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ON THE LENGHT OF THE INTERPOLATION INTERVAL FOR A CLASS OF LINEAR DIFFERENTIAL OPERATORS

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Summary. Let $L_{n,p}$ be a differential operator of the form

$$L_{n,p}=(d^n/dx^n)+\sum_{k=1}^{n}a_k(x)(d^{n-k}/dx^{n-k}) \quad \text{where} \quad a_k\in L^p[a,a+h],$$

$k=1, 2,\dots,n$; $h>0$; $1\leqq p\leqq+\infty$. If we consider the differential equation (1) $L_{n,p}[y]=0$, we denote by Y_n the set of the solutions (in the sense of Carathéodory) of the equation (1), that is the set of functions $y:[a,a+h]\to\mathsf{R}$ with the properties: a) y has a derivative of order $(n-1)$ which is absolutely continuous on $[a,a+h]$; b) y verifies the equation (1) almost everywhere on $[a,a+h]$. A differential operator has the interpolation property $H_n[a,a+h]$ if for each $P=(x_1,x_1,\dots,x_1,x_2)\in(a,a+h)^n$, $x_1\neq x_2$, and for each system of values $V=(y_1,y_2,\dots,y_{n-1},y_n)\in\mathsf{R}^n$ there is an unique solution $\overline{y}=\overline{y}(P,V)\in Y_n$ such that $\overline{y}(x_1)=y_1, \overline{y}'(x_1)=y_2,\dots, \overline{y}^{(n-2)}(x_1)=y_{n-1}, \overline{y}^{(w)}(x_2)=y_n$, where w is one of the numbers $0, 1,\dots, n-2$.

The purpose of this paper is to give some upper bounds for h, that is for the length of the interpolation interval, in the cases $n=5$, $p=+\infty$ and $n=5$, $p=2$. In the cases $n\leqq 4$, different p, we refer to O. Aramă [1], O. Aramă and D. Ripianu [2], P. Bailey and P. Waltman [3], Ph. Hartman and A. Wintner [6], A. Lupas [7] and Z. Nehari [9].

1. Some integral inequalities. *Lemma 1.1* (Wirtinger Inequality). Let $y:[a,b]\longrightarrow\mathsf{R}$ be an absolutely continuous function on $[a,b]$. Let ξ_1 and ξ_2 be real numbers such that $a\leqq\xi_1\leqq\xi_2\leqq b$ and $y(\xi_1)=y(\xi_2)$. Then

$$\int_a^b |y(x)-y(\xi_1)|^2dx\leqq\frac{4}{\pi^2}C\int_a^b |y'(x)|^2dx$$

where

$$C=\max\left[(\xi_1-a)^2, (b-\xi_2)^2, \left(\frac{\xi_2-\xi_1}{2}\right)^2\right].$$

For an extensive bibliography on this inequality see J. B. Diaz and F. T. Metcalf [5] and D. S. Mitrinović and P. M. Vasić [8].

Lemma 1.2 (Opial's Inequality). If $y:[a,b]\longrightarrow R$ verifies the conditions of lemma 1.1, with similar ξ_1 and ξ_2, then

$$\int_a^b |y'(x)|\,|y(x)-y(\xi_1)|\,dx \leq \frac{1}{2}C_0\int_a^b |y'|^2 dx,$$

where the best constant is defined by

$$C_0=\max\left[(\xi_1-a),(b-\xi_2),\left(\frac{\xi_2-\xi_1}{2}\right)\right].$$

This inequality is a generalization of the inequality obtained by Z. Opial [10]. The proof is the same as the proof of Wirtinger's inequality; see [5].

Lemma 1.3. If $y:[a,b]\longrightarrow R$ is absolutely continuous on $[a,b]$ and if $y(a)=0$, then the following inequalities are valid:

$$\int_a^b |y(x)y'(x)|^2 dx \leq \frac{4}{\pi^2}(b-a)\left[\int_a^b |y'(x)|^2 dx\right]^2, \tag{1}$$

$$\int_a^b |y(x)|^4 dx \leq \frac{3\,.\,2^8\pi^2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^8}(b-a)^3\left[\int_a^b |y'(x)|^2 dx\right]^2. \tag{2}$$

If moreover, $y:[a,b]\longrightarrow R$ has a second derivative which is absolutely continuous on $[a,b,]$ and if $y(a)=y'(a)=y''(a)=0$, $y''(b)=0$, then

$$\int_a^b |y(x)y''(x)|^2 dx \leq \frac{4}{\pi^4}(b-a)^4\int_a^b |y'''(x)|^2 dx. \tag{3}$$

For the proof see [7], where these inequalities are obtained by using the results of D. W. Boyd [4].

