Труды Международной Конференции по конструктивной теории функций Варна, 19—25 мая 1970

Proceedings of the International Conference on Constructive Function Theory, Varna, May 19—25, 1970

APPROXIMATION BY MEANS OF BIMODULAR NORM

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Summary. A functional $\tilde{\varrho}(x)$ defined in a real linear space X is called a convex pseudomodular, if $0 \leqq \tilde{\varrho}(x) \leqq \infty$, $\tilde{\varrho}(0)=0$, $\tilde{\varrho}(-x)=\tilde{\varrho}(x)$, $\tilde{\varrho}(\alpha x+\beta y) \leqq \alpha\tilde{\varrho}(x)+\beta\tilde{\varrho}(y)$ for $\alpha, \beta \geqq 0$, $\alpha+\beta=1$. Let X be a real linear space, Y a real normed linear space, and let $\varrho(x, y)$ be a functional defined in $X \times Y$ such that it is convex pseudomodular both in X and Y separately, and $\varrho(\lambda x, y)=\varrho(x, \lambda y)$ for real λ. Let $\|(x, y)\|_\varrho = \inf\{\gamma>0 : \varrho(\gamma^{-1}x, y) \leqq 1\}$, then $\|x\|^0 = \sup\{\|(x,y)\|_\varrho : \|y\| \leqq 1\}$ is a norm in the linear space $X^0=\{x : \|x\|^0<\infty, x \in X\}$, called the bimodular norm. Necessary and sufficient conditions for approximation of elements $x \in X^0$ by means of a subset X_1 of X^0 are expressed by means of the functional ϱ.

1. Let X be a real linear space. An extended real-valued functional $\tilde{\varrho}$ defined on X is called a convex pseudomodular, if $0 \leqq \tilde{\varrho}(x) \leqq \infty$, $\tilde{\varrho}(0)=0$, $\tilde{\varrho}(-x)=\tilde{\varrho}(x)$, $\tilde{\varrho}(\alpha x+\beta y) \leqq \alpha\tilde{\varrho}(x)+\beta\tilde{\varrho}(y)$ for $\alpha, \beta \geqq 0$, $\alpha+\beta=1$. If, additionally $\tilde{\varrho}(x)=0$ implies $x=0$, $\tilde{\varrho}$ is called a convex modular. The linear subspace $X_{\tilde{\varrho}}=\{x : \tilde{\varrho}(\lambda x) \to 0 \text{ as } \lambda \to 0+\}$ of X is called a modular space. The functional

$$\|x\|_{\tilde{\varrho}} = \inf\{\varepsilon>0 : \tilde{\varrho}(x/\varepsilon) \leqq 1\}$$

is a homogeneous pseudonorm in $X_{\tilde{\varrho}}$ (if $\tilde{\varrho}$ is a convex modular, $\|\cdot\|_{\tilde{\varrho}}$ is a norm in $X_{\tilde{\varrho}}$), and $\|x\|_{\tilde{\varrho}}<1$ implies $\tilde{\varrho}(x) \leqq \|x\|_{\tilde{\varrho}}$ (see [1] and [2]).

The pseudomodular $\tilde{\varrho}$ will be called

(i) left-continuous, if $\lim_{\lambda \to 1-} \tilde{\varrho}(\lambda x)=\tilde{\varrho}(x)$ for every $x \in X$,

(ii) right-continuous, if $\lim_{\lambda \to 1+} \tilde{\varrho}(\lambda x)=\tilde{\varrho}(x)$ for every $x \in X$;

(iii) continuous, if it is both left-continuous and right-continuous.

Proposition 1.1. If $\tilde{\varrho}$ is left-continuous, then the inequality $\|x\|_{\tilde{\varrho}} \leqq 1$ is equivalent to $\tilde{\varrho}(x) \leqq 1$. If $\tilde{\varrho}$ is right-continuous, then the inequality $\|x\|_{\tilde{\varrho}}<1$ is equivalent to $\tilde{\varrho}(x)<1$.

P r o o f. Let $\tilde{\varrho}$ be left-continuous and let $\|x\|_{\tilde{\varrho}} \leqq 1$. Then $\tilde{\varrho}(\lambda x) \leqq 1$ for all $0<\lambda<1$, and so $\tilde{\varrho}(x) \leqq 1$. If $\|x\|_{\tilde{\varrho}}<1$, then $\tilde{\varrho}(x)<1$ always. Hence $\|x\|_{\tilde{\varrho}} \leqq 1$ implies $\tilde{\varrho}(x) \leqq 1$. The converse implication is true always. Now, let

ϱ be right-continuous and let $\tilde{\varrho}(x)<1$. Then $\|x\|_{\tilde{\varrho}}\leq 1$. Let us suppose that $\|x\|_{\tilde{\varrho}}=1$, then $\tilde{\varrho}(\lambda x)>1$ for all $\lambda>1$. Consequently, $\tilde{\varrho}(x)\geq 1$, a contradiction. The converse implication is true always.

