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APPROXIMATION OF LINEAR OPERATORS

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Summary. We give some applications of the theory of interpolation spaces to questions of approximation of linear operators. Let X_0 and X_1 be two Banach spaces embedded in a topological vector space and Y a third Banach space (not related to the previous ones). Given a linear operator $T: X_0+X_1 \to Y$ we study the quantity $K(t,T)=K(t,T;L_0,L_1)$ where $L_i=L(X,Y_i)=$the set of all operators $X \to Y_i$ $(i=0, 1)$ and $0<t<\infty$. In particular we introduce the class $\Omega_\theta=\Omega_\theta(X_0,X_1,Y)$, where $0<\theta<1$, of operators T such that $K(t,T)=O(t^\theta)$. This gives a new way of looking at the work of Stečkin and associates, devoted to the problem of best numerical differentiation. We also consider a more general setting. When specialized Ω_θ reduces to the entropy and diameter quasi-norm ideals introduced by Triebel.

Introduction. The theory of interpolation spaces has applications to many branches of analysis, but in particular to questions pertaining to approximation theory and (related to that) numerical analysis. My talk to-day is in a way a continuation of my lecture at the Budapest meeting in Aug. 1969 (see [4]). I shall now point out still another possibility of applying the ideas of the theory of interpolation spaces in this area. More precisely, I shall consider some questions on approximation of linear operators, which are intimately related to the work of Stečkin and his associates (Arestov, Gabušin, Subbotin, Taikov) on the so-called problem of best numerical differentiation. (See the issues of the journal *Matematičeskie Zametki*, in particular the article by Stečkin [28]; see also Sard [26].) To a great extent what I am doing has been inspired by the lecture of Prof. Stečkin at that meeting (see [29]).

The plan is as follows. In Section 1 we give the necessary background on interpolation spaces. In Section 2 we present a general problem of approximation of linear operators. This leads us to consider interpolation of operator couples, which we accordingly treat in Section 3. We think that what we do will clarify certain points in Stečkin [29]. Let us also remark that in a way we develop the dual of the theory given in Peetre [21]. In Section 4 we consider the special case of Hilbert space. Here more explicit results can be obtained. We note that our results when specialized cover those of Taikov [31], Gabušin [7], Subbotin–Taikov [30]. Finally, in Section 5 we indicate a generalization of the problem of Section 3. It turns out that we then can cover various results on (guasi-) norm ideals of operators in Banach space due to Triebel [32], [33]. Full details of this portion will be published elsewhere, within the framework of the interpolation theory for normed Abelian groups (Peetre-Sparr [23]).

1. Brief review of the theory of interpolation spaces. By a Banach couple $\vec{A}=\{A_0, A_1\}$ over $\boldsymbol{R}$ (or $\boldsymbol{C}$) we mean an entity which consists of two Banach spaces A_0 and A_1 over $\boldsymbol{R}$ (or $\boldsymbol{C}$) both continuously embedded in a Hausdorff topological vector space $\mathbb{A}$ over $\boldsymbol{R}$ (or $\boldsymbol{C}$). Since the set of subspaces of a vector space forms a lattice, we can speak of the sum (or linear hull) of $\sum(\vec{A})=A_0+A_1$ and the intersection $\Delta(A)=A_0\cap A_1$ of $\vec{A}$. Given $a\in A_0+A_1$ (so that $a=a_0+a_1$ for some $a_0\in A_0$, $a_1\in A_1$) and $0<t<\infty$ we set

$$(1.1)\qquad K(t, a;\vec{A})=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+t\|a_1\|_{A_1}).$$

For a fixed this is a norm in A_0+A_1. (It is very suggestive to denote the vector space A_0+A_1 provided with this norm by A_0+tA_1: $\|a\|_{A_0+tA_1}=K(t,a;\vec{A})$. Generally speaking, if A is any Banach space, we denote by tA the same vector space A with the old norm replaced by a multiple: $\|a\|_{tA}=t\|a\|_A$.) With the aid of this family of (equivalent) norms in A_0+A_1 we define spaces $\vec{A}^K_{\theta q}$ (K-interpolation spaces) as follows:

$$(1.2)\qquad \|a\|_{\vec{A}^K_{\theta q}}=\{a\mid a\in A_0+A_1;\ \|a\|_{\vec{A}^K_{\theta q}}=\left(\int_0^\infty (t^{-\theta}K(t,a))^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\},$$

where the parameters θ, q are such that $0<\theta<1$, $1\leq q\leq\infty$ or $0\leq\theta\leq 1$, $q=\infty$. For technical reasons we will also consider a more general form of (1.1) namely

$$(1.3)\qquad K_p(t,a;\vec{A})=\inf_{a=a_0+a_1}(\|a_0\|^p_{A_0}+t^p\|a_1\|^p_{A_1})^{1/p},$$

for any p with $1\leq p\leq\infty$. Clearly $K_1=K$. It is obvious that for fixed t the norms K_p are all uniformly equivalent: $2^{-1/p}K\leq K_p\leq K$. (For a more detailed study of the relationship between K_p for various p, see Holmstedt-Peetre [10].) Another quantity related to K is

$$(1.4)\qquad E(t,a;\vec{A})=\inf_{\|a_0\|_{A_0}\leq t}\|a_1\|_{A_1}.$$

Note that $E(t,a)$ is not a norm. One can show that

$$E(t,a)=0 \Longleftrightarrow a\in\vec{A}^K_{0\infty},\ \|a\|_{\vec{A}^K_{0\infty}}\leq t,$$

$$E(t,\gamma a)=|\gamma|E(t,a).$$

$$E(t_1+t_2, a_1+a_2)\leq E(t_1,a_1)+E(t_2,a_2)$$

(subadditivity in (t,a)!). Note that the latter replaces the usual triangle inequality. We have the following inversion formulas relating K and E:

$$(1.5)\qquad K(t)=\inf_s (s+tE(s)),$$

$$(1.6)\qquad E(s)=\sup_t \left(\frac{K(s)}{s}-\frac{t}{s}\right),$$

which are at least implicitly contained in e. g. Holmstedt [9], Section 5. There is also a dual theory starting with a family of norms in $A_0 \cap A_1$ defined by

$$(1.7)\qquad J(t, a; \vec{A})=\max(\| a \|_{A_0}, t\| a \|_{A_1})$$

leading to spaces $\vec{A}^J_{\theta q}$ (J-interpolation spaces) with $0<\theta<1$, $1<q\leqq\infty$ or $0\leqq\theta\leqq 1$, $q=1$. (Actually if $0<\theta<1$ we have $\vec{A}^K_{\theta q}=\vec{A}^J_{\theta q}$ up to an equivalence of norm.) We also need corresponding to (1.4)

$$1.8)\qquad J_p(t, a; \vec{A})=(\| a \|^p_{A_0}+t^p \| a \|^p_{A_1})^{1/p}.$$

Clearly $J_\infty = J$.

For a more detailed treatment of K- and J-interpolation spaces we refer to the book of Butzer-Berens [2], notably chap. III and also the survey articles Peetre [15], [16].

