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A CONSTRAINED RATIONAL APPROXIMATION PROBLEM IN FILTER DESIGN*

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Summary. Convergence of an iterative algorithm for constructing a rational function R with minimal sup-norm in a pass band $[0, 1]$ and $|R(x)| \geqq A$ (with prescribed attenuation constant A) in a stop band $[a, \infty)$ is analyzed by studying the existence, uniqueness, and characterizing properties of an appropriate constrained rational approximation problem. The algorithm has been used to construct low pass electric filters.

1. Introduction. This paper is concerned with a constrained rational approximation problem which is equivalent to a certain electric filter design problem. The approximation problem was suggested by an iterative algorithm of McGee [1] for synthesizing filters. We shall establish the existence, uniqueness, and characterizing properties of the best approximation, but the bulk of our effort will be devoted to proving the convergence of the computational algorithm.

Before stating the basic approximation problem, we need to introduce some notation. Let $1<a<\infty$ and $A>0$ be prescribed, and let l, m, n be positive integers satisfying $m<n+l$. Let Π_n denote the class of real polynomials of degree n, and define $\Pi_n^0=\{p: p(x)=x^n+u(x), u\in\Pi_{n-1}\}$. Let

$$(1.1)\qquad \boldsymbol{R}=\boldsymbol{R}(l, m, n, a, A)=\{r(x)=cx^l p(x)/q(x): p\in\Pi_n^0, q\in\Pi_m^0,\ c>0,$$
$$\text{where } p \text{ has } n \text{ simple zeros in } (0,1),\ q \text{ has } m \text{ simple zeros in } (a,\infty) \text{ and } |r(x)|\geqq A \text{ for } a\leqq x<\infty\}.$$

Problem 1.1. Determine $R(x)\in\boldsymbol{R}$ such that $\|R\|_{0,1}=\inf\{\|r\|_{0,1},\ r\in\boldsymbol{R}\}$, where in general $\|\varphi\|_{\alpha,\beta}=\max_{\alpha\leqq x\leqq\beta}|\varphi(x)|$.

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The solution of Problem 1.1 corresponds to the systhesis of a "best" low-pass filter with prescribed attenuation A in the stop band $[a, \infty)$ and minimal ripple in the pass band $[0, 1]$.

2. Existence, uniqueness and characterization. The solution of Problem 1.1 is uniquely characterized by the following alternation theorem.

Theorem 2. 1. Let $R \in \mathbf{R}$ satisfy

$$(2.1)\qquad \begin{aligned} &(-1)^{m+n+j+1}R(x_j)=\|R\|_{0,1}; \quad j=1, 2, \dots, n+1, \\ &(-1)^{m+j+1}R(y_j)=A, \quad j=1, 2, \dots, m+1 \end{aligned}$$

for some $0<x_1<\dots<x_{n+1}=1$ and $a=y_1<\dots<y_{m+1}$. Then R is the unique solution of Problem 1.1.

Proof. Suppose $r \in \mathbf{R}$ and that $\|r\|_{0,1} \leq \|R\|_{0,1}$. Let $R=cx^l p/q$, $r=\tilde{c}x^l\tilde{p}/\tilde{q}$ and set $\Delta=R-r=x^l(cp\tilde{q}-\tilde{c}\tilde{p}q)/q\tilde{q}$. Now in view of the alternating properties (2.1) of R, Δ must have l zeros at 0, at least n zeros on $(0, 1]$, and at least m zeros on $[a, \infty)$ (Counting double zeros twice). This is a total of $n+m+l$ zeros while $N(\Delta)$ = numerator of Δ is of class Π_{n+m+l}. But we claim Δ must have yet another zero, and thus must be identically zero. To see this, consider the following cases:

Case I: $(-1)^m(r-R)>0$ in $(1, a)$. Then Δ has an extra zero at 1.

Case II: $(-1)^m(r-R)<0$ in $(1, a)$. In this situation Δ has an extra zero at a.

Case III: $r=R$ somewhere in $(1, a)$, providing the extra zero.

Finally, if $\Delta \equiv 0$, then $cp\tilde{q} \equiv \tilde{c}\tilde{p}q$ and it follows that $c=\tilde{c}$. Moreover, since q, $\tilde{q}$ have no zeros in $(0, 1)$ we must have $p \equiv \tilde{p}$ and thus also $q \equiv \tilde{q}$. We conclude that $r=R$.

