i$). Hence

$$p(F(x)-\Phi(x))=p\left(\sum_{i=1}^{m}\varphi_i(x)[F(x)-F(z_i)]\right)\leq\sum_{i=1}^{m}\varphi_i(x)p(F(x)-F(z_i))<\eta.$$

2°. In the second part we prove that $FA(K, E)$ is dense in $FC(K, E)$ with the topology, induced from $C(K, E)$. Let $\Phi\in FC(K, E)$ and let $p\in P$, $\eta>0$. If $\Phi(x)=\varphi_1(x)e_1+\dots+\varphi_m(x)e_m$ we will choose $a_1, \dots, a_m\in A$ such that $p(\Phi(x)-\Psi(x))<\eta$, $x\in K$, where

$$\Psi(x)=a_1(x)e_1+\dots+a_m(x)e_m,\quad \Psi\in FA(K, E).$$

For this purpose we estimate the difference $\Phi-\Psi$:

$$p(\Phi(x)-\Psi(x))=p\left(\sum_{i=1}^{m}\varphi_i(x)e_i-\sum_{i=1}^{m}a_i(x)e_i\right)=p\left(\sum_{i=1}^{m}[\varphi_i(x)-a_i(x)]e_i\right)$$

$$\leq\sum_{i=1}^{m}|\varphi_i(x)-a_i(x)|\,p(e_i)\leq\sum_{i=1}^{m}\sup_{x\in K}|\varphi_i(x)-a_i(x)|\,p(e_i).$$

By the theorem of Stone—Weierstrass we can choose $a_i\in A$ such that

$$\sup_{x\in K}|\varphi_i(x)-a_i(x)|<\eta/m(\max_{1\leq i\leq m}p(e_i)+1).$$

Then $p(\Phi(x)-\Psi(x))<\eta$, $x\in K$. Now the proof of the theorem is complete.

Let $K$ be a compact subset of the locally convex space $X$. Let $P$ denote the following algebra of real-valued continuous functions on $K$:

$$P=\left\{\sum_{i=1}^{m} k_i+k \quad k_i=l_1^i l_2^i \dots l_{n(i)}^i,\ l_j^i \in X',\ k \in \mathrm{R},\ m \geqq 1\right\}.$$

It follows from the Hahn-Banach theorem that $P$ separates the points of $K$ and that $P$ is non-vanishing in any point of $K$. Hence the following is true:

Corollary. The vector space $FP(K, E)$ is dense in $C(K, E)$.

The functions of $FP(K, E)$ are polynomial operators in a sense similar to those of the note [1], but they are polynomial operators of very special kind.

## REFERENCES

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*Received on June 25, 1970*