2. A problem of best approximation of linear operators. Let there be given a Banach couple $\vec{A}=\{A_0, A_1\}$ which is regular in the sense that $A_0 \cap A_1$ is dense in both A_0 and A_1, a Banach space B, and a continuous linear operator $T: A_1 \to B$. Note that since $A_0 \cap A_1$ is dense in A_1, T is uniquely determined by its restriction to $A_0 \cap A_1$! Let there also be given an element $a \in A$, an "appoximation" to a denoted by $\tilde{a} \in A_0+A_1$, and a continuous linear operator $S: \vec{A} \to B$, i. e. S is defined in A_0+A_1 and its restriction to A_0 and to A_1, and hence to $A_0 \cap A_1$, is continuous; note again that S is uniquely determined by its restriction to $A_0 \cap A_1$. We want to use Sa as an "approximation" to Ta:

$$(2.1)\qquad Ta=S\tilde{a}.$$

How shall we choose S so that the approximation becomes optimal? Before we can answer this question, we must make the situation still more precise. First we assume that for a the following bound is known

$$(2.2)\qquad \| a \|_{A_1} \leqq t_1.$$

Secondly we assume that the error committed in approximating a by $\tilde{a}$ is subject to

$$(2.3)\qquad \| a-\tilde{a} \|_{A_0} \leqq t_0.$$

Here t_0 and t_1 are given numbers. Now we can estimate the total error in the approximation (2.1). Write

$$Ta-S\tilde{a}=(T-S)a+S(a-\tilde{a}).$$

Then follows by the triangle inequality and by (2.2) and (2.3)

$$\varepsilon=\|Ta-S\tilde{a}\|_B\leq\|(T-S)a\|_B+\|S(a-\tilde{a})\|_B\leq\|T-S\|_{L(A_1,B)}\|a\|_{A_1}$$
$$+\|S\|_{L(A_0,B)}\|a-\tilde{a}\|_{A_0}\leq\|T-S\|_{L(A_1,B)}t_1+\|S\|_{L(A_0,B)}t_0.$$

Here, generally speaking, $L(A,B)$ denotes the Banach space of continuous linear operators $T:A\rightarrow B$, with norm defined by

$$\|T\|_{L(A,B)}=\sup_{a\neq 0}\|Ta\|_B/\|a\|_A.$$

Let us now write $T_0=S,\ T_1=T-S,\ t=t_1/t_0$. Then we get

$$\varepsilon\leq t_0(\|T_0\|_{L(A_0,B)}+t\|T_1\|_{L(a_1,B)}).$$

If we compare this with (1.1) we see that for the optimal approximation error $\varepsilon_{\text{opt}}=\inf_S\varepsilon$ we obtain

$$\varepsilon_{\text{opt}}\leq t_0K(t,T;\ \vec{L}(\vec{A},B), \tag{2.4}$$

where we have put $\vec{L}(\vec{A},B)=\{L(A_0,B),L(A_1,B)\}$.

Since $\vec{A}$ is regular by assumption, it is clear how $L(A_0,B)$ and $L(A_1,B)$ define a Banach couple, namely we embed them of course in the space $L(A_0\cap A_1,B)$, which thus plays the role of A in Section 1. An answer to our question is thus by (2.4) that one should choose S so that

$$\|S\|_{L(A_0,B)}+(t_1/t_0)\|T-S\|_{L(A_1,B)}$$

comes as close to $K(t_1/t_0,T;\vec{L}(\vec{A},B))$ as possible — or even equal to it, in the case when the minimum is attained.

Example 2.1. The reader might find it useful to think of the following concrete situation, corresponding thus to the problem of best numerical differentiation referred to in the Introduction:

$A_0=C^0$ — continuous bounded functions on a closed interval $I\subset R$;
$A_1=C^2$ — functions, whose second derivatives are in C^0;
$B=C^0$;
$Ta=Da$ — derivative;
$Sa=\Delta_ha/h=$ finite difference quotient with increment h.

The relation to the work of Stečkin and his associates now should be clear, too. The main difference is that in place of K they use E. In view of (1.5)—(1.6) this is, however, really not important.

3. Interpolation of operator couples: An inequality and its converse. Let again $\vec{A}=\{A_0,A_1\}$ be a regular Banach couple. The considerations of Section 2 (see in particular (2.4)) lead us to consider the quantity

$$K(t,T)=K(t,T;\vec{A},B)=K(t,T;\vec{L}(\vec{A},B)) \tag{3.1}$$

which now will be studied more closely. We shall compare $K(t,T)$ with the following related quantity:

(3.2) $$\mathsf{K}(t,T)=\mathsf{K}(t,T;\vec{A},B)=\sup_{a\neq 0}\|Ta\|_B/J(t^{-1},a;\vec{A}).$$

More generally, corresponding to $K_p(t,T)$ we set

(3.3) $$\mathsf{K}_p(t,T)=\mathsf{K}_p(t,T;\vec{A},B)=\sup_{a\neq 0}\|Ta\|_B/J_{p'}(t^{-1},a;\vec{A}),\quad \frac{1}{p}+\frac{1}{p'}=1.$$

Clearly $\mathsf{K}_1=\mathsf{K}$. Our first result (see S t e č k i n [28], formula (6)) is almost trivial:

Proposition 3.1. For all p holds $K_p(t,T)\geq\mathsf{K}_p(t,T)$.

P r o o f. To fix the ideas, let us consider the case $p=1$ only. Let $T=T_0+T_1$ be a general "decomposition" of T. Using the triangle inequality we get for any $a\in A_0\cap A_1$

$$\|Ta\|_B\leq\|T_0a\|_B+\|T_1a\|_B\leq\|T_0\|_{\mathsf{L}(A_0,B)}\|a\|_{A_0}$$

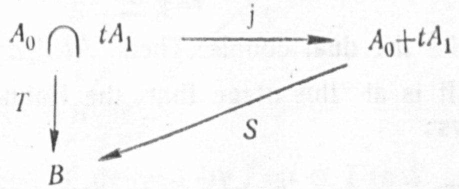
$$+\|T_1\|_{\mathsf{L}(A_1,B)}\|a\|_{A_1}\leq J(t^{-1},a)\,(\|T_0\|_{\mathsf{L}(A_0,B)}+t\|T_1\|_{\mathsf{L}(A_1,B)}).$$

This clearly implies

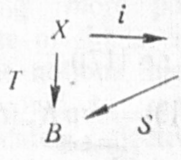
$$\mathsf{K}(t,T)=\sup\|Ta\|_B/J(t^{-1},a)$$

$$\leq\inf(\|T_0\|_{\mathsf{L}(A_0,B)}+t\|T_1\|_{\mathsf{L}(A_1,B)})=K(t,T).\ \#$$

Now we ask under which circumstances converse inequality holds true. In other words we are looking for special "decompositions" $T=T_0+T_1$ with nice properties. But this can also be reinterpreted as an extension problem: To find an operator $S:A_0\oplus tA_1$ (direct sum!) $\to B$ such that the following diagram is commutative:



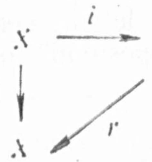
where j stands for the "natural" embedding $j:A_0\cap tA_1\to A_0\oplus tA_1:a\to(a,a)$. Indeed, if we define S by $S:A_0\oplus tA_1\to B:(a_0,a_1)\to T_0a_0+T_1a_1$, it is readily seen that the requirement $T=T_0+T_1$ is equivalent to precisely $T=S\circ j$. Again this is just a special case of the following general extension problem in Banach space theory: Given any two Banach spaces X and Y, X being embedded in Y, and an operator $T:X\to B$, to find an operator $S:Y\to B$ such that the following diagram is commutative:



Classically, there are two answers given to this problem:

1^0 (a condition on B). B is c-injective ($c \geq 1$), i. e. the extension problem has a solution S for all X, Y and T whatsoever satisfying $\|S\| \leq c\|T\|$. Examples are: $B = \mathbf{R}$(of $\mathbf{C}$) (for $c=1$, by the Hahn-Banach theorem), $B = L_\infty$ (for $c=1$), $B = \mathrm{Lip}\,\alpha$ (for some c, by the results of Cisielski [3] and Bonic-Frampton-Tromba [1], saying that L_∞ and $\mathrm{Lip}\,\alpha$ are isomorphic). For a complete discussion, see Day [5, p. 94—96].