Lemma 1.4. If $y:[a,b]\longrightarrow R$ has a third derivative which is absolutely continuous on $[a,b]$, and if

$$y(a)=y'(a)=y''(a)=y'''(a)=0,\quad y'''(b)=0, \tag{4}$$

then the following inequalities hold

$$\int_a^b |y(x)y'''(x)|\,dx \leq \frac{8(b-a)^5}{\pi^5}\int_a^b |y^{(IV)}(x)|^2 dx, \tag{5}$$

$$\int_a^b |y(x)y'''(x)|^2 dx \leq \frac{(b-a)^9}{40\pi^4}\left[\int_a^b |y^{(IV)}(x)|^2 dx\right]^2. \tag{6}$$

Proof. We have

$$\int_a^b |y(x)y'''(x)|\,dx \leq \left[\int_a^b |y(x)|^2 dx\right]^{\frac{1}{2}} \left[\int_a^b |y'''(x)^2 dx\right]^{\frac{1}{2}}$$

$$\leq \left[\frac{4(b-a)^2}{\pi^2}\int_a^b |y'(x)|^2 dx\right]^{\frac{1}{2}} \left[\frac{(b-a)^2}{\pi^2}\int_a^b |y^{(IV)}(x)|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \frac{2(b-a)^2}{\pi^2}\left[\frac{4(b-a)^2}{\pi^2}\int_a^b |y''(x)|^2\,dx\right]^{\frac{1}{2}} \left[\int_a^b |y^{(IV)}(x)|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \frac{4(b-a)^3}{\pi^3}\left[\frac{4(b-a)^2}{\pi^2}\int_a^b |y'''(x)|^2\,dx\right]^{\frac{1}{2}} \left[\int_a^b |y^{(IV)}(x)|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \frac{8(b-a)^5}{\pi^5}\int_a^b |y^{(IV)}(x)|^2 dx.$$

In conclusion, the inequality (5) is proved. From

$$\int_a^b |y(x)y'''(x)^2 dx = \int_a^b |y'''(x)|^2 \left[\int_a^x ds \int_a^s dt \int_a^t y'''(r)dr\right]^2 dx$$

$$= \int_a^b |y'''(x)|^2 \left[\int_a^x \frac{(x-s)^2}{2} y'''(s) ds\right]^2 dx$$

$$\leq \int_a^b |y'''(x)|^2 \left[\int_a^x \frac{(x-s)^4}{4} ds\right] \left[\int_a^x |y'''(s)|^2 ds\right] dx$$

$$\leq \int_a^b |y'''(x)|^2 \frac{(x-a)^5}{20} \left[\int_a^x |y'''(s)|^2 ds\right] dx,$$

we may write

$$\int_a^b |y(x)y'''(x)|^2 dx \leq \frac{(b-a)^5}{20}\int_a^b |y'''(x)|^2 \left[\int_a^x |y'''(s)|^2 ds\right] dx$$

$$= \frac{(b-a)^5}{40}\int_a^b \int_a^b |y'''(x)|^2 |y'''(s)|^2 ds dx = \frac{(b-a)^5}{40}\left[\int_a^b |y'''(x)|^2 dx\right]^2,$$

and from Wirtinger's inequality we conclude with

$$\int_a^b |y(x)y'''(x)|^2 dx \leq \frac{(b-a)^9}{40\pi^4}\left[\int_a^b |y^{(IV)}(x)|^2 dx\right]^2,$$

that is the inequality (6).

2. The cases $n=5$, $p=+\infty$ and $n=5$, $p=2$.

Theorem 2.1. If

$$\frac{8h^5}{\pi^5}\|a_5\|_\infty+\frac{4h^4}{\pi^4}\|a_4\|_\infty+\frac{h^3}{2\pi^2}\|a_3\|_\infty+\frac{h^2}{\pi^2}\|a_2\|_\infty+\frac{h}{4}\|a_1\|_\infty\leq 1,$$

then the operator $L_{5,\infty}$ has the property $H_5[a, a+h)$.

