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ON THE DIVERGENCE PROPERTIES OF LAGRANGE INTERPOLATION

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Summary. Theorem 1. Let $f(x)$ be a continuous function with modulus of continuity $\omega(\delta)$, and $S_n(x)$ be quasi-orthogonal or asymptotically orthogonal polynomials, belonging to a weight function $w_0(x)$, then $|f(x)-L_n(f,x,S_n)| \leqq C\lambda_n(P_n)\omega(f, 1/n)$, where λ_n are the Lebesgue-constants and $L_n(f, x, P_n)$ the Lagrange interpolating polynomials.

Theorem 2. There exist two continuous functions $f(x)$ and $h(x)$ for which the arithmetical means of Lagrange interpolation polynomials based on the zeros of first kind Tchebisheff polynomials are divergent in the points of the set H resp. almost everywhere in $[-1; +1]$, where $H: \{x_0; x_0 = \cos(p/q), p \equiv q \equiv 1 \pmod 2\}$.

Throughout this paper we shall use the well-known notation for the Lagrange interpolation:

$L_n(f, x, S_n)$, $n=1, 2, \ldots$, are the Lagrange interpolating polynomials of degree n, belonging to $f(x)$ and based on the zeros of the polynomials S_n x_k^n, $k=1, 2, \ldots, n$: $n=1, 2, \ldots$, are the found-points of the polynomials $\lambda_n(S_n)$ are the Lebesgue-constants, belonging to the Lagrange interpolation based on the zeros of S_n.

1. Let $P_n(x)$ be orthogonal polynomials with the weight function, $w_0(x)$ in the interval $[a, b]$, where $w_0(x)$ is a classical weight function, e. g.

if $w_0(x)=(1-x)^\alpha(1+x)^\beta$; $[a, b]\equiv[-1, +1]$ the $P_n(x)$ are Jacobi polynomials;

if $\omega_0(x)=e^{x^2/2}$; $[a, b]\equiv(-\infty, +\infty)$ the $P_n(x)$ are Hermite polynomials;

if $w_0(x)=x^\alpha e^x$; $[a, b]\equiv[0, +\infty]$ the $P_n(x)$ a Laguerre polynomials etc.

Let us denote by $Q_n(x)$ and call quasi-orthogonal polynomials relave to $P_n(x)$ the polynomials, for which

$$Q_n(x)=P_n(x)+AP_{n-1}(x)-BP_{n-2}(x).$$

holds, where A and B are nonnegative constants.

The investigations of such polynomials are connected with the names of L. Fejér and G. Freud.

Further, let us denote by $R_n(x)$ and call assymptotically orthogonal polynomials belonging to $P_n(x)$ the polynomials orthogonal respectively to the weight-function $w(x)$ for which

$$A'w_0(x) \leq w(x) \leq B'w_0(x)$$

holds, where A' and B' are positive constants.

Let us consider the approximation errors of Lagrange Interpolation based on the zeros of $Q_n(x)$ and $R_n(x)$ polynomials, i. e. $S_n \equiv Q_n$ or $S_n \equiv R_n$. In these both cases we can prove the following theorem.

Theorem 1. Let $f(x)$ be a continuous function with modulus of continuity ω and S_n as above, then

$$|f(x) - L_n(f, x, S_n)| \leq C\lambda_n(P_n)\omega\left(f, \frac{1}{n}\right)$$

holds, where C is a positive constant depending on A and B.

In the special case, if $P_n(x)$ are the Jacobi polynomials, this estimate is correct considering the whole function-class, while in the first case $(S_n \equiv Q_n)$ if $A = B = 0$, then $Q_n(x)$ are equal to $P_n(x)$ and in the second case $(S_n \equiv R_n)$ $R_n(x)$ are equal to $P_n(x)$ and we gave an example for a continuous function for which the lower and upper estimates are equal in order in the case of Jacobi polynomials.

For the proof of these theorems we used in the first case some results of J. Shohat and in the second case a method which is originated by G. Grünwald and P. Turán.

2. In this part of our paper we should like to present two results of ours about a problem which is connected with the paper of G. Grünwald and P. Erdös. That paper was published in 1938 and contains an error.

The statement of the theorem of that paper was as follows:

There exists a continuous function $f(x)$ for which the arithmetical means of the Lagrange Interpolation polynomials based on the zeros of first kind Tchebisheff polynomials are divergent almost everywhere in $[-1, +1]$, i. e

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{i=1}^{N} L_k(f, x, T_n) = \infty.$$

Having found the error in 1950 P. Erdös corrected the above result to the following weaker one:

There exists a continuous function $f(x)$ for which the arithmetical means of the absolute value of Lagrange interpolation polynomials based on the zeros of the first kind Tchebisheff polynomials are divergent almost everywhere in $[-1, +1]$, i. e.

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{i=1}^{N} | L_k(f, x, T_n) | = \infty.$$

Considering this subject we prove the following theorem.

Theorem 2. There exists a continuous function $f(x)$ for which the arithmetical means of Lagrange interpolation polynomials based on the zeros of first kind Tchebisheff polynomials are divergent in the points of the set, H i. e.

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{i=1}^{N} L_k(f, x_0, T_n) = \infty, \quad x_0 \in H,$$

where $H = \left\{x_0 : x_0 = \cos\frac{p}{q},\ p \equiv q \equiv 1 \pmod 2\right\}$.

For the proof of this theorem we used some Lemmas of P. Erdös. On the other hand we have succeeded in changing the construction of the function from the quoted Erdös Grünwald paper in such a way that it satisfies the original, stronger statement too, i e. we can prove the following statement:

There exists a continuous function $f(x)$ for which

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{i=1}^{N} L_k(f, x, T_n) = \infty$$

almost everywhere in $[-1, +1]$.

The paper about the construction of the function of Theorem 2 and the modification of the Erdös-Grünwald function will be submitted to the *Acta Math. Acad. Sci. Hung.*

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