2^0 (a condition on X and Y). X is a c-retract ($c \geq 1$) of Y, i. e. there is an operator $r: Y \to X$ (c-retraction) with $\|r\| \leq c$ such that we have the commutative diagram:



In other words $r \circ i = id$. In this case we simply take $S = T \circ r$, because then follows $S \circ i = T \circ r \circ i = T \circ id = T$.

The insight thus gained from this look at the general situation we now exploit in our special case. We start with case 1^0. First we make use of the Hahn-Banach-theorem.

Theorem 3.1. Let $B = \mathbf{R}$ (or $\mathbf{C}$). Then for all p holds $K_p(t, T) = \mathsf{K}_p(t, T)$.

Proof. Again we consider the case $p=1$ only. For any Banach space A we have

$\mathsf{L}(A, B) = A'$ (dual space);

$Ta = \langle a, a' \rangle$ (duality);

$$\|T\|_{\mathsf{L}(A,B)} = \|a'\|_{A'} \quad (= \sup_{a \neq 0} |\langle a, a' \rangle| / \|a\|_A).$$

Let $\vec{A}' = \{A_0', A_1'\}$ be the dual couple. Then $J(t^{-1}, a; \vec{A})$ and $K(t, a'; \vec{A}')$ are conjugate norm. (It is at this stage that the Hahn-Banach theorem enters.) Therefore follows:

$$\mathsf{K}(t, T) = \sup |\langle a, a' \rangle| / J(t^{-1} a; \vec{A}') = K(t, a'; \vec{A}') = K(t, T). \ \#$$

Next we consider the case $B = L_\infty(M) = L_\infty(M, \mathbf{R}$ (or $\mathbf{C}$)) = bounded measurable functions on the measure space M.

Theorem 3.2. Let $B = L_\infty(M, \mathbf{R}$ (or $\mathbf{C}$)). Then holds $K_\infty(t, T) = \mathsf{K}_\infty(t, T)$.

Proof. In this case we have

$\mathsf{L}(A, B) = L_\infty(M, A')$ = bounded measurable functions with values in A';

$Ta(m) = \langle a, a'(m) \rangle$;

$$\|T\|_{\mathsf{L}(A,B)} = \|a'\|_{L_\infty(M,A')} = \sup_{m \in M} \|a'(m)\|_{A'}.$$

Moreover holds (e. g. Peetre [17])

$$K_\infty(t, a'; L_\infty(M, \vec{A}')) = \sup_{m \in M} K_\infty(t, a'(m); \vec{A}').$$

Therefore follows $K_\infty(t, T) = \sup\limits_{m \in M} K_\infty(t, a'(m); \vec{A}')$. On the other hand, we get

$$\mathsf{K}_\infty(t, T) = \sup_{a \neq 0} \| Ta \|_B / J_1(t^{-1}, a) = \sup_{a \neq 0} \sup_{m \in M} |\langle a, a'(m) \rangle| / J_1(t^{-1}, a)$$

$$= \sup K_\infty(t, a'(m); \vec{A}'). \quad \#$$

Remark 3.1. Let us also note the following general formula, valid for any p:

$$\mathsf{K}_p(t, T) = \sup_{b \neq 0} K_p(t, T^* b'; \vec{A}') / \| b' \|_{B'}, \tag{3.4}$$

where $T^*: B' \to A'$ is the transpose of T. We could alternatively use (3.4) (with $p = \infty$) in the above proof of Th. 3.2.

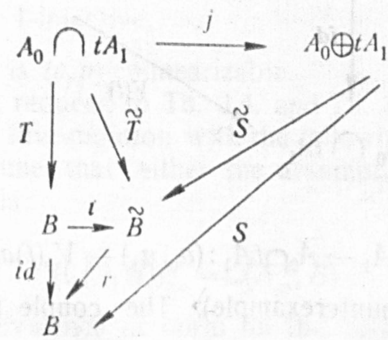
Now let B be varied.

Theorem 3.3. Let B be a c-retract of $\widetilde{B}$ and assume that

$$K_\infty(t, T; \vec{A}, \widetilde{B}) \leq \widetilde{d} \mathsf{K}_\infty(t, T; \vec{A}, \widetilde{B}).$$

Then the same inequality holds with $\widetilde{B}$ replaced by B and $\widetilde{d}$ by $d = c\widetilde{d}$.

Proof. The proof becomes obvious once we have written down the relevant commutative diagram:



Namely, we first extend T defined by $\widetilde{T} = i \circ T$ to $\widetilde{S}$ and then set $S = r \circ \widetilde{S}$. Since $r \circ i = id$ we get $S \circ j = r \circ \widetilde{S} \circ j = r \circ \widetilde{T} = r \circ i \circ T = id \circ T = T$.#

Since every c-injective Banach space B is an c-retract of some space $L^\infty(M)$ (see Day [5, p. 94]), we obtain as a combination of Th. 3.2 and Th. 3.3.

Theorem 3.4. Let B be c-injective. Then holds $K_\infty(t, T) \leq c\mathsf{K}_\infty(t, T)$. In particular if B is 1-injective, then $K_\infty(t, T) = \mathsf{K}_\infty(t, T)$.

Remark 3.2. Th. 3.4 could also have been derived more directly using just injectiveness. However, we found the road followed above more instructive. One reason for writing down all this in such a detail was indeed to make "diagram thinking" more popular among analysts. We refer to Peetre [18] for further use of similar ideas.

Remark 3.3. Actually the notions injective, rectract etc. are too strong for the present purpose and could be replaced by less restrictive notions that might be called approximately injective, approximate retract, etc. The reason is that we are only interested in inequalities for less norms, not actual identities for the operators involved. E. g. we may say that X is an approximate c-retract of Y if there exists a family of operators $r_n : Y \to X$ with $\overline{\lim\limits_{n\to\infty}} \| r_n \| \leq c$ such that $r_n \circ i \to id$ in the strong operator topology. Now it becomes clear that e. g. in Th. 3.2. we could substitute $L_\infty(M)$ for $C(K)=$ continuous functions on a compact topological space K. (See also Peetre [19].)

