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SOLUTION OF LANDAUS' PROBLEM CONCERNING HIGHER DERIVATIVES ON THE HALFLINE*

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For J. L. Walsh on his 75th birthday

Summary. Let $f(x)$ be defined in the interval $I=[0,\infty)$ and let $M_\nu=\sup_{x\in I}|f^{(\nu)}(x)|$, $\nu=1, 2, \ldots, n$; $n\geqq 2$. It is assumed that M_0 and M_n are finite, more precisely that $f^{(n-1)}(x)$ is absolutely continuous in I and that M_n is the essential supremum of $f^{(n)}(x)$. Then also M_ν, $0<\nu<n$, is finite and it is known that inequalities of the form $M_\nu\leqq C_{n,\nu}M_0^{(\nu/n)}M_n^{1-(\nu/n)}$ hold. In the present paper we solve the so called Landau problem of determining the best (least) constants $C_{n,\nu}$ in the inequality.

1. Formulation of the problem. We are here concerned with real-valued functions $f(x)$ defined on a closed interval I, finite or infinite. We denote by $\|f\|=\|f\|_\infty$ the supremum, or essential supremum, of $f(x)$ on I. If the nth derivative $f^{(n)}(x)$ is mentioned, then we assume that $f^{(n-1)}(x)$ is absolutely continuous on I and that $\|f^{(n)}\|_\infty$ is finite.

In 1913 Landau [3] formulated and solved some extremum problems concerning twice differentiable functions. From among his results we single out the following.

Theorem I (Landau). If

$$\|f\|=1,\ \|f''\|=4 \text{ in } I=[0,\infty), \tag{1}$$

then

$$\|f'\|\leqq 4 \text{ in } [0,\infty). \tag{2}$$

The constant 4 in (2) is best possible.

Landau derives Theorem I from the following

Theorem II (Landau). If

$$\|f\|\leqq 1,\ \|f''\|\leqq 4 \text{ in } I_1=[0,1], \tag{3}$$

then

$$|f'(0)|\leqq 4. \tag{4}$$

The interval I_1 can not be replaced by any smaller interval $[0,\theta]$, nor can the constant 4 in (4) be replaced by any smaller number.

Theorem II implies Theorem I, for if we apply Theorem II to the function $f(x+x_0)$, $x_0 \geqq 0$, we obtain that $|f'(x_0)| \leqq 4$, hence (2) holds.

In 1955 Matorin [4] generalized Landau's Theorem I as follows

Theorem III (Matorin). Let $T_n(x)$ denote the Chebyshev polynomial. If $\|f\|=1$, $\|f^{(n)}\|=2_n^{n-1}n!$ in $I=[0,\infty[$, then

(5) $$\|f^{(\nu)}\| \leqq T_n^{(\nu)}(1) \text{ in } [0,\infty), \quad \nu=1, 2, \ldots, n-1.$$

Since $T_n^{(n)}(x)=2^{n-1}n!$ and $T_n'(1)=n^2$, we see that Theorem III reduces to Theorem I if $n=2$. Matorin does not state a similar generalization of, Landau's Theorem II. This we do now and state the result as

Theorem 1. Let $\alpha=\cos(\pi/n)$ be the largest zero of $T_n'(x)$. If $\|f\| \leqq 1$ $\|f^{(n)}\| \leqq 2^{n-1}n!$ in $[0, 1+\alpha]$ then

(6) $$|f^{(\nu)}(0)| \leqq T_n^{(\nu)}(1), \quad \nu=1,\ldots, n-1.$$

The interval $[0, 1+\alpha]$ can not be replaced by any smaller interval $[0, 1+\alpha']$ $(\alpha'<\alpha)$, nor can the constant on the right side of (6), for any ν, be replaced by any smaller constant. Moreover, if we have the equality sign in (6) for some ν, then we have equality for all ν, and this happens if and only if $f(x)=\pm T_n(x-1)$ in $[0, 1+\alpha]$.

For $n=2$ our Theorem 1 reduces to Landau's Theorem II with some additional information. Clearly, Theorem 1 implies Matorin's Theorem III, just as Theorem II implies Theorem I.

Matorin observes that his Theorem III gives the best constants in (5) if $n=2$ and also if $n=3$. The main concern of the present paper is to find the best constants on the right side of (5) if $n \geqq 4$. We reformulate this question as follows.