2. Let X, Y be two real linear spaces. An extended real-valued functional ϱ defined on $X\times Y$ is called a bipseudomodular on $X\times Y$, if $\varrho(x, y)$ is a pseudomodular on X for each $y\in Y$ and a pseudomodular on Y for each $x\in X$, and moreover, $\varrho(\lambda x, y)=\varrho(x, \lambda y)$ for every real λ. If, additionally, $\varrho(x, y)=0$ for all $y\in Y$ implies $x=0$, ϱ is called a bimodular (see [1]).

For all $(x, y)\in X\times Y$ one can define

$$\|(x, y)\|_\varrho=\inf\left\{\varepsilon>0:\varrho\left(\frac{x}{\varepsilon}, y\right)\leq 1\right\}=\inf\left\{\varepsilon>0:\varrho\left(x, \frac{v}{\varepsilon}\right)\leq 1\right\};$$

in case when the set under the sign of infimum is void, we put $\|(x, y)\|_\varrho=\infty$.

Now, let Y be a normed linear space, and let us write

$$\|x\|^0=\sup\{\|(x, y)\|_\varrho:\|y\|\leq 1\},$$

$$X^0=\{x:\|x\|^0<\infty,\ x\in X\}.$$

Then $\|\cdot\|^0$ is a pseudonorm in X^0. If ϱ is a bimodular, $\|\cdot\|^0$ is a norm in X^0, called the bimodular norm (see [1]).

Proposition 2.1. Let $x_1, x\in X^0$ and let $\varepsilon>0$. Then

(i) if $\|x_1-x\|^0<\varepsilon$, then $\varrho(x_1-x, y)<1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$;

(ii) if $\varrho(x_1-x, y)\leq 1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$, then $\|x_1-x\|^0\leq\varepsilon$.

P r o o f. Let $\|x_1-x\|^0<\varepsilon$, then $\varrho(x_1-x, y/\varepsilon)\leq 1$ uniformly in the ball $\|y\|\leq 1$, i. e. $\varrho(x_1-x, y)<1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$. Conversely, let $\varrho(x_1-x,y)\leq 1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$, then $\varrho(x_1-x, y/\varepsilon)\leq 1$ uniformly in the ball $\|y\|\leq 1$, and so $\|x_1-x\|^0\leq\varepsilon$.

Proposition 2.2. Let $x_1, x\in X$ and let $\varepsilon>0$;

(i) if $\varrho(x,y)$ is left-continuous in X for every $y\in Y$, then $\|x_1-x\|^0\leq\varepsilon$, if and only if, $\varrho(x_1-x, y)\leq 1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$,

(ii) if $\varrho(x,y)$ is right-continuous in X for every $y\in Y$, then $\|x_1-x\|^0<\varepsilon$, if and only if, $\varrho(x_1-x, y)<1$ uniformly in the ball $\|y\|\leq 1/\varepsilon$.

This follows from Propositions 1.1 and 2.1, immediately. From Proposition 2.1 follows also

Proposition 2.3. Let $x_n, x_0\in X^0$ for $n=1, 2, \ldots$ There holds $x_n\to x_0$ in X^0, if and only if, for each $r>0$, $\varrho(x_n-x_0, y)\to 0$ as $n\to\infty$ uniformly in the ball $\|y\|\leq r$.

3. If $\emptyset\neq X_1\subset X^0$, $x\in X^0$, we call

$$E_{X_1}(x)=\inf\{\|x_1-x\|^0: x_1\in X_1\}$$

the best approximation of the element x by means of the set X_1.

We shall consider for a given $\delta\geq 0$ the problem under which conditions $E_{X_1}(x)=\delta$. We shall treat the cases $\delta=0$ and $\delta>0$ separately. If $E_{X_1}(x)=0$, we shall say that x may be approximated arbitrarily by elements from X_1 in the bimodular norm $\|\cdot\|^0$.

Theorem 3.1. Let ϱ be a bimodular in $X\times Y$ and let $\emptyset \neq X_1 \subset X^0$, $x\in X^0$, $x\bar{\in} X_1$. The element x may be approximated arbitrarily by elements from X_1 in the bimodular norm $\|\cdot\|^0$, if and only if, there is a sequence of elements $x_n\in X_1$ such that $\varrho(x_n-x, y)<1$ uniformly in the ball $\|y\|\leq n$ (or equivalently, if and only if, there is a sequence of elements $x_n\in X_1$ such that $\varrho(x_n-x, y)\leq 1$ uniformly in the ball $\|y\|\leq n$).