Proof. If $L_{5,\infty}$ has not this property $H_5[a, a+h)$ then for all points $x_1<x_2$ there is at least one nontrivial solution y_0 of the equation

$$y^{(V)}+a_1(x)y^{(IV)}+a_2(x)y'''+a_3(x)y''+a_4(x)y'+a_5(x)y=0,$$

$$a_k\in L^\infty[a, a+h],\ k=1,2,\ldots,5,$$

with the property

$$y_0(x_1)=y_0'(x_1)=y_0''(x_1)=y_0'''(x_1)=0,$$

and $y_0^{(w)}(x_2)=0$, where w is one of the numbers 0, 1, 2, 3. If $w=0$ there are points t_1, t_2, t_3, $x_1<t_3<t_2<t_1<x_2$, such that $y_0'(t_1)=y_0''(t_2)=y_0'''(t_3)=0$. In this situation we put $\alpha=x_1$, $\beta=t_3$. If $w=1$ there exist points s_2, s_3 $x_1<s_3<s_2<x_2$, such that $y_0''(s_2)=y_0'''(s_3)=0$, and then we set $\alpha=x_1, \beta=s_3$ For $w=2$ there exists a point r_3 such that $x_1<r_3<x_2$ and $y_0'''(r_3)=0$. In this case we select $\alpha=x_1$ and $\beta=r_3$. When $w=3$ we assume that $\alpha=x_1$ and $\beta=x_2$. Therefore we observe that in all the cases there are points α, β such that

$$y_0(\alpha)=y_0'(\alpha)=y_0''(\alpha)=y_0'''(\alpha)=0,\quad y_0'''(\beta)=0,\quad \beta-\alpha<h.$$

In order to prove the theorem we write

$$\int_\alpha^\beta |y_0^{(IV)}|^2 dx=\int_\alpha^\beta y_0'''[a_1(x)y_0^{(IV)}+a_2(x)y_0'''+a_3(x)y_0''+a_4(x)y_0'+a_5(x)y_0]dx.$$

If $\Omega=\int_\alpha^\beta |y_0^{(IV)}|^2 dx$, then we have

$$\Omega\leq\|a_1\|_\infty\int_\alpha^\beta|y_0'''y_0^{(IV)}|\,dx+\|a_2\|_\infty\int_\alpha^\beta|y_0'''|^2dx \tag{7}$$

$$+\|a_3\|_\infty\int_\alpha^\beta|y_0''y_0'''|dx+\|a_4\|_\infty\int_\alpha^\beta|y_0'y_0'''|\,dx+\|a_5\|_\infty\int_\alpha^\beta|y_0y_0'''|dx.$$

From Opial's inequality

$$(8)\qquad \int_\alpha^\beta |y_0''' y_0^{(\mathrm{IV})}|\, dx \leqq \frac{(\beta-\alpha)}{4}\,\Omega < \frac{h}{4}\,\Omega$$

and from Wirtinger's inequality

$$(9)\qquad \int_\alpha^\beta |y_0'''|^2 dx \leqq \frac{(\beta-\alpha)^2}{\pi^2}\,\Omega < \frac{h^2}{\pi^2}\,\Omega.$$

Further

$$\int_\alpha^\beta |y_0'' y_0'''|\, dx \leqq \frac{(\beta-\alpha)}{2}\int_\alpha^\beta |y_0'''|^2 dx \leqq \frac{(\beta-\alpha)^3}{2\pi^2}\int_\alpha^\beta |y_0^{(\mathrm{IV})}|^2 dx$$

or

$$(10)\qquad \int_\alpha^\beta |y_0'' y_0'''|\, dx < \frac{h^3}{2\pi^2}\,\Omega.$$

Likewise from Lemma 1.3, inequality (3), we have

$$(11)\qquad \int_\alpha^\beta |y_0' y_0'''|\, dx \leqq \frac{4}{\pi^4}(\beta-\alpha)^4\Omega < \frac{4h^4}{\pi^4}\,\Omega.$$

Thus if we take into account the inequalities (8)—(11), together with inequality (5) from Lemma 1.4, we conclude from (7) that

$$1 < \frac{h}{4}\,\|a_1\|_\infty + \frac{h^2}{\pi^2}\,\|a_2\|_\infty + \frac{h^3}{2\pi^2}\,\|a_3\|_\infty + \frac{4h^4}{\pi^4}\,\|a_4\|_\infty + \frac{8h^5}{\pi^5}\,\|a_5\|_\infty,$$

which is a contradiction; thus the theorem is proved.