Problem 1 (Generalized Landau Problem). For the class of functions $\mathfrak{F}_n=\{f(x);\ \|f\|=1, \|f^{(n)}\|=2^{n-1}n! \text{ in } I=[0,+\infty)\}$ we are to determine the Landau constants $L_n^{(\nu)}=\sup_{f\in F_n}\|f^{(\nu)}\|$, $\nu=1, 2, \ldots, n-1$, and we are to find the extremizing functions $f(x)$, if such exist, for which $\|f^{(\nu)}\|=L_n^{(\nu)}$. A more familiar (but only apparently more general) formulation of this problem is as follows. If we write $\|f^{(n)}\|=M_n$, $\nu=0,1,\ldots n$, in $I=[0,\infty)$, and if we assume M_0 and M_n to be finite, then inequalities of the form

(7) $$M_\nu \leqq C_{n,\nu} M_0^{1-\frac{\nu}{n}} M_n^{\frac{\nu}{n}}, \quad \nu=1,\ldots, n-1,$$

hold, with appropriate numerical constants $C_{n,\nu}$ as first shown by A. Gorny (for references see [7]). The generalized Landau problem is to determine in (7) the best constants $C_{n,\nu}$ and also the functions $f(x)$, if any, for which we have equality in (7).

The problem obtained if in either of the two equivalent formulations we replace the interval $I=[0,\infty)$ by $R=(-\infty,\infty)$, was solved in 1939 by Kolmogorov [2]. He showed that the best constants $C_{n,\nu}=C_{n,\nu}(R)$ are obtained with equality in (7) and this for all ν simultaneously, for certain functions $\mathfrak{E}_n(x)$ which we propose to call Euler splines of degree n. This term seems justified because $\mathfrak{E}_n(x)$ is a spline function whose polynomial components are essentially the Euler polynomials $E_n(x)$.

The question of unicity of the extremizing functions in Kolmogorov's case of R has to our knowledge not been investigated before. Using spline function theory, the first named author will publish elsewhere a new proof of Kolmogorov's result that will also show that in a certain appropriate stricter sense the functions

$$f(x)=a\mathfrak{E}_n(bx+c)$$

are the only extremizing functions if $n \geqq 3$.

2. The Chebyshev-Euler splines and their properties. We return to Landau's Problem 1. For its solution we need the following new spline generalization of the Chebyshev polynomials.

Let n and k be natural numbers, $-1=\xi_0<\xi_1<\ldots<\xi_k<\xi_{k+1}=+1$ and let

$$S(x)=S_{n,k}(x)=S_{n,k}(x;\xi_i) \tag{8}$$

be a spline function in $[-1,1]$, of degree n, and having the simple knots $\xi_1,\ldots,\xi_k$. This means

1° $S(x)\in C^{n-1}[-1,1]$,

2° The restriction of $S(x)$ to the interval $[\xi_i, \xi_{i+1}]$ $(i=0,\ldots, k)$ is a polynomial of degree at most n.

We shall also assume that $S(x)$ is a perfect spline, using a term introduced by G. Glaeser [1]. A spline function $S(x)$ is said to be perfect, provided that the terms of degree n, in each polynomial component, are of the form $\pm cx^n$ (with the same constant $c>0$), the sign of the coefficient changing from one interval (ξ_i, ξ_{i+1}) to the next. The Euler spline $\mathfrak{E}_n(x)$ is a perfect example of a perfect spline having infinitely many knots. Here we are discussing its finite analogues in $[-1, 1]$.

We assume concerning the function (8) that $S^{(n)}(x)=(-1)^i\, 2^{n-1}n!$ in $\xi_i<x<\xi_{i+1}$, $i=0,\ldots, k$. If we define

$$u_+=\begin{cases} u & \text{if } u\geqq 0,\\ 0 & \text{if } u<0,\end{cases}$$

we can incorporate all the requirements so far mentioned by writing

$$S(x)=2^{n-1}x^n+\sum_{i=1}^{k}(-1)^i\, 2^n(x-\xi_i)_+^n+\sum_{0}^{n-1}a_\nu x^\nu \text{ in } -1\leqq x\leqq 1. \tag{9}$$

This function depends on a total of $n+k$ parameters, linearly on the a_ν, non-linearly on the ξ_i.