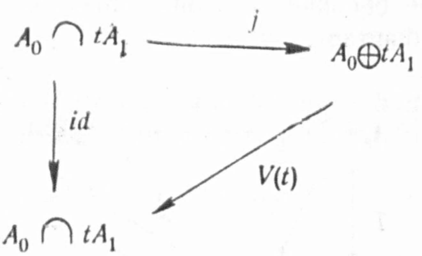
We have now used up case 1^0 and turn accordingly to case 2^0. Thus B will be "general" and instead we have to impose a condition on $\vec{A}$.

Definition 3.1. A Banach couple $\vec{A}$ is called (c, p)-colinearizable iff there exist two families of operators $V_0(t) : A_0 \to \vec{A}$ and $V_1(t) : A_1 \to \vec{A}$ such that

$$id = V_0(t) + V_1(t), \tag{3.5}$$

$$J_p(t, V_0(t)a_0 + V_1(t)a_1) \leq C(\| a_0 \|^p_{A_0} + t^p \| a_1 \|^p_{A_1})^{1/p}. \tag{3.6}$$

We can express Def. 3.1 also with the aid of the following commutative diagram



where we have defined

$$V(t) : A_0 \oplus tA_1 \to A_0 \cap tA_1 : (a_0, a_1) \to V_0(t)a_0 + V_1(t)a_1.$$

Example 3.1 (counterexample). The couple $\{L_{p_0}, L_{p_1}\}$, $p_0 \neq p_1$, is not colinearizable.

Example 3. 2. On the other hand, e. g. $\{C^0, C^1\}$ is (c, p)-colinearizable for some c (depending on p). There arises the problem to determine the "best" constant c (as a function of p). In particular the extremal case $p = \infty$ is here of importance. See also [14].

Theorem 3.5. Let $\vec{A}$ be $(c, 1)$-colinearizable. Then holds:

$$K_\infty(t, T) \leq c\mathsf{K}_\infty(t, T).$$

Proof. Write $T = T_0 + T_1$ with $T_0 = TV_0(t)$, $T_1 = TV_1(t)$. Then we get

$$K_\infty(t, T) \leq \max(\| T_0 \|_{L(A_0, B)}, t \| T_1 \|_{L(A_1, B)})$$

$$\leq \max(\| T \| \| V_0(t) \|_{L(A_0, A_0 \cap \frac{1}{t} A_1)}, t \| T \| \| V_1(t) \|_{L(A_1, A_0 \cap \frac{1}{t} A_1)})$$

$$= \| T \| \max(\| V_0(t) \|_{L(A_0, A_0) \cap \frac{1}{t} A_1)}, t \| V_1(t) \|_{L(A_1, A_0 \cap \frac{1}{t} A_1)}).$$

Here we use in $A_0 \cap \frac{1}{t} A_1$ the ∞-norm. But on one hand

$$\|T\|=\|T\|_{\mathsf{L}(A_0\cap\frac{1}{t}A_1,B)}=\mathsf{K}_\infty(t,T)$$

and on the other hand ((3.6) with $p=1$)

$$\max(\|V_0(t)\|_{\mathsf{L}(A_0,A_0\cap\frac{1}{t}A_1)},t\|V_1(t)\|_{\mathsf{L}(A_1,A_0\cap\frac{1}{t}A_1)})\leq c.\#$$

Remark 3.4. Why is there a restriction on p in Th. 3.2 — Th. 3. 4, but not in Th. 3.1? A way of saying this is that the property that given b_1 and $b_2\in B$ there exists ε with $|\varepsilon|=1$ such that $\|b_1+\varepsilon b_2\|_B=\|b_1\|_B+\|b_2\|_B$ is true only in the scalar case, $B=\boldsymbol{R}$ (or $\boldsymbol{C}$). So there is a limitation of principal nature. A way of circumvening the difficulty is to use in place of $K_p(t,T)$ the modified quantity

$$\bar{K}_p(t,T)=\inf_{T=T_0+T_1}\ \sup_{(\|a_0\|^{p'}_{A_0}+\frac{1}{t^{p'}}\|a\|^{p'}_{A_1})^{1/p_1}\leq 1}\|T_0a_0+T_1a_1\|_B,$$

the introduction of which is also natural from the view point of the consideration of Section 2. We have in general $\mathsf{K}_p\leq\bar{K}_p\leq K_p\leq 2^{1/p}\bar{K}_p$.

It is also easy to show that

i) $\bar{\mathsf{K}}_p=K_p$ if B is 1-injective,

ii) $\mathsf{K}_p\leq cK_p$ if $\vec{A}$ is (c,p)-colinearizable.

Taking $p=\infty$ this reduces to Th. 3.4. and Th. 3.5 respectively.

We conclude our investigation with the following interpolation theorem.

Theorem 3.6. Assume that either the assumptions of Th. 2.4 or Th. 3.5 are fulfilled. Then holds:

$$(\vec{\mathsf{L}}(\vec{A},B))^K_{\theta\infty}=\mathsf{L}(\vec{A}^J_{\theta 1},B)$$

(possibly up to an equivalence of norm on the case of Th. 3.5).

Proof. We have $\mathsf{K}\leq K\leq c'\mathsf{K}$ with $c'=1$ in the case of Th. 3.4 and $c'=c$ in case of Th. 3.5. Thus $T\in(\vec{\mathsf{L}}(\vec{A},B))^K_{\theta\infty}$ iff $\mathsf{K}(t,T)\leq Ct^\theta$ for some C But this means precisely that

$$\|Ta\|_B\leq Ct^{-\theta}J(t,a;\vec{A})$$

which again is equivalent to

$$\|Ta\|_B\leq c\,|a|_{\vec{A}^J_{\theta 1}}$$

and thus $T\in\mathsf{L}(\vec{A}^J_{\theta 1},B)$. For the norms holds

$$\|T\|_{\mathsf{L}(\vec{A}^J_{\theta 1},B)}\leq\|T\|_{(\vec{\mathsf{L}}(\vec{A},B))^K_{\theta\infty}}\leq c'\|T\|_{\mathsf{L}(\vec{A}^J_{\theta 1},B)}.\#$$

Remark 3.5. Finally let us point out that the dual theory, dealing thus with $\vec{\mathsf{L}}(\vec{A},B)$ in place of $\mathsf{L}(\vec{A},\vec{B})$, was developped in Peetre [21], where notably an application to absolutely summing operators was given (see also [20]). Note in particular that the dual definition of our Def. 3.1. was used there (leading to so-called linearizable couples). However the treatment in [21] is not so complete, but using our present new ideas it can certainly receive a much more elegant form.