P r o o f. Let $\varrho(x_n-x, y)\leq 1$ uniformly in the ball $\|y\|\leq n$, where $x_n\in X_1$, then $\varrho(x_n-x, y)r/n\leq 1$ uniformly in the ball $\|y\|\leq 1$, and so $\|x_n-x\|^0\leq 1/n$. Hence $\|x_n-x\|^0\to 0$. Conversely, let $E_{X_1}(x)=0$, i. e. $\|x'_n-x\|^0\to 0$ for a sequence of elements $x'_n\in X_1$. By Proposition 2.3, this implies $\varrho(x'_n-x, y)\to 0$ as $n\to\infty$ uniformly in each ball $\|y\|\leq r$. In particular, there exists an increasing sequence of indices n_k such that $\varrho(x'_n-x, y)<1$ for $n\geq n_k$ uniformly in the ball $\|y\|\leq k$. Writing $x_k=x'_{n_k}$, we get $\varrho(x_k-x, y)<1$ uniformly in the ball $\|y\|\leq k$.

Theorem 3.2. Let ϱ be a right-continuous bimodular in $X\times Y$ and let $\emptyset\neq X_1\subset X^0$, $x\in X^0$, $x\bar{\in} X_1$, $\delta>0$. There holds $E_{X_1}(x)=\delta$, if and only if,

(i) for every $x_1\in X_1$ there exists $y_1\in Y$ such that $\|y_1\|\leq 1/\delta$ and $\varrho(x_1-x, y_1)\geq 1$;

(ii) for every $\eta>0$ there exists $x_\eta\in X_1$ such that $\varrho(x_\eta-x, y)<1$ uniformly in the ball $\|y\|\leq 1/(\delta+\eta)$. An equivalent necessary and sufficient condition for $E_{X_1}(x)=\delta$ is obtained, if we replace the inequality $\varrho(x_\eta-x, y)<1$ in (ii) by the inequality $\varrho(x_\eta-x, y)\leq 1$.

P r o o f. First, let us see that $E_{X_1}(x)=\delta$ is equivalent to the conditions

1° $\|x_1-x\|^0\geq\delta$ for every $x_1\in X_1$;

2° for every $\eta>0$ there exists $x_\eta\in X_1$ such that $\|x_\eta-x\|^0<\delta+\eta$.

Condition 2° may be replaced by an equivalent condition replacing the inequality $\|x_\eta-x\|^0<\delta+\eta$ by $\|x_\eta-x\|^0\leq\delta+\eta$.

Now, let us suppose that there holds 2°, then, by Proposition 2.1, $\varrho(x_\eta-x, y)<1$ uniformly in the ball $\|y\|\leq 1/(\delta+\eta)$, and we obtain (ii). Conversely, let us suppose (ii), then, again by Proposition 2.1, $\|x_\eta-x\|^0\leq\delta+\eta$. Since $\eta>0$ is arbitrary, this gives 2°. Now, applying Proposition 2.2 we find that 1° is equivalent to the following condition: there exists $y_1\in Y$ such that $\|y_1\|\leq 1/\delta$, but $\varrho(x_1-x, y_1)\geq 1$, i. e. we get the condition (i).

4. Let us yet remark that just as in the case of modular spaces, besides norm convergence one may consider also modular convergence ([2], 1.04). A sequence of elements $x_n\in X^0$ is called ϱ-convergent (bimodular convergent) to $x\in X^0$, if there exists an $r>0$ such that $\varrho(x_n-x, y)\to 0$ as $n\to\infty$ uniformly in the ball $\|y\|\leq r$. ϱ-convergence of $\{x_n\}$ to x will be denoted by $x_n\xrightarrow{\varrho}x$. From Proposition 2.3 follows immediately that $x_n\to x$ (in bimodular norm $\|\cdot\|^0$) implies $x_n\xrightarrow{\varrho}x$. Converse implication does not hold in general. In fact, it is easily seen that $x_n\xrightarrow{\varrho}x$ implies $x_n\to x$ for every sequence of elements $x_n\in X^0$, if and only if, for every $r>0$ uniform convergence to 0 of a sequence $\varrho(x_n, y)$ in the ball $\|y\|\leq r$ implies uniform convergence to 0 of this sequence in the ball $\|y\|\leq 2r$ (this generalizes Theorem 1.31 in [2]). To problems of ϱ-convergence in X^0 we shall return in another paper.

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Received on May 20, 1970