Definition 1. We define the function $T_{n,k}(x)$ as the function of the form (9) having least L_∞-norm in $[-1,1]$ all $n+k$ parameters being free to vary. Thus $\|T_{n,k}\|\leqq\|S\|$ for any S of the form (9). $T_{n,k}(x)$ being a perfect spline of least deviation from zero, we propose to call it the Chebyshev-Euler spline of degree n with k knots.

Observe that if k should be zero, then the first sum in (9) drops out and we obtain the Chebyshev polynomial. We may therefore write $T_{n,0}(x)=T_n(x)$. Existence, unicity and a few descriptive properties of $T_{n,k}(x)$ are as follows.

Theorem 2. There is a unique $T_{n,k}(x)$ and it is a perfect spline of degree n with k simple knots. $T_{n,k}(x)$ has precisely $n+k+1$ points of equioscillation. This means that if

$$\varrho_{n,k}=\| T_{n,k} \|, \tag{10}$$

then there are $n+k+1$ points

$$-1=x_1<x_2<\ldots<x_{n+k}<x_{n+k+1}=+1 \tag{11}$$

such that

$$T_{n,k}(x_i)=(-1)^{n+i-1}\varrho_{n,k}, \quad i=1,\ldots, \ n+k+1, \tag{12}$$

and $|T_{n,k}(x)|<p_{n,k}$ if $x\neq x_i$ for all i.

Moreover, the equioscillation property (10), (11), (12) characterizes $T_{n,k}(x)$ within the class of perfect splines of the form (9).

Theorem 3. $T_{n,k}(x)$ is an even function if $n+k$ is even, and it is an odd function if $n+k$ is odd.

Les $\mu=[n/2]$ have its usual arithmetic meaning. Between the extreme points (11) of $T_{n,k}(x)$ and its knots $\xi_1,\ldots,\xi_k$, the following inequalities hold

$$\begin{aligned} &x_{\mu+1}<\xi_1<x_{n+2-\mu}, \\ &x_{\mu+2}<\xi_2<x_{n+3-\mu}, \\ &\quad\cdots\cdots\cdots \\ &x_{\mu+k}<\xi_k<x_{n+k+1-\mu}. \end{aligned} \tag{13}$$

The deviations (10) satisfy $1=\varrho_{n,0}>\varrho_{n,1}>\ldots>\varrho_{n,k}>\varrho_{n,k+1}>\ldots>0$ and $\lim_{k\to\infty}\varrho_{n,k}=0$.

Just as the Chebyshev monosplines of degree n and k knots in [-1, 1 were easily constructed for $n=2$, 3 and 4, and all values of k (see [6, §1]), so the Chebyshev-Euler splines are readily found for $n=2$ and $n=3$ and all values of k. Without further details we refer the reader to Figure 1 for

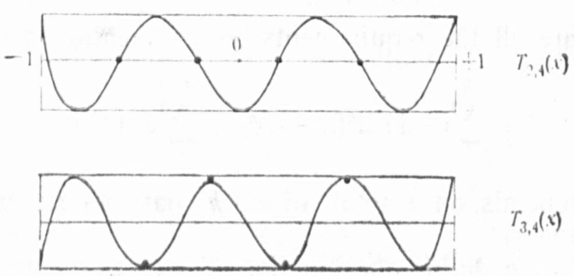

Fig. 1

sketches of the graphs of $T_{2,4}(x)$ and $T_{3,4}(x)$. In their central portions they are identical, except for changes of scale and origin, to corresponding sections of the graphs of the Euler splines $\mathfrak{E}_2(x)$ and $\mathfrak{E}_3(x)$, respectively. The knots are marked by heavy dots. However, the construction of $T_{4,k}(x)$ is already difficult (see §6 below).

3. Solution of the generalized Landau problem. We shall now apply the Chebyschev-Euler spline $T_{n,k}(x)$ to Landau's Problem 1. This will be done by first generalizing our Theorem 1 from $T_n(x)=T_{n,0}(x)$ to $T_{n,k}(x)$. For this purpose it is convenient to perform a preliminary change of scale and origin. We consider $f(x)=aT_{n,k}(bx)$, $a>0$, $b>0$, and determine a and b such that $\|f\|=1$ and $\|f^{n)}\|=2^{n-1}n!$ in $[-b^{-1}, b^{-1}]$. This is evidently achieved by choosing