4. The Hilbert space case. Let $\vec{A}=\{A_0, A_1\}$ be a regular Hilbert couple (i. e. both A_0 and A_1 are Hilbert spaces). In this case the "optimal" decomposition $a=a_0+a_1$ as well as "explicit" expressions for $K_p(t, a)$ (or $E(t, a)$) are very easy to determine. (See e. g. Peetre [17]; see also Lions [11], which is the first paper to deal with this special case, and with abstract interpolation on the whole, and Foias-Lions [6].) It suffices in principle to take $p=2$. For simplicity let us also assume that one of the spaces is embedded in the other, say:

$$A_0 \supset A_1. \tag{4.1}$$

From standard theorems in Hilbert space theory (see [11]) we deduce that there exists a (strictly) positive self adjoint operator Λ in A_0 with

$$D(\Lambda)=A_1, \quad \|\Lambda a\|_{A_0}=\|a\|_{A_1}.$$

Again for simplicity, let us assume that the spectral multiplicity of Λ is 1. Then by the spectral theorem of von Neumann Λ is unitary equivalent to multiplication with λ in the Hilbert space $L_2(\boldsymbol{R}, \mu)$ where μ is some positive measure on $\boldsymbol{R}=(-\infty, \infty)$ (spectral measure of Λ). (In the general case of arbitrary spectral multiplicity so-called direct integrals of Hilbert spaces intervene!) In other words we may restrict attention to the special case

$$A_0=L_2(\boldsymbol{R}, \mu), \quad \|a\|_{A_0}=\left(\int_{-\infty}^{\infty}|a(\lambda)|^2 d\mu(\lambda)\right)^{1/2}, \quad \Lambda a(\lambda)=\lambda a(\lambda).$$

Then holds

$$A_1=L_2(\boldsymbol{R}, \lambda\mu), \quad \|a\|_{A_1}=\left(\int_{-\infty}^{\infty}|\lambda a(\lambda)|^2 d\mu(\lambda)\right)^{1/2}.$$

Now follows easily by the case of equality in Schwarz's inequality that

$$(K_2(t, a))^2=\inf_{a=a_0+a_1}\left(\|a_0\|_{A_0}^2+t^2\|a_1\|_{A_1}^2\right)\inf_{a=a_0+a_1}\int_{-\infty}^{\infty}\left(|a_0(\lambda)|^2+t^2|\lambda a_1(\lambda)|^2\, d\mu(\lambda)\right)$$

$$=\int_{-\infty}^{\infty}\inf_{a(\lambda)=a_0(\lambda)+a_1(\lambda)}\left(|a_0(\lambda)|^2+t^2|a_1(\lambda)|^2\right)d\mu(\lambda)=\int_{-\infty}^{\infty}\frac{1}{1+\frac{1}{(t\lambda)^2}}|a(\lambda)|^2 d\mu(\lambda)$$

where the optimal decomposition $a=a_0+a_1$ is determined by

$$a_0(\lambda)=\frac{1}{1+\frac{1}{(t\lambda)^2}}a(\lambda), \quad a_1(\lambda)=\frac{\frac{1}{(t\lambda)^2}}{1+\frac{1}{(t\lambda)^2}}a(\lambda).$$

Switching back again to the general abstract case we can now express this in terms of the spectral calculus of von Neumann:

(4.2) $$a_0=\left(1+\frac{1}{(t\Lambda)^2}\right)^{-1}a,\quad a_1=\frac{1}{(t\Lambda)^2}\left(1+\frac{1}{(t\Lambda)^2}\right)^{-1}a.$$

We also obtain

(4.3) $$K_2(t,a)=\left\|\left(1+\frac{1}{(t\Lambda)^2}\right)^{-1/2}a\right\|_{A_0}=\left(\left(1+\frac{1}{(t\Lambda)^2}\right)^{-1}a\middle|a\right)_{A_0}^{1/2}$$

where $(\,|\,)_{A_0}$ stands for the scalar product in A_0.

Remark 4.1. We could also have derived (4.2) and (4.3) more directly using a variational argument in Hilbert space. And this irrespective of what is the spectral multiplicity of Λ. An important observation about (4.2) is further that a_0 and a_1 depend linearly on a. This is of course typical for the Hilbert space case.

We now use the above results to the computation of $\mathsf{K}_2(t,a)$. Since (cf. remark 3.1, formula (3.4) with $p=2$)

(4.4) $$\mathsf{K}_2(t,T)=\sup_{b\neq 0}\frac{K_2(t,T^*b';\vec{A}')}{\|b\|_B}$$

we can actually restrict ourselves to the scalarvalued case: $B=\boldsymbol{R}$ (or $\boldsymbol{C}$) i. e. the computation of $K_2(t,a';\vec{A}')$. If we identify A_0 and A_0' in the usua way made in Hilbert space theory we see that we can take

$$A_1'=\text{completion of } D\left(\frac{1}{\Lambda}\right),\ \|a\|_{A_1'}=\left\|\frac{1}{\Lambda}a\right\|_{A_0}.$$

Thus from (4.2) and (4.3) we get (for the optimal decomposition $a'=a_0^a+a_1'$)

(4.5) $$a_0'=\frac{1}{1+\left(\frac{\Lambda}{t}\right)^2}a,\quad a_1'=\frac{\left(\frac{\Lambda}{t}\right)^2}{1+\left(\frac{\Lambda}{t}\right)^2}a$$

and also

(4.6) $$K_2(t,a')=\left\|\left(1+\left(\frac{\Lambda}{t}\right)^2\right)^{-1/2}a'\right\|_{A_0}=\left(\left(1+\left(\frac{\Lambda}{r}\right)^2\right)^{-1}a'\middle|a'\right)_{A_0}^{1/2}.$$

When (4.6) is applied to the general case, we finally obtain by (4·4)

(4.7) $$\mathsf{K}_2(t,T)=\sup_{b\neq 0}\frac{\left(\left(1+\left(\frac{\Lambda}{t}\right)^2\right)^{-1}T^*b\middle|T^*b\right)_{A_0}^{1/2}}{\|b\|_{B'}}.$$

Note that $(1+(\Lambda/t)^2)^{-1}$ essentially is the resolvent of Λ^2!

Now we illustrate the above considerations in some concrete cases. (Actually strictly speaking the theory is not directly applicable here, since $\|a\|_{A_1}$ will be a semi-norm, not a norm, in general; however, as can be readily seen, all conclusions are true nevertheless.)

Example 4.1. Let

$A_0=L_2=L_2(I)$, where $I=[\alpha,\beta]$ is a closed interval $\subset\boldsymbol{R}$;

$A_1 = W_2^n = W_2^n(I)$;
$B = C^0$;
$Ta = D^k a$,

Now Λ^2 is easy to describe: It is precisely the "realization" in the Hilbert space L_2 of the differential operator D^{2n} corresponding to Neumann's boundary conditions: $a^{(n)}(\alpha) = \cdots = a^{(2n-1)}(\alpha) = 0$, $a^{(n)}(\beta) = \cdots = a^{(2n-1)}(\beta) = 0$. (If α or β is infinite, the corresponding set of boundary conditions is left out.) Therefore $G(\Lambda) = (1 + (\Lambda/t)^2)^{-1}$ is of the form

$$G(t)a(x) = \int_\alpha^\beta G(t, x, \xi) a(\xi) d\xi$$

where $E(t, x, \xi)$ is a Green type kernel. Applying (4.7) we obtain the inequality

$$\sup_{\alpha \leq x \leq \beta} |D^k a(x) \leq C\left(\int_\alpha^\beta |a(x)|^2 dx + \frac{1}{t^2} \int_\alpha^\beta |D^n a(x)|^2 dx \right)^{1/2} \tag{4.8}$$

with the "best" constant given by

$$C = \sup_{\substack{\alpha \leq x \leq \beta \\ \alpha \leq \xi \leq \beta}} G_k(t, x, \xi))^{1/2} \tag{4.9}$$