Theorem 2.2. If h satisfies the following inequality

$$\frac{\sqrt{10}}{20\pi^2}\, h^5 M_2(a_5, h) + \frac{\sqrt{6}}{6\pi^2}\, h^4 M_2(a_4, h) + \frac{2}{\pi^3}\, h^3 M_2(a_3, h)$$
$$+ \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4}\, h^2 M_2(a_2, h) + \frac{2}{\pi}\, h M_2(a_1, h) \leqq 1,$$

$M_2(a_k, h)$ being $h^{-1/2}\,\|a_k\|_2$, then $L_{5,2}$ has the interpolatory property $H_5[a, a+h]$.*

P r o o f. We assume that $L_{5,2}$ has not the property $H_5[a, a+h)$; then there is a nontrivial solution from Y_5 such that

* We note that $16\pi\sqrt{3}/[\Gamma(1/4)]^4 = 0{,}2063\ldots$

$y_0(\alpha)=y_0'(\alpha)=y_0''(\alpha)=y_0'''(\alpha)=0,\ y_0'''(\beta)=0,$ where $\alpha,\ \beta$ are defined as in the proof of Theorem 2.1. Then, with the same Ω, we receive

$$\Omega=\int_\alpha^\beta a_1(x)y_0'''y_0^{(IV)}dx+\int_\alpha^\beta a_2(x)\,|\,y_0'''\,|^2dx+\int_\alpha^\beta a_3(x)y_0''y_0'''dx$$

$$+\int_\alpha^\beta a_4(x)y_0'y_0'''dx+\int_\alpha^\beta a_5(x)y_0y_0'''dx.$$

Thus, we have the inequality

$$(12)\qquad \Omega\leq\|\,a_1\,\|_2\left[\int_\alpha^\beta|\,y_0'''y_0^{(IV)}\,|^2dx\right]^{\frac{1}{2}}+\|\,a_2\,\|_2\left[\int_\alpha^\beta|\,y_0'''\,|^4dx\right]^{\frac{1}{2}}$$

$$+\|\,a_3\,\|_2\left[\int_\alpha^\beta|\,y_0''y_0'''\,|^2dx\right]^{\frac{1}{2}}+\|\,a_4\,\|_2\left[\int_\alpha^\beta|\,y_0'y_0'''\,|^2dx\right]^{\frac{1}{2}}$$

$$+\|\,a_5\,\|_2\left[\int_\alpha^\beta|\,y_0y_0'''\,\|^2dx\right]^{\frac{1}{2}}.$$

From Lemma 1.3, inequality (2),

$$\int_\alpha^\beta|\,y_0'''y_0^{(IV)}\,|^2dx\leq\frac{4}{\pi^2}(\beta-\alpha)\left[\int_\alpha^\beta|\,y_0^{(IV)}\,|^2dx\right]^2$$

therefore

$$(13)\qquad \int_\alpha^\beta|\,y_0'''y_0^{(IV)}\,|^2dx<\frac{4h}{\pi^2}\,\Omega^2.$$

Further, by using the second inequality from Lemma 1.3, we have

$$(14)\qquad \int_\alpha^\beta|\,y_0'''\,|^4dx<\frac{3\cdot2^8\pi^2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^8}\,h^3\,\Omega^2.$$

Also from (2)

$$\int_\alpha^\beta|\,y_0''y_0'''\,|^2dx\leq\frac{4}{\pi^2}(\beta-\alpha)\left[\int_\alpha^\beta|\,y_0'''\,|^2dx\right]^2$$

and then Wirtinger's inequality yields

$$(15)\qquad \int_{\alpha}^{\beta} |y_0''y_0'''|^2 dx < \frac{4h^5}{\pi^6}\Omega^2.$$

According to an inequality of O. Aramă [1] (see inequality (23)) we may write

$$(16)\qquad \int_{\alpha}^{\beta} |y_0'y_0'''|^2 dx < \frac{h^7}{6\pi^4}\Omega^2.$$

Finally (6) furnishes the following one

$$(17)\qquad \int_{\alpha}^{\beta} |y_0 y_0'''|^2 dx < \frac{h^9}{40\pi^4}\Omega^2.$$

Now, if we insert the inequalities (13)—(17) in (12), after deleting Ω we obtain

$$1 < \frac{2}{\pi} hM_2(a_1, h) + \frac{16\pi\sqrt{3}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^4} h^2 M_2(a_2, h) + \frac{2}{\pi^3} h^3 M_2(a_3, h)$$
$$+ \frac{\sqrt{6}}{6\pi^2} h^4 M_2(a_4, h) + \frac{\sqrt{10}}{20\pi^2} h^5 M_2(a_5, h),$$

an inequality which is a contradiction, and the theorem is proved.

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