(14) $$a=a_{n,k}=(\varrho_{n,k})^{-1},\quad b=b_{n,k}=(\varrho_{n,k})^{1/n}.$$

We finally translate $f(x)$ by b^{-1} and define

(15) $$S_{n,k}(x)=a\,T_{n,k}(bx-1)$$

in the interval $0\leq x\leq 2b^{-1}$. This is a perfect spline of degree n and k knots in $[0, 2b^{-1}]$. Using the same symbols x_i as in Theorem 2 but with a different meaning, we denote by

(15a) $$0=x_1<x_2<\ldots<x_{n+k}<x_{n+k+1}=2(b_{n,k})^{-1}$$

the $n+k+1$ points of equioscillation of $S_{n,k}(x)$. By construction we therefore have $S_{n,k}(x_i)=(-1)^{n+i-1}$, $i=1,\ldots,n+k+1$, and $|S(x)|<1$ in all other points of $[0, 2b^{-1}]$.

Our generalization of Theorem 1 is as follows.

Theorem 4. If

(16) $$\|f\|\leq 1,\quad \|f^{(n)}\|\leq 2^{n-1}n!$$

in the interval $0\leq x\leq x_{n+k}$, then

(17) $$|f^{(\nu)}(0)|\leq |S^{(\nu)}_{n,k}(0)|,\quad \nu=1,\ldots,n-1.$$

Here the interval $[0, x_{n+k}]$ can not be replaced by any shorter interval $[0, \theta x_{n+k}]$, $0<\theta<1$, nor can any of the constants on the right side of (17) be replaced by smaller constants.

What follows applies only if $n\geq 3$: If in one of the inequalities (17) we have the equality sign, then we have equality in all of them, and this happens if and only if

(18) $$f(x)=\pm S_{n,k}(x) \text{ in } [0, x_{n+k}].$$

Theorem 4 is a key result : Just as Theorem 1 implied Matorin's Theorem III so Theorem 4 now yields

Theorem 5. If

(19) $$\|f\|=1,\quad \|f^{(n)}\|=2^{n-1}n! \text{ in } [0, \infty],$$

which means that $f(x)\in\mathfrak{F}_n$, then

(20) $$\|f^{(\nu)}\|\leq |S^{\nu}_{n,k}(0)|,\quad \nu=1,\ldots,n-1.$$

The bounds in (20) are easily computed from (14) and (15) and we find that

(21) $$|S^{(\nu)}_{n,k}(0)|=ab^{\nu}|T^{(\nu)}_{n,k}(1)|=\varrho_{n,k}^{-1+(\nu/n)}|T^{(\nu)}_{n,k}(1)|,$$

and in particular, for $k=0$ when $\varrho_{n,0}=1$, we find Matorin's bounds $|S_{n,0}^{(\nu)}(0)|=|T_{n,0}^{(\nu)}(1)|=T_n^{(\nu)}(1)$. However, now we are in a position to improve on Matorin's bounds indefinitely, by letting k increase, and to obtain the Landau constants in the limit.

We dispose first of the simple cases when $n=2$ or $n=3$. In each of these cases the $S_{n,k}(x)$, $k=0, 1, 2, \ldots$, are successively extensions of each other. More precisely $S_{n,k+1}(x)=S_{n,k}(x)$ in $[0,x_{n+k}^{(k)}]$ if $n=2$ or 3, where $x_{n+k}^{(k)}$ is the $(n+k)$th extreme point of $S_{n,k}(x)$. This can be nicely seen in Figure 1 if we interpret the two curves as the graphs of $S_{2,4}(x)$ and $S_{3,4}(x)$, respectively: In passing from $S_{2,4}$ to $S_{2,5}$, the last zero of $S_{2,4}$ simply becomes the last knot of $S_{2,5}$. In passing from $S_{3,4}$ to $S_{3,5}$, the last maximum of $S_{3,4}$ becomes the last knot of $S_{3,5}$.

These results show that

(22) $$L_2'=|S_{2,k}'(0)|,\ k\geqq 0,$$

$$L_3'=|S_{3,k}'(0)|,\quad L_3''=|S_{3,k}''(0)|,\quad k\geqq 0,$$

are the values of the corresponding Landau constants. We also see, for $k=0$, why Matorin's bounds gave their exact values.

Matters are different if $n\geqq 4$. Here we have

Theorem 6. If $n\geqq 4$, $0<\nu<n$, then the sequence $|S_{n,k}^{(\nu)}(0)|$, $k=0,1,\ldots$, s strictly decreasing, and the value of the Landau constant $L_n^{(\nu)}$ of Problem is given by

(23) $$L_n^{(\nu)}=\lim_{k\to\infty}|S_{n,k}^{(\nu)}(0)|.$$

As one would expect, the functions $S_{n,k}(x)$ converge, as $k\to\infty$, to the extremizing function.

