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A CONVERSE THEOREM IN RATIONAL APPROXIMATION

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Summary. Let $f(x)$ be a continuous positive function of the real variable x in $[0, \infty)$ and Π_n the class of polynomials of degree n. We define the minimal distance

$$\varrho_n(f) = \inf_{P_n \in \Pi_n} \sup_{0 \leq x < \infty} \left| \frac{1}{P_n(x)} - \frac{1}{f(x)} \right|.$$

There are some analytic conditions on f for the geometrical convergence of $\varrho_n(f)$ to zero that is $\overline{\lim\limits_{n\to\infty}} (\varrho_n(f))^{1/n} = q^{-1}$ with $q > 1$. Now we are able to prove a converse theorem: In case of geometric convergence there exists an entire function $f^*(z)$ of finite order such that $f^*(x) = f(x)$ for all $x \in [0, \infty)$. Furthermore there is a connection between q and the order of $f^*(z)$.

Let Π_n the space of polynomials of degree at most n, $C[-1, +1]$ the space of continuous functions on the real interval $[-1, +1]$ with the uniform norm and

$$\varrho_n(f) = \inf_{P_n \in \Pi_n} \| f - P_n \|$$

the minimal distance in approximating the function $f \in C[-1, +1]$ by polynomials $P_n \in \Pi_n$. The classical theorem of S. Bernstein states that there is geometric convergence of the minimal distance, that is

$$\lim\sup (\varrho_n(f))^{1/n} = \frac{1}{q} < 1$$

if and only if the function f is the restriction of an analytic function f^* of the complex variable z which is holomophic in a region G containing the interval $[-1, +1]$. More specific, if f^* is holomorphic in an ellipse $E(q)$ with $q < 1$ and foci ± 1, where q is the sum of the half axis, then

$$\lim\sup (\varrho_n(f))^{1/n} \leq \frac{1}{q}.$$

On the other hand, if $f \in C[-1, +1]$ is given and

$$\lim\sup (\varrho_n(f))^{1/n} \leq \frac{1}{q} < 1$$

then there exists an analytic function $f^*(z)$, holomorphic in $F(q)$ such that

$$f^*(x)=f(x), \quad \forall x \in [-1, +1].$$

It would be of interest to generalize this connection between analyticity and geometric convergence of the minimal distance to rational approximations. The general case of rational approximation seems to be very difficult. But in a special case it is possible to get some analogues.

Let $\Pi_{m,n}$ be the set of all rational functions

$$r_{m,n}(x)=\frac{P_m(x)}{Q_n(x)}$$

with degree $P_m \leq m$ and degree $Q_n \leq n$.

We consider continuous functions $f \in C[0, \infty]$. For $m \leq n$, $f \in C[0, \infty)$ and $f(x)>0$ for $x \in [0, \infty)$, we denote by

$$\lambda_{m,n}(f)=\inf_{r_{m,n} \in \Pi_{m,n}} \sup_{x \geq 0} \left| \frac{1}{f(x)} - r_{m,n}(x) \right|$$

the minimal deviation of f. We are looking for sufficient and for necessary conditions for f that the minimal deviation converges geometrically to zero. The following investigations have been done in cooperation with W. Y. Cody [1], R. S. Varga [2] and A. Taylor [3].

In the special case $f(x)=e^x$ it has been proved that

$$\frac{1}{6} \leqq \lim_{n\to\infty} (\lambda_{0,n})^{1/n} \leq \frac{1}{2} e^{-\alpha}, \tag{1}$$

where α is the real solution of the equation

$$2ae^{2a+1}=1.$$

Because of

$$\lambda_{0,n} \geq \lambda_{1,n} \geq \ldots \geq \lambda_{n,n}$$

the upper bound is also correct for the

$$\overline{\lim_{n\to\infty}} (\lambda_{m,n})^{1/n}.$$

In the proof of (1) one uses for the upper bound estimates for the transformed truncated Taylor series, and for getting the lower bound a certain linear functional. From this method one comes to the assumptions of the following

Theorem 1. Let $f(z)$ be an entire function of finite order $\varrho>0$. Let all coefficients a_γ of the Taylor series

$$f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$$

be real and positive. Furthermore the limit

$$\lim_{n\to\infty} \frac{\log M_f(r)}{r^\varrho}=B$$

may exist (here $M_f(r)$ is the maximum modulus of f):

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Then

$$\frac{1}{2^{2+1/\varrho}} \leqq \overline{\lim_{n\to\infty}} (\lambda_{0,n}(f))^{1/n} \leqq \frac{1}{2^{1/\varrho}}.$$

In looking for necessary conditions for geometric convergence we need some notations:

1) We denote by $E(N, s)$, where $N>0$ and $s>1$ are fixed numbers, the ellipse with foci 0 and N and the ratio $(s^2-1)(s^2+1)$ of half axis.

2) If $g(z)$ is an entire function we introduce the modified maximum modulus $\hat{M}g(N, s)$ by the formula

$$\hat{M}_g(N, s) = \max_{z \in E(N,s)} |g(z)|.$$

Now we are able to formulate

Theorem 2. Let $f \in C[0, \infty)$ and $f(x)>0$ for $x \in [0, \infty)$. Furthermore let

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n}(f))^{1/n} = \frac{1}{q} < 1.$$

Then there exists an entire function $f^*(z)$ of finite order ϱ such that $f(x)$ is the restriction of $f^*(z)$ to the real half line $x \geqq 0$.

The order satisfies the inequality

$$\varrho \leqq \frac{\log 2}{\log\left(1+\frac{1}{2}\left(q+\frac{1}{2}\right)\right) - \log 2}. \tag{2}$$

For any s with $1<s<q$ there exists a constant B not depending on f^* and N such that for all $N>0$ the growth condition

$$\hat{M}_{f^*}(N, s) \leqq (B \sup_{0 \leqq x \leqq N} |f(x)|)^2 \tag{3}$$

is satisfied

The condition (3) gives for instance for the entire function

$$f(z) = (1+z)(2+\cos z)$$

of order 1 that there is no geometric convergence of $\lambda_{0,n}(f)$, because

$$\sup_{0 \leqq x \leqq N} |f(x)| \leqq 3(N+1)$$

but there are positive constants C, γ such that

$$\hat{M}_f(N, s) > Ce^{\gamma N}.$$

Using the growth condition (3) we can prove another sufficient condition for geometric convergence. This condition does not give very good estimates for the geometric factor.

Theorem 3. Let $f(z)$ an entire function. All coefficients a_γ of the Taylor series

$$f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$$

may be real and positive. There may exist two numbers $B>0$ and $s>1$ such that for every $N>0$ the inequality

$$\hat{M}_f(N,s)\leq \left(B \sup_{0\leq x\leq N} |f(x)|\right)^2$$

is satisfied. Then

$$\overline{\lim_{n\to\infty}}(\lambda_{0,n}(f))^{1/n}\leq\frac{1}{s^{1/3}}.$$

The function

$$f(z)=\int_1^\infty e^{-t}t^{z-1}dt$$

is an example of an entire function of order $\varrho=1$ for which the Theorem 3 but not the Theorem 1 can be applied to prove geometric convergence of the minimal deviation $\lambda_{0,n}(f)$.

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