where we have put $G_k(t, x, \xi) = D_x^k D_\xi^k G(t, x, \xi)$. It is clear that the "sup" in (4.9) is attained when $x = \xi = \alpha$ or $x = \xi = \beta$. When specialized, (4.8) yields as a particular case the results of Taikov [31] ($\alpha = -\infty$, $\beta = \infty$) and Gabušin [7] ($\alpha = 0$, $\beta = \infty$). We can also get "pointwise" astimates ($B = \boldsymbol{R}$ (or $\boldsymbol{C}$), $Ta = D^k a(x_0)$):

$$|D^k a(x_0)| \leq C(x_0)\left(\int_\alpha^\beta |a(x)|^2 dx + \frac{1}{t^2} \int_\alpha^\beta |D^n a(x)|^2 dx \right)^{1/2} \tag{4.10}$$

with the "best" constant

$$C(x_0) = (G_k(t, x_0, x_0))^{1/2}. \tag{4.11}$$

As an illustration of (4.10) let us quote the inequality (corresponding to the special case $n = 0$),

$$a|(x_0)|^2 \leq \frac{\cosh t(x_0 - \alpha) \cosh t(\beta - x_0)}{\sinh t(\beta - \alpha)} \left(\int_\alpha^\beta |a(x)|^2 dx + \frac{1}{t^2} \int_\alpha^\beta |Da(x)|^2 dx \right),$$

as well as its limiting case ($\beta = \infty$)

$$|a(x_0)| \leq \cosh t(x_0 - \alpha) e^{-t(x_0 - \alpha)} \left(\int_\alpha^\beta |a(x)|^2 dx + \frac{1}{t^2} \int_\alpha^\beta |Da(x)|^2 dx \right).$$

We can, as a further (really more typical!) application of (4.7), write down similar estimates in the L_q-metric ($B = L_q = L_q(I)$):

$$(4.12)\qquad \left(\int_\alpha^\beta |D^k a(x)|^q dx\right)^{1/q} \leq C_q\left(\int_\alpha^\beta |a(x)|^2 dx + \frac{1}{t^2}\int_\alpha^\beta |D^n a(x)|^2 dx\right)^{1/2}$$

with the "best" constant

$$(4.13)\qquad C_q = \sup_b \left(\int_\alpha^\beta\int_\alpha^\beta G_k(t, x, \xi)\overline{b(x)}b(\xi)dx\,d\xi\right)^{1/2} / \| b\|_{L_{q'}}, \qquad \frac{1}{q}+\frac{1}{q'}=1.$$

Note that

$$C_2 \leq \left(\sup_{\alpha\leq x\leq\beta}\int_\alpha^\beta G_k(t, x, \xi)d\xi\right)^{1/2}.$$

(The case $q=2$, $\alpha=-\infty$, $\beta=\infty$ is essentially contained in [30].)

Example 4.2. The same example in several variables!

Example 4.3 (discrete norms). Here we have no detailed results. We exhibit a typical situation to indicate in which direction further work should be done. Note in particular that what we do is somehow related to spline type approximation. See notably Schoenberg [27], where reference to interesting work by Whittaker is made. The interval is taken to be the whole axis $\boldsymbol{R}=(-\infty, \infty)$. The "norms" are:

$$\| a\|_{A_0} = \left(\sum_{\nu=-\infty}^{\infty} |a(\nu)|^2\right)^{1/2}, \quad \| a\|_{A_1} = \left(\int_{-\infty}^{\infty} |D^n a(x)|^2 dx\right)^{1/2}.$$

"Dualization" leads to the following norms for distributions:

$\| a'\|_{A'_0} = \left(\sum_{\nu=-\infty}^{\infty} |b_\nu|^2\right)^{1/2}$, defined for distributions of the form

$$a'(x) = \sum_{\nu=-\infty}^{\infty} b_\nu \delta(x-x_\nu) \text{ only};$$

$$\| a'\|_{A'_1} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\mathsf{F}a'(\lambda)^2|}{|\lambda|^{2n}}d\lambda\right)^{1/2} \quad (\mathsf{F}\text{—Fourier transform}).$$

Let P be the orthogonal projection in the A'_1-metric onto the subspace generated by the deltafunctions $\delta(x-\nu)$, $\nu=0, \pm 1, \pm 2, \ldots$, and write $Q=id-P$. Then if

$$Pa' = \sum_{\nu=-\infty}^{\infty} b_\nu\delta(x-x_\nu), \quad \mathsf{F}Pa' = b$$

follows

$$\| a'\|_{A'_1}^2 = \| Qa'\|_{A_1}^2 + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|b(\lambda)|^2}{(Z_n(\lambda))^2}d\lambda$$

with

$$Z_n(\lambda)=\left(\sum_{\nu=-\infty}^{\infty}\frac{1}{|\lambda+2\pi\nu|^{2n}}\right)^{-1/2}\qquad (\text{"zetafunction"}).$$

We also have

$$\|a'\|_{A_0'}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|b(\lambda)|^2 d\lambda\right)^{1/2}.$$

Thus we are lead to consider norms

$$\|b\|_{B_0}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|b(\lambda)|^2 d\lambda\right)^{1/2},\quad \|b\|_{B_1}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|b(\lambda)|^2}{(Z_n(\lambda))^2}\,d\lambda\right)^{1/2}.$$

Now we have

$$K_2(t,b;\vec{B})=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|b(\lambda)|^2}{1+\left(\frac{Z_n(\lambda)}{t}\right)^2}\,d\lambda\right)^{1/2}.$$

Switching back from $\vec{B}$ to $\vec{A'}$ we get

$$K_2(t,a';\vec{A'})=\left(\|Qa\|^2_{A_1'}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|b(\lambda)|^2}{1+\left(\frac{Z_n(\lambda)}{t}\right)^2}\,d\lambda\right)^{1/2}.$$

It woult be interesting to have some more explicit instances of these formulas, displaying thus also the best approximation of some concrete operators.

5. Various generalizations. In Section 3 we studied the quantity $K(t, T)$ (see (3.1)). In view of (1.5)—(1.6) this is equivalent to the study of the quantity $E(t, T)=E(t, T; \vec{A}, \vec{B})=E(t, T; \vec{L}(\vec{A}, B))$. Now we generalize these guantities in the respect that we replace $L(A_0, B)$ be any normed Abelian group G of non-linear operators S (with values in B). That G is an Abelian group means that if $S_1, S_2 \in G$ then S_1+S_2, $S_1-S_2 \in G$. Moreover G should carry a norm, i. e. a function $G\to R_+ : S\to\|S\|_G$ about which we assume that it is (positive) definite:

$$\|S\|_G=0 \Longleftrightarrow S=0 \tag{5.1}$$

and satisfies the triangle inequality:

$$\|S_1+S_2\|_G\leq\|S_1\|_G+\|S_2\|_G \tag{5.2}$$

(but it need not be homogeneous

$$\|\gamma S\|_G=|\gamma|\ \|S\|_G \tag{5.3}$$

in general!) Since the previous space A_0 will not intervene anymore we write simply $A_1=A$. From now on accordingly we consider the quantities:

$$K(t, T)=K(t, T; A, B, G)=K(t, T; \{G, F(A, B)\}) = \inf_{T=T_0+T_1}(\|T_0\|_G+t\|T_1\|_{F(A,B)}) \tag{5.4}$$

and

$$(5.5)\qquad E(t, T)=E(t, T; A, B, \mathsf{G})=E(t, T; \{\mathsf{G}, \mathsf{F}(A, B)\})$$
$$= \inf_{\|S\|_{\mathsf{G}}\leq t} \| T-S \|_{\mathsf{F}(A, B)}.$$