Theorem 7. The limit relation $\lim_{k\to\infty} S_{n,k}(x)=\mathfrak{E}_n^+(x)$ holds uniformly in every finite interval $[0, A]$. The limit function $\mathfrak{E}_n^+(x)$ is a perfect spline of degree n having infinitely many knots η_i such that $0<\eta_1<\eta_2<\ldots$, $\eta_i\to\infty$. Moreover $\mathfrak{E}_n^+(x)\in\mathfrak{F}_n$ and we may express the Landau constants (23) by

(24) $$L_n^{(\nu)}=|\mathfrak{E}_n^{+(\nu)}(0)|,\quad 0<\nu<n.$$

Theorem 8. We now assume that $n\geqq 3$. If $f(x)\in\mathfrak{F}_n$ and if the equality $\|f^{(\nu)}\|=L_n^{(\nu)}$ holds for one value of ν, $0<\nu<n$, then it holds for all such ν and this happens if and only if

(25) $$f(x)=\pm\mathfrak{E}_n^+(x)\quad \text{in } [0,\infty).$$

We propose to call the functions $\mathfrak{E}_n^+(x)$ the one-sided Euler splines

Our expressions (24) for the Landau constants being far from explicit Stečkin's asymptotic estimates retain their full importance (see [7]). Possibly these estimates can now be improved on the basis of some of our results.

4. A kinematic interpretation. In Theorem 8 we may evidently relax our condition $f(x)\in\mathfrak{F}_n$, hence (19), and replace it by the inequalities

(26) $$\|f\|\leqq 1,\ \|f^{(n)}\|\leqq 2^{n-1}n! \text{ in } [0,\infty).$$

Following Newton, we think of the derivatives as "fluxions" by interpreting the variable x as time in the interval $[0, \infty)$. We may then think of the function $f(x)$, satisfyng (26), as representing the motion of a point $f=f(x)$. The point f is to move in the interval $-1 \leq f \leq 1$ such as to have continuous ν-velocities $f^{(\nu)}(x)$ $(0<\nu<n)$, while its n-velocity $f^{(n)}(x)$ should never exceed $2^{n-1}n!$ in absolute value. The problem then asks for the largest ν-velocity $\|f^{(\nu)}\|$ consistent with these restrictions. Our results (Theorems 7 and 8) show that this largest ν-velocity is the entry-ν-velocity $L_n^{(\nu)}$. Moreover, that this maximal entry-ν-velocity $L_n^{(\nu)}$ can be realized only by the two motions (25): The point then moves at all times with the maximal n-velocity $\pm 2^{n-1}n!$ that switches sign at the times $x=\eta_1, \eta_2, \ldots$

We owe to Walter Rudin the following interesting remark made orally during a seminar lecture on the subject by one of the authors. Rudin observed that all our results, and their proofs, very likely also hold in the complex field. As a matter of fact they do, and without any additional complications of any kind. This means that $f(x)$ is now assumed to be complex-valued throughout our discussion. Then all results remain valid with the difference that now the extremizing functions $f(x)$ in Theorem 8 are not only the two functions (25) but the entire family

(27) $$f(x)=\alpha \mathfrak{E}_n^{+}(x), \ \alpha \text{ constant}, \ |\alpha|=1.$$

Returning to our kinematic interpretation, we may now interpret the complex-valued $f(x)$ as the motion of a point in the unit circle

(28) $$|f| \leq 1;$$

during the time interval $[0, \infty)$ during which its vectorial n-velocity should never exceed $2^{n-1}n!$ in magnitude. Our results show that the motion of largest entry-ν-velocity $L_n^{(\nu)}$ corresponds to the motion (27) on a diameter αu, $-1 \leq u \leq 1$, of the circle (28). It seems surprising that the moving point can make no use of its enlarged domain of variability to further increase its entry-ν-velocity.

Very likely the same result holds if we should replace the circle (28) by any plane closed convex domain having a diameter $AB=2$, the term "diameter" being taken in the distance-theoretic sense. This is evidently so if the closed circle U, of diameter AB, should contain the given domain.