The existence of a solution to Problem 1.1 follows immediately from Theorem 2.1 if a rational function R satisfying (2.1) can be produced. This will be accomplished constructively in the following section by showing that an appropriate algorithm produces a sequence of rational functions in $\mathbf{R}$ converging to a rational function satisfying (2.1).

3. A computational algorithm. The following is the algorithm suggested by McGee [1] for synthesizing filters. We shall show that it produces a sequence of rational functions converging to the unique solution of Problem 1.1.

Algorithm 3.1.

(a) Choose $q_0 \in \Pi_m^0$ with m zeros on (a, ∞). Set $i=1$.

(b) Determine $p_i \in \Pi_n^0$ and $c_i>0$ such that $R_i=c_i x^l p_i/q_{i-1}$ satisfies $\|R_i^{-1}\|_{a,\infty}=A^{-1}$ and $(-1)^{m+n+j+1}R_j(x_j^i)=\|R_i\|_{0,1}$, $j=1, 2, \dots, n+1$, for some $0<x_1^i<\dots<x_{n+1}^i=1$.

(3.1) (c) Determine $q_i \in \pi_m^0$ and $d_i>0$ such that $Q_i=d_i x^l p_i/q_i$ satisfies $\|Q_i^{-1}\|_{a,\infty}=A^{-1}$ and $(-1)^{m+j+1}Q_i(y_j^i)=A$, $j=1, 2, \dots, m+1$ for some $a=v_1^i<\dots<y_{m+1}^i<\infty$.

(d) Set i to $i+1$ and repeat (b) and (c).

Steps (b) and (c) can be accomplished by standard algorithms for producing equialternating polynomials with respect to a weight function. They can also be done by a Remez algorithm (see e. g. [2]) for computing the best approximation of 0 by a polynomial with respect to a weight function.

For example step (b) essentially amounts to the determination of the best approximation to zero in the $\|\cdot\|_{0,1}$ norm by a polynomial $p \in \Pi_n^0$ with respect to the weight function $w_i(x)=x^l/q_{i-1}(x)$.

McGee has successfully tested Algorithm 3.1 on a number of examples with rapid convergence, but no proof of convergence is included in [1]. We shall establish convergence of the algorithm by showing that every subsequence of the sequence $R_i(x)$ produced by (3.1) contains a further subsequence converging to a rational function $R(x)$, which is the unique solution of the constrained rational approximation problem 1.1.

We elaborate the proof in a series of lemmas. First, we notice that algorithm (3.1) is well defined and produces infinite sequences of polynomials $\{p_i\}$ and $\{q_i\}$ without breaking down. Indeed, since p_i alternates n times on $[0, 1]$ it must have n simple zeros in $(0, 1)$ and thus has no other zeros. Similarly, $q_i(x)$ has m simple zeros in (a, ∞) and thus is non-zero on $[0, 1]$. We denote by $\{z_j^i\}_{j=1}^n$ and $\{\theta_j^i\}_{j=1}^m$ the zeros of p_i and q_i, respectively.

Lemma 3.2. $\|R_{i+1}\|_{0,1} \leq \|Q_i\|_{0,1} \leq \|R_i\|_{0,1}$. Strict inequality persists on the left unless $p_{i+1} \equiv p_i$.

Proof. To establish the first inequality, suppose $\|R_{i+1}\|_{0,1} \geq \|Q_i\|_{0,1}$, and consider $\delta = R_{i+1} - Q_i = x^l(c_{i+1}p_{i+1} - d_i p_i)/q_i$. Because R_{i+1} alternates on $[0, 1]$, δ has n zeros on $[0, 1]$ (counting double zeros twice). Moreover, since $\|R_{i+1}^{-1}\|_{a,\infty} = \|Q_i^{-1}\|_{a,\infty} = A^{-1}$ and δ^{-1} vanishes precisely at the zeros of q_i on $[a, \infty)$, we conclude that δ also has a zero in $[a, \infty)$. This implies that $c_{i+1}p_{i+1} \equiv d_i p_i$ and $\|Q_i\|_{0,1} = \|R_{i+1}\|_{0,1}$. This can only happen if $c_{i+1} = d_i$ and $p_{i+1} \equiv p_i$.