(Here $\mathsf{F}(A, B)$ denotes the Banach spaces of homogeneous bounded nonlinear operators $T: A \to B$ with norm given by $\| T\|_{\mathsf{F}(A, B)}=\sup_{a\neq 0} \| Ta \|_B / \| a \|_A$.) In particular we can then define the interpolation spaces

$$(5.6)\qquad \mathsf{O}_\alpha=\mathsf{O}_\alpha(A, B, \mathsf{G})=(\mathsf{G}, \mathsf{F}(A, B))^K_{\theta\infty}=\{T \mid \sup t^{-\theta} K(t, T)<\infty\}$$
$$=\{T \mid \sup t^\alpha E(t, T)<\infty\}, \quad \alpha=\frac{\theta}{1-\theta}.$$

For a more detailed treatment of interpolation in the case of normed Abelian groups, see Peetre-Sparr [23].

Let us now consider an important special instance of spaces G, which really gave us the original motivation of introducing this general framework. Let there be given a setfunction κ in B, by which we mean a mapping κ: $\mathsf{P}(B)\setminus\phi \to \bar{R}_+$ (i. e. κ (N) is a positive extended real number for every nonempty subset N of B). Here is a list of properties that κ may (or may not) enjoy:

(5.7) If $\kappa(\mathsf{N})$ is sufficiently small then N consists of just one point (i. e. card $\mathsf{N}=1$)

$$(5.8)\qquad \kappa(\mathsf{N}_1+\mathsf{N}_2)\leq\kappa(\mathsf{N}_1)+\kappa(\mathsf{N}_2) \text{ (subadditivity)}$$

where

$$\mathsf{N}_1+\mathsf{N}_2=\{b \mid \exists b_1 \in \mathsf{N}_1, b_2 \in \mathsf{N}_2 : b=b_1+b_2\} \text{ (Minkowski sum)};$$

$$(5.9)\qquad \kappa(\gamma\mathsf{N})=\kappa(\mathsf{N}) \text{ if } \gamma\neq 0 \text{ (homogeneity)}$$

where

$$\gamma\mathsf{N}=\{b \mid \exists b' \in \mathsf{N} : b=\gamma b'\}.$$

We define $\mathsf{G}=\mathsf{G}_\kappa$ by

$$\| S \|_{\mathsf{G}}=\kappa(S\{\| a \|_A\leq 1\}).$$

Note that the above properties (5.1) — (5.2) follow from (5.7)—(5.8) while as (5.3) follows from (5.9). Let us write

$$E(t, T)=E(t, T; A, B, \kappa)=E(t, T; A, B, \mathsf{G}_\kappa).$$

$$\mathsf{O}_\alpha=\mathsf{O}_\alpha(A, B, \kappa)=\mathsf{O}_\alpha(A, B, \mathsf{G}_\kappa).$$

Note that the definition of $E(t, T)$ (see (5.5)) here can be rephrased as follows

$$(5.10)\qquad E(t, T)= \inf_{\kappa(\mathsf{N})\leq t} \sup_{\|a\|_A\leq 1} \inf_{b\in N} \| Ta-b \|_B.$$

Example 5.1. $\kappa(\mathsf{N})=\log_2 \operatorname{card} \mathsf{N}$ if N is finite; $\kappa(\mathsf{N})=\infty$ otherwise.

Example 5.2. $\kappa(\mathsf{N})=\dim$ linear hull N.

Example 5.3. $\kappa(\mathsf{N})=\sup_{b\in\mathsf{N}} G(b)$ where $G: B\to \boldsymbol{R}_+$ is a given subadditive function (i. e. $G(b_1+b_2)\leq G(b_1)+G(b_2)$).

All properties (5.7)—(5.10) are fulfilled in the case of Ex. 5.1—5.2, property (5.8) only in the case of Ex. 5.3. Note also that the setfunctions of Ex. 5.1—5.2 are defined artomatically once B is given while as the oneof Ex. 5.3 requires an auxiliary choice of G. As a readily seen, in the case of Ex. 5.1 we have $\mathcal{O}_\alpha=E_\alpha$ = entropy ideal (in the notation of Triebel [32]) and in the case of Ex. 5 2 $\mathcal{O}_\alpha=\kappa_\alpha$ = diameter ideal (in the same notation).

We can now generalize various results in [32]. First we note that under a suitable additional assumption with the ideal property.

Theorem 5.1. Choose for every Banach space B a setfunction $\kappa=\kappa_B$ satisfying properties (5.7)—(5.9) in a such a way that for every other Banach space B and every continuous linear mapping $V: B\to\widetilde{B}$ holds (with $\kappa=\kappa_B$, $\widetilde{\kappa}=\kappa_{\widetilde{B}}$)

$$\widetilde{\kappa}(V\mathrm{N})\leq\kappa(\mathrm{N}). \tag{5.11}$$

Then the class of sets $\mathcal{O}_\alpha(A, B, \kappa_B)$ (where A and B are running through all Banach spaces) forms a guasi-norm ideal (in the sense of Triebel [32]; see also Pietsch [24], [25] for the case of norm ideals).

Next consider Banach couples $\vec{A}=\{A_0, A_1\}$ and mappings $T\colon \vec{A}\to B$. We have the following interpolation result, which obviously contains Triebel [32], Satz 3 (see also Peetre [22]) and Satz 6.

Theorem 5.2. Assume that $T\in\mathcal{O}_{\alpha_0}(A_0, B)\cap\mathcal{O}_{\alpha_1}(A_1, B)$ and let A be any space of class $\mathcal{C}^K(\theta;\vec{A})$ $(0<\theta<1)$. Then, under the hypothesis of (5.8)—(5.9), follows that $T\in\mathcal{O}_\alpha(A, B)$ where $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$.

Proof. Let $a\in A$ with $\|a\|_A\leq 1$. By definition for every $s>0$ we can write

$$a=a_0+a_1,\ \|a_0\|_{A_0}\leq Cs^\theta,\ \|a_1\|_{A_1}\leq Cs^{\theta-1},$$

where C is a constant. From (5.10) follows that given $t>0$ we can find sets N_0, N_1 with $\kappa(\mathrm{N}_0)\leq t/2$, $\kappa(\mathrm{N}_1)\leq t/2$ and elements $b_0\in\mathrm{N}_0$, $b_1\in\mathrm{N}_1$ such that

$$\left\|\frac{Ta_0}{Cs^\theta}-b_0\right\|_B\leq 2E(t/2, T; A_0, B)\leq Ct^{\alpha_0}.$$

$$\left\|\frac{Ta_1}{Cs^{\theta-1}}-b_1\right\|_B\leq 2E(t/2, T; A_1, B)\leq Ct^{\alpha_1}.$$

Set

$$b=Cs^\theta b_0+Cs^{\theta-1}b_1,\quad \mathrm{N}=Cs^\theta\mathrm{N}_0+Cs^{\theta-1}\mathrm{N}_1.$$