Mentioning the above results concerning the unit circle to W. Walter, he asked if they could be generalized to the unit sphere of the higher dimensional euclidean space E_m. In fact our theorems 4 and 5 immediately generalize to vector functions $f(x)$ from $[0, \infty)$ into E_m. The only difference is that the extremizing functions (25) are obtained by replacing the factor ± 1 by $\alpha \in E_m$ such that its magnitude $|\alpha|=1$. The answer to Walter's question is therefore affirmative.

There is however a very simple geometric argument for extending our results from E_1 to E_m. For simplicity let $m=2$. Let U be again the unit circle in the plane and let us assume that we have a motion in U such that

(29) $$|f| \leq 1, \ |f^{(n)}| \leq 2^{n-1} n!.$$

Furthermore let us assume that the entry-velocity ν_0 is such that

(30) $$|\nu_0| \geq L_n' = |\mathfrak{E}_n'(0)|.$$

We claim that 1. The vector ν_0 must be orthogonal to the circle, 2. That in (30) we must have equality.

Indeed, let us assume that ν_0 does not point towards the center 0 and let e be the line carrying ν_0. Let $f_1(x)$ be the orthogonal projection of $f(x)$ on e. Now $f_1(x)$ is a motion on the segment A_1B_1 of length 2 whose velocities satisfy all our conditions. However, since ν_0 is not the entry-1-velocity at A_1 we have a contradiction, because A is in the interior of A_1B_1. This proves both statements and ν_0 lies on the diameter AB.

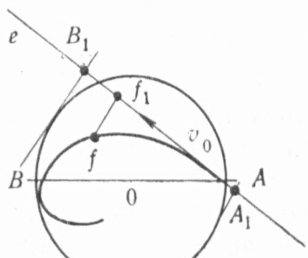

Fig. 2

Let $f(x)=f_1(x)+ih(x)$, $0\leq x<\infty$, represent our motion. But then we must have

$$f_1(x)=\mathfrak{E}_n^+(x)\quad [0,\infty). \tag{31}$$

However, we know that $|\mathfrak{E}_n^{+(n)}(x)|=2^{n-1}n!$ for all $x\neq\eta_i$. This and the second inequality (29) imply that $h^{(n)}(x)=0$ almost everywhere, hence $h(x)$ can only be a polynomial of degree $\leq n-1$. On the other hand, from Figure 2 it is clear that $h(x)=0$ for such x where (31) reaches its extreme values ± 1. These are infinitely many district times and therefore $h(x)=0$ for all x. This proof applies as well for the other ν-velocities and all dimensions m.

5. Proofs of a few key results are sketched. Complete proofs of our theorems will appear elsewhere. Here we sketch the main ideas concerning a few of the theorems.

We study perfect splines of the form (9) and observe that they can have at most $n+k-1$ distinct relative extrema. We say that $S(x)$ is an oscillatory perfect spline (O. P. S.) provided that it has exactly this maximal number of $n+k-1$ relative extrema. Let $e_1, e_2, \ldots, e_{n+k-1}$ be the critical values of $S(x)$, i. e. the values of $S(x)$ at its extreme points in their natural order of increasing x. It is shown that the O. P. S. $S(x)$ is uniquely defined, up to horizontal translations, if these critical values e_i are preassigned at will. Of course, we must have $e_1<e_2$, $e_2>e_3$, $e_3<e_4, \ldots$ if n is even, and the opposite inequalities if n is odd.

The structure of O. P. S. of degree $n=2$ is especially simple and proofs go by induction starting with $n=2$ when the result concerning preassigned extra values is perfectly trivial.

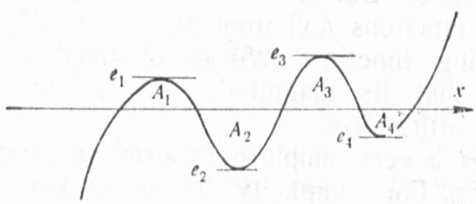

Fig. 3

Let us assume that we already know that for a certain value of n the extreme values can be preassigned. A lemma then shows that these extreme values e_i can be so chosen with alternating signs, that the areas $A_1, A_2, \ldots,$

A_{n+k-1} of the arches of $S(x)$ (Figure 3) have preassigned positive values. This lemma is established by a topological argument equivalent to a version of the fixed-point theorem. By one integration we then obtain the theorem concerning preassigned extreme values for the next degree $n+1$.