The proof of the second inequality is only slightly more complicated. Indeed, suppose to the contrary that $\|Q_i\|_{0,1} > \|R_i\|_{0,1}$. Setting $\Delta = (R_i - Q_i)$ and $N(\Delta)$ = numerator of Δ, we notice that $N(\Delta) \in \Pi_{n+m+l}$. Clearly Δ has l zeros at 0, then n common zeros of p_i, and since Q_i alternates on $[a, \infty)$, also m zeros in $[a, \infty)$. Now we know that $(-1)^{m+2}Q_i(a) = A \leq R_i(a)(-1)^{m+2}$. If inequality holds, then Δ must exhibit an additional zero in $(0, a)$ in view of the assumption that $\|Q_i\|_{0,1} > \|R_i\|_{0,1}$. On the other hand, if $Q_i(a) = R_i(a)$, then as above Δ will have an extra zero in $(0, a]$ if $[R_i'(a) - Q_i'(a)](-1)^{m+2} < 0$, or an extra zero in $[a, y_2^i]$ if $[R_i'(a) - Q_i'(a)](-1)^{m+2} \geq 0$. In either case (counting double zeros twice), Δ has a total of $l+n+m+1$ zeros which contradicts the fact that $N(\Delta) \in \Pi_{n+m+l}$.

Let $\gamma = (i_1 < i_2 < \ldots)$ be any subsequence of 1, 2, ...

Lemma 3.3. There exists a subsequence $\nu \subset \gamma$ such that $p_i(x)$ converges uniformly to some $p(x) \in \Pi_n^0$ on compact (bounded) subsets of $(-\infty, \infty)$ as $i \to \infty$, $i \in \nu$. (Equivalently, the coefficients of $p_i(x)$ converge to those of $p(x)$ as $i \to \infty$, $i \in \nu$.)

Proof. As noted above, $p_i(x)$ possesses n zeros $0 < z_1^i < \ldots < z_n^i < 1$; i. e. $p_i(x) = \prod_{j=1}^{n}(x - z_j^i)$. Since the $\{z_j^i\}$ lie in the compact set $[0, 1]$ there exists a subsequence $\nu \subset \gamma$ such that $z_j^i \to z_j$, $1 \leq j \leq n$, for some $0 \leq z_1 \leq \ldots \leq z_n \leq 1$ as $i \to \infty$, $i \in \nu$. Clearly p_i converges uniformly to $p(x) = \prod_{j=1}^{n}(x - z_j)$ on compact sets as $i \to \infty$, $i \in \nu$.

Applying the proof of the above lemma to the sequence $\{p_{i+1}\}$, $i\in\nu$, we can find a further subsequence $\nu^*\subset\nu$ such that simultaneously $p_{i+1}(x)$ converges uniformly to some $\widetilde{p}(x)\in\Pi_n^0$ on compact subsets of $(-\infty,\infty)$ as $i\to\infty$, $i\in\nu^*$.

Lemma 3.4. There exists a subsequence $\mu\subset\nu^*$ such that $q_i(x)$ converges uniformly to a $q(x)\in\Pi_m^0$ as $i\to\infty$, $i\in\mu$, on compact subsets of $(-\infty,\infty)$.

Proof. We first notice that since the $q_i(x)$ produced by (3.1) is the best approximation to 0 in Π_m^0 with respect to the weight function $1/x^l p_i(x)$ on $[a,\infty)$, we have for $i\in\nu^*$

$$\left\|\frac{q_i(x)}{x^l p_i(x)}\right\|_{a,a+1}\leqq\left\|\frac{q_i(x)}{x^l p_i(x)}\right\|_{a,\infty}\leqq\left\|\frac{x^m}{x^l p_i(x)}\right\|_{a,\infty}<K<\infty.$$

(The last inequality follows since $p_i(x)\to p(x)$ as $i\to\infty$, $i\in\nu^*$ while $m<n+l$.) But then

$$\|q_i(x)\|_{a,a+1}=\left\|\frac{q_i(x)x^l p_i(x)}{x^l p_i(x)}\right\|_{a,a+1}\leqq K\|x^l p_i(x)\|_{a,a+1}<\infty \tag{3.2}$$

for all $i\in\nu^*$. The inequality (3.2) implies that the coefficients of $q_i(x)$ for $i\in\nu$ are uniformly bounded, and hence there exists a subsequence $\mu\subset\nu^*$ sueh that the coefficients of $q_i(x)$ converge as $i\to\infty$, $i\in\mu$.