Clearly $b\in\mathrm{N}$ and, by (5.8)—(5.9), $\kappa(\mathrm{N})\leq t/2+t/2=t$. Moreover we ge' $\|Ta-b\|_B\leq Ct^{\alpha_0}s^\theta(1+t^{\alpha_1-\alpha_0}s^{-1})$ or, if we choose s and t so that $s=t^{\alpha_1-\alpha_0}t$ $\|Ta-b\|_B\leq Ct^{\alpha_0}t^{\theta(\alpha_1-\alpha_0)}=Ct^\alpha$. Thus, by (5.10), $E(t, T; A, B)\leq Ct^\alpha$ and

$$T\in\mathcal{O}_\alpha(A, B).\#$$

There is also a linearized version of the above theory. We now take

$$\mathrm{G}=\mathrm{G}_\kappa^\#=\mathrm{G}_\kappa\cap\mathrm{L}(A, B)$$

and write

$$E^{\#}(t, T) = E^{\#}(t, T; A, B, \kappa) = E(t, T; A, B, \mathfrak{G}^{\#}_{\kappa}),$$

$$\mathfrak{O}^{\#}_{\alpha} = \mathfrak{O}^{\#}_{\alpha}(A, B, \kappa) = \mathfrak{O}_{\alpha}(A, B, \mathfrak{G}^{\#}_{\kappa}).$$

In the case of Ex. 5.2 we now have $\mathfrak{O}^{\#}_{\alpha} = S_{\alpha}$ (in the notation of Triebel [32]), while as in the case of Ex. 5.1 $\mathfrak{O}^{\#}_{\alpha} = 0$. The sets $\mathfrak{O}^{\#}_{\alpha}$ are in many respects analogous to the $\mathfrak{O}_{\alpha}$. Thus we have the full analogue of Th. 5.1. As for the interpolation result (see Th. 5.2) some restrictions must be composed. To this end let us investigate the relation between $\mathfrak{O}_{\alpha}$ and $\mathfrak{O}^{\#}_{\alpha}$. The inclusion $\mathfrak{O}^{\#}_{\alpha} \subset \mathfrak{O}_{\alpha}$ is obvious, since we have

$$E(t, T) \leq E^{\#}(t, T). \tag{5.12}$$

For the other direction we may imitating Triebel [32] make the following admitedly artificial looking assumption: (5.13) There exist numbers C and $\varrho \geq 0$ such that for every sufficiently large t and every set $\mathfrak{N}$ with $\kappa(\mathfrak{N}) \leq t$ there exists a (linear) projection P of norm $\leq Ct^{\varrho}$ onto a subspace containing $\mathfrak{N}$ such that $\kappa(PT\{\| a \|_A \leq 1\}) \leq t$ for every T.

Taking now $S = PT$ we obtain every $\mathfrak{N}$ and every $b \in \mathfrak{N}$, using $Pb = b$;

$$\| Ta - Sa \|_B \leq \| Ta - b \|_B + \| P(b - Ta) \|_B$$

$$\leq \| Ta - b \|_B + Ct^{\varrho} \| Ta - b \|_B \leq C_1 t^{\varrho} \| Ta - b \|_B.$$

It follows that

$$E^{\#}(t, T) \leq Ct^{\varrho} E(t, T) \tag{5.14}$$

which implies $\mathfrak{O}_{\alpha} \subset \mathfrak{O}^{\#}_{\alpha-\varrho}$ if $\alpha \geq \varrho$. It is now easy to prove an interpolation theorem of the type of th. 5.2 which then as a special case will cover Triebel [32], Satz 8: Starting with $T \in \mathfrak{O}^{\#}_{\alpha}(A_0, B) \cap \mathfrak{O}^{\#}_{\alpha}(A_1, B)$ we arrive at the conclusion $T \in \mathfrak{O}^{\#}_{\alpha-\varrho}(A, B)$ if $\alpha \geq \varrho$. Instead of (5.13) we may also make the assumption (5.15) $\vec{A}$ is linearizable (in the sense of Peetre [21]; see remark 3.5 above) we then get as the end result $T \in \mathfrak{O}^{\#}_{\alpha}(A, B)$ (instead of the previous $T \in \mathfrak{O}^{\#}_{\alpha-\varrho}(A, B)$).

There is furthermore a dual theory. The role of A and B will now be interchanged. Instead of κ we give a setfunction λ in A. The property corresponding to (5.8) is

$$\lambda(\mathfrak{M}_1 \cap \mathfrak{M}_2) \leq \lambda(\mathfrak{M}_1) + \lambda(\mathfrak{M}_2). \tag{5.16}$$

Corresponding to our previous $\mathfrak{O}_{\alpha}$ one can define sets $Q_{\alpha} = Q_{\alpha}(A, B, \lambda)$ and one can prove analogous of Th. 5.1 and Th. 5.2. When specializing one can get $Q_{\alpha} = G_{\alpha}$ (in the notation of Triebel [32]).

Remark 5.1. We have been able to include in our discussion all interpolation results in Triebel [32] with the exception of Satz 4 (and Satz 5, which is, however, just the conjunction of Satz 3 and Satz 4), which does net seem to fit into the present framework.

Finally let us point out that the families $\mathfrak{O}_{\alpha}$ and $\mathfrak{O}^{\#}_{\alpha}$ themselves are closed for interpolation, e. g. we have

$$(5.17) \qquad (\mathsf{O}_{\alpha_0}, \mathsf{O}_{\alpha_1})^{\kappa}_{\tau\infty} = \mathsf{O}_{\alpha} \text{ if } \frac{\alpha}{1+\alpha} = (1-\tau)\frac{\alpha_0}{1-\alpha_0} + \frac{\alpha_1}{1+\alpha_1}$$

$$\text{or } \theta = (1-\tau)\theta_0 + \tau\theta_1, \text{ with } \theta = \frac{\alpha}{1+\alpha} \text{ etc.}$$

and similarly for $\mathsf{O}^{\#}_{\alpha}$. This follows from general results on interpolation of normed Abelian groups (Peetre-Sparr [23]). The same result holds also for the more general classes $\mathsf{O}^{\#}_{\alpha p} = (\mathsf{G}^{\#}_{\kappa}, \mathsf{F}(A, B))^{\kappa}_{\theta q}$ and $\mathsf{O}^{\#}_{\alpha p} = (\mathsf{G}^{\#}_{\kappa}, \mathsf{F}(A, B))^{\kappa}_{\theta q}$ with $\theta = \frac{\alpha}{1+\alpha}$, $q = \theta p$. The special case $\alpha = 1/p$ is of particular interest. Let us write $\mathsf{T}_p = \mathsf{O}_{1/p,p}$ and $\mathsf{T}^{\#}_p = \mathsf{O}^{\#}_{1/p,p}$. If κ is as in Ex. 5.2 and if $A = B =$ Hilbert space, then $\mathsf{T}_p = \mathsf{T}^{\#}_p =$ trace class (see Gohberg-Krein [8]). That the trace classes are closed for interpolation was proven by Triebel [33] (see also Mitiagin [12], Cotlar [4]). Thus we have here a farreaching generalization of that result. This indicates that there may be hopes of extending parts of the general theory of normed ideals in Hilbert space (as presented e. g. in Gohberg-Krein [8]) to the more general setting of Banach space. To what extent this is possible, the future will reveal.

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