Another key result is Theorem 4 (of which Theorem 1 is a special case corresponding to $k=0$). Let us sketch the proof of the inequality (17) for $\nu=1$. We also assume that $n\geqq 4$, as the case $n=3$ requires a special discussion. We summarize the relevant properties of the function $S(x)=S_{n,k}(x)$ defined by (15): Its extreme points are given by (15a) and let $\xi_1,\ldots,\xi_k$ denote its knots. Then

$$S(x_i)=(-1)^{n+i-1},\quad i=1,\ldots,n+k, \tag{32}$$

$$S^{(n)}(x)=(-1)^j 2^{n-1}\, n! \text{ in } \xi_j<x<\xi_{j+1},\quad j=0,\ldots,k, \tag{33}$$

where we write $\xi_0=0=x_1$, and $\xi_{k+1}=x_{n+k}$.

We now consider the family $\mathfrak{S}=\{s(x)\}$ of spline functions $s(x)$ of degree $n-1$ having as simple knots the k knots $\xi_1,\ldots,\xi_k$ of $S_{n,k}(x)$. This family depends on $n+k$ parameters and one would expect that there exists a unique $s(x)\in\mathfrak{S}$ which is a solution of the interpolation problem

$$s(x_i)=y_i,\quad i=1,\ldots,n+k. \tag{34}$$

This is indeed the case: In the inequalities (13) we have $\mu=[n/2]\geqq 2$ and (13) imply a fortiori the inequalities

$$\begin{aligned} &x_1<\xi_1<x_{n+1},\\ &x_2<\xi_2<x_{n+2},\\ &\cdots\cdots\cdots\\ &x_k<\xi_k<x_{n+k}. \end{aligned} \tag{35}$$

These inequalities (35) are the necessary and sufficient conditions for the unique solution $s(x)\in\mathfrak{S}$ of the problem (34), the y_i being preassigned (see [5, Theorem 2, 258]).

Let

$$f(x)=\sum_{1}^{n+k} f(x_\nu)\,L_\nu(x)+Rf \tag{36}$$

be the corresponding spline interpolation formula, the $L_\nu(x)$ being its fundamental functions. By construction we see that the interpolation formula (36) is exact if $f(x)\in\mathfrak{S}$. This means that

$$Rf=0 \text{ if } f(x)\in\mathfrak{S} \tag{37}$$

and in particular

$$Rf=0 \text{ if } f(x)\in\pi_{n-1}. \tag{38}$$

From (36) we obtain the approximate differentiation formula

$$f'(0)=\sum_{1}^{n+k} f(x_\nu)L'_\nu(0)+Rf, \tag{39}$$

whose remainder Rf also enjoys the properties expressed by (37) and (38). Setting $A_\nu=L'_\nu(0)$ we may apply Peano's theorem and write (39) in the exact form

$$(40) \qquad f'(0)=\sum_{1}^{n+k} A_\nu f(x_\nu)+\int_{0}^{x_{n+k}} K(x) f^{(n)}(x)\, dx.$$

Here $K(x)$ is a certain spline function of degree $n-1$ having as knots the nodes $x_1, x_2, \ldots, x_{n+k}$ of our formula.

At this point we must start omitting proofs and simply state the following properties of the coefficients and kernel of the formula (40):

$$(41) \qquad (-1)^\nu A_\nu > 0, \quad \nu=1, \ldots, n+k;$$

$$(42) \qquad (-1)^{n+j-1} K(x) > 0 \text{ in } \xi_j < x < \xi_{j+1}, \quad j=0, \ldots, k,$$

where, as in (33), $\xi_0=0$, $\xi_{k+1}=x_{n+k}$. That $K(\xi_j)=0$ follows very simply from (37). However, the precise sign properties (42) require the variation diminishing properties of spline functions if represented in terms of the so-called B-splines.