We remark that since p_i, q_i and p_{i+1} all converge, the c_i and d_i are uniformly bounded and hence μ can be chosen so that also $c_i\to c>0$ and $d_i\to d>0$ as $i\to\infty$, $i\in\mu$. Let

$$R(x)=\frac{cx^l\widetilde{p}(x)}{q(x)}\quad\text{and}\quad Q(x)=\frac{dx^l p(x)}{q(x)}.$$

Lemma 3.5. As $i\to\infty$, $i\in\mu$,

$$\|R_{i+1}(x)-R(x)\|_{0,1}\to 0,\quad \|Q_i(x)-Q(x)\|_{0,1}\to 0 \tag{3.3}$$

and

$$\left\|\frac{1}{Q_i(x)}-\frac{1}{Q(x)}\right\|_{a,\infty}\to 0,\quad \left\|\frac{1}{R_{i+1}(x)}-\frac{1}{R(x)}\right\|_{a,\infty}\to 0. \tag{3.4}$$

Proof. For $i\to\infty$, $i\in\mu$, we have

$$\left\|\frac{c_{i+1}x^l p_{i+1}(x)}{q_i(x)}-\frac{cx^l\widetilde{p}(x)}{q(x)}\right\|_{0,1}=\left\|\frac{c_{i+1}x^l p_{i+1}(x)q(x)-cx^l\widetilde{p}(x)q_i(x)}{q_i(x)q(x)}\right\|_{0,1} \tag{3.5}$$

$$\leqq\max_{0\leqq x\leqq 1}\frac{1}{q_i(x)q(x)}\|c_{i+1}p_{i+1}(x)q(x)-c\widetilde{p}(x)q_i(x)\|_{0,1}\to 0$$

which establishes (3.3). The proof of the analogous result for Q is identical. To obtain (3.4), we note that the coefficients of p_i and q_i are uniformly bounded as $i\to\infty$, $i\in\mu$. Since $m<n+l$ we infer that for every $\varepsilon>0$ there exists $a<b<\infty$ such that

$$\left|\frac{q_i(x)}{d_i x^l p_i(x)}\right|\leqq\varepsilon\quad\text{and}\quad\left|\frac{q(x)}{dx^l p(x)}\right|\leqq\varepsilon\quad\text{for}\quad x\geqq b,\ i\in\mu. \tag{3.6}$$

Arguing as in (3.5) we obtain

(3.7) $$\left\| \frac{q_i(x)}{d_i x^l p_i(x)} - \frac{q(x)}{dx^l p(x)} \right\|_{a,b} \leq \varepsilon$$

for $i \in \mu$ sufficiently large, and since ε is arbitrary (3.4) follows on combining (3.6) and (3.7). The proof of the second part of (3.4) proceeds in the same way.

Lemma 3.6. The rational functions R and Q have the properties

(3.8) $$R(x_j)(-1)^{m+n+j+1} = \| R \|_{0,1}, \ j=1, 2, \ldots, n+1$$

for some $0 < x_1 < \ldots < x_{n+1} = 1$ and

(3.9) $$Q(x_j)(-1)^{m+j+1} = A, \ j=1, 2, \ldots, m+1$$

for some $a = y_1 < \ldots < y_{m+1} < \infty$, where $|Q(x)| \geq A$ for $a \leq x < \infty$.

Proof. We recall

(3.10) $$(-1)^{m+n+j+1} R_{i+1}(x_j^{i+1}) = \| R_{i+1} \|_{0,1}, \ j=1, 2, \ldots, n+1$$

for some $0 < x_1^{i+1} < \ldots < x_{n+1}^{i+1} = 1$. Since the x_j^{i+1} lie in the compact set $[0, 1]$, there exists a subsequence $\theta \in \mu$ such that x_j^{i+1} converges to x_j, $j=1, 2, \ldots, n+1$, with $0 \leq x_1 \leq \ldots \leq x_{n+1} = 1$. Now in view of (3.10),

$$|R(x_j)(-1)^{m+n+j+1} - \| R \|_{0,1}| \leq |R(x_j) - R(x_j^{i+1})| + |R(x_j^{i+1}) - R_{i+1}(x_j^{i+1})|$$
$$+ | \| R_{i+1} \|_{0,1} - \| R \|_{0,1}|.$$

Letting $i \to \infty$, $i \in \theta$ and applying Lemmas 3.3, 3.4 and 3.5 yields the result. Property (3.9) is established analogously. In this case we have

$$\frac{(-1)^{m+j+1}}{Q_i(y_j^i)} = \| 1/Q_i \|_{a,\infty}$$

for some $a = y_1^i < \ldots < y_{m+1}^i < \infty$. Since the coefficients of p_i and q_i are uniformly bounded for $i \in \mu$ it follows that $y_{m+1}^i \leq \text{Con} < \infty$, and Lemmas 3.3, 3.4 and 3.5 apply as before.

Lemma 3.7. $\| R \|_{0,1} = \| Q \|_{0,1}$.

Proof. By Lemma 3.2 and the continuity of the norm, $\| R_{i+1} \|_{0,1} \downarrow \| R \|_{0,1}$ and $\| Q_i \|_{0,1} \downarrow \| Q \|_{0,1}$ as $i \to \infty$, $i \in \theta$. In view of the inequalities of Lemma 3.2, these limits must be equal.

Lemma 3.8. The rational functions $R(x) = \dfrac{cx^l \tilde{p}(x)}{q(x)}$ and $Q(x) = \dfrac{dx^l p(x)}{q(x)}$ coincide.

Proof. Consider $\Delta(x) = R(x) - Q(x)$. By Lemma 3.6 R alternates n times on $[0, 1]$ and Q alternates m times on $[a, \infty)$. Moreover, by Lemma 3.7, $\| Q \|_{0,1} = \| R \|_{0,1}$ while $\| 1/Q \|_{a,\infty} = \| 1/R \|_{a,\infty} = A$. In addition to the l common zeros at 0, Δ is also zero n times on $[0, 1]$ and m times on $[a, \infty)$ as a consequence of the alternating properties of R and Q. This is a total of $m+n+l$ zeros while $N(\Delta) \in \Pi_{n+l}$, and thus $N(\Delta) \equiv 0$, i. e., $R(x) \equiv Q(x)$. This means in particular that R alternates n times on [1] and m times on $[a, \infty)$.

The main result of this paper is as follows:

Theorem 3.9. Let $\{p_i\}$ and $\{q_i\}$ be the sequences of polynomials produced by the algorithm (3.1) and let p, q, R_i and R be as above. Then the coefficients of p_i converge to those of p, the coefficients of q_i converge to those of q, and the rational function R satisfies (1.1) and is thus the unique solution of Problem 1.1. Moreover, either $\|R_i\|_{0,1} < \|R_{i-1}\|_{0,1}$, or the process becomes stationary with $R_i \equiv R$.

Proof. The above sequence of lemmas shows that for any subsequence γ of 1, 2, ..., there exists a subsequence $\theta \subset \gamma$ such that p_i and q_i converge uniformly on compact sets as $i \to \infty$, $i \in \theta$. By Lemma 3.8 the limits of p_i and q_i for $i \in \theta$ must be p and q, and hence the sequences $\{p_i\}$, $\{q_i\}$ themselves converge fo p and q, respectively, as $i \to \infty$.

Remark 1. The constrained rational approximation Problem 1.1 corresponds to synthesizing a low pass filter. A more important design problem is to compute filters with several pass and stop-bands. A version of McGee's algorithm can also be used here, and he reports [1] success in using it on multi-band problems. It is not hard to formulate a corresponding rational approximation problem, but analysis of convergence is more difficult, and will be done elsewhere.

Remark 2. I wish to thank Dr. Mike Sablatash for bringing this problem to my attention. I would also like to express my appreciation to Professor J. Barkley Rosser for his careful reading of an earlier version of this manuscript and for his helpful suggestions.

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