If we substitute $S(x)=S_{n,k}(x)$ into (40) and use the properties (32), (33), (41) and (42) we obtain the relation

$$|S'(0)|=\sum_{1}^{n+k} |A_\nu|+2^{n-1} n! \int_{0}^{x_{n+k}} |K(x)|\, dx.$$

For this function $f(x)=S(x)$ the formula (40) is "taut". If $f(x)$ is any function satisfying the conditions (16) then (40) shows that

$$(43) \qquad |f'(0)| \leq \sum_{1}^{n+k} |A_\nu|+2^{n-1} n! \int_{0}^{x_{n+k}} |K(x)|\, dx=|S'(0)|$$

which proves (17) for $\nu=1$. Moreover, from (16) we see that we can have equality in (43) only if $f(x)$, or perhaps $-f(x)$, satisfies the conditions

$$(44) \qquad f(x_i)=(-1)^{n+i-1}, \quad i=1, \ldots, n+k,$$

$$(45) \qquad f^{(n)}(x)=(-1)^j 2^{n-1} n! \text{ in } \xi_j < x < \xi_{j+1}, \quad j=0, \ldots, k.$$

Now (45) shows that $f(x)$ is not only a spline function of degree n, but even a perfect spline with the same knots and highest degree terms as $S(x)$. Therefore $f(x)$ could differ from $S(x)$ only by a polynomial of degree $n-1$ which, however, must vanish in view of the relations (44) and (32). This establishes the relation (18). Similarly we establish (17), for any ν, by working with an approximate spline differentiation formula for $f^{(\nu)}(0)$.

6. Numerical upper bounds for the Landau constants for $n=4, 5$ and 6. We conclude the paper with numerical upper bounds for the constants $L_4^{(\nu)}$, $L_5^{(\nu)}$ and $L_6^{(\nu)}$. Matorin's Theorem III shows that

$$L_4' \leq 16, \quad L_4'' \leq 80, \quad L_4''' \leq 192;$$

$$L_5' \leq 25, \quad L_5'' \leq 200, \quad L_5''' \leq 840, \quad L_5^{(4)} \leq 1920;$$

$$L_6' \leq 36, \quad L_6'' \leq 420, \quad L_6''' \leq 2688, \quad L_6^{(4)} \leq 10368, \quad L_6^{(5)} \leq 23040.$$

Let us improve them by using our Theorem 5 for $n=4$, 5 and 6.

Dennis Kuba, of the Mathematics Research Center Computing Staff, using linear programming methods combined with Newton's method for finding the knots, has determined approximations to the Chebyshev-Euler splines

(46) $$T_{n,k}(x) \text{ for } n=4, 5, 6 \text{ and } k=1, 2, 3, \ldots, 10.$$

The accuracy of these approximations was a little better than 5 significant figures. These were used to compute the values of

(47) $$|S^{(\nu)}_{n,k}(0)|$$

by our relation (21). By Theorem 6, the Landau constants $L^{(\nu)}_n$ are the limits of (47) at $k\to\infty$.

The results obtained are summarized in the Tables 1, 2 and 3.

Table 1

k	$\lvert S'_{4,k}(0)\rvert$	$\lvert S''_{4,k}(0)\rvert$	$\lvert S'''_{4,k}(0)\rvert$
0	16	80	192
1	15.9653	79.6937	191.291
2	15.9632	79.6784	191.258

Table 2

k	$\lvert S'_{5,k}(0)\rvert$	$\lvert S''_{5,k}(0)\rvert$	$\lvert S'''_{5,k}(0)\rvert$	$\lvert S^{(4)}_{5,k}(0)\rvert$
0	25	200	840	1920
1	24.8796	198.212	830.909	1903.52
2	24.8680	198.043	830.057	1901.99
3	24.8673	198.028	829.978	1901.84
4	24.8666	198.024	829.963	1901.82

Table 3

k	$\lvert S'_{6,k}(0)\rvert$	$\lvert S''_{6,k}(0)\rvert$	$\lvert S'''_{6,k}(0)\rvert$	$\lvert S^{(4)}_{6,k}(0)\rvert$	$\lvert S^{(5)}_{6,k}(0)\rvert$
0	36	420	2688	10368	23040
1	35.7493	414.443	2641.33	10184.1	22749.7
2	35.7118	413.628	2634.57	10157.7	22708.7
3	35.7069	413.507	2633.54	10153.6	22702.3
4	35.7063	413.485	2633.34	10152.9	22701.1

We recall that the convergence in the limit relation (23) was "instantaneous" for $n=2$ and $n=3$, since all terms of the sequence had the same value by (22). As a matter of fact, the rate of convergence is still very fast for $n=4$. That is why, for $n=4$, the available accuracy allowed only the use of the first two approximations (46). In any case, the last entries in our tables describe the corresponding Landau constants to five significant figures. At least this is the naive computer's point of view. Actually, in the absence of equally accurate lower bounds for the Landau constants, we can only conclude with certainty that the last entries in our tables are upper bounds for these constants.

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