

ON THE APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY MEANS OF A CLASS OF LINEAR OPERATORS

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Summary. In this paper we develop a theory concerning the approximation of functions of two variables $f \in C(D)$, where $D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$, by means of the linear positive operator $P_m^{[\alpha]}$ defined as follows

$$P_m^{[\alpha]} f(x, y) = \sum_{i=0}^m \sum_{j=0}^{m-i} w_{m,i,j}^{[\alpha]}(x, y) f(i/m, j/m),$$

where

$$w_{m,i,j}^{[\alpha]}(x, y) = 1^{(-m, -\alpha)} \binom{m}{i} \binom{m-i}{j} x^{(i, -\alpha)} y^{(j, -\alpha)} (1-x-y)^{(m-i-j, -\alpha)},$$

α being a parameter which may depend only on m , while by $u^{[n, h]}$ one denotes the generalized factorial of degree n and increment h of u .

1. In this paper we consider a sequence of positive linear operators $(P_m^{[\alpha]})$, depending on a real parameter α , defined on the vector space $C(D)$ of functions f , of two variables, continuous on the isosceles right triangle $D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$, and we give some results concerning the approximation of functions f by means of $P_m^{[\alpha]} f$.

Preliminary we point out that $(P_m^{[\alpha]} f)(x, y)$ can be considered as originating in an identity derived from the Vandermonde formula for the factorial power and then we present a probabilistic method for obtaining it.

The results of the present paper represent extensions to two variables of those given in [9]. It should be mentioned that in our previous paper [11] we have considered the operators $P_m^{[\alpha]}$ in the case $\alpha \geq 0$ as a generalization to the two-dimensional case of the polynomial operator of Bernstein type which we have introduced and investigated in [7]. In [8] and [10] we have discussed some probabilistic methods for constructing and investigating positive linear operators.

2. We denote by $u^{(n, h)}$ the factorial power of order n (: natural number) and increment h (: real number) of u , that is:

$$u^{(n, h)} = u(u-h) \cdots (u - \overline{n-1}h),$$

where $n-1 h$ represents the product $(n-1)h$.

If $h=1$, for brevity, we shall use the notation $u^{(n)}$ instead of $u^{(n,1)}$. When $n=0$ and $u \neq 0$ we set $u^{(0,h)}=1$. Evidently for $h=0$ we have $u^{(n,0)}=u^n$.

We shall make use of the following notations

$$u^{(-n,h)} = \frac{1}{u^{(n,h)}}, \quad u^{(-n,-h)} = \frac{1}{u^{(n,-h)}} = \frac{1}{(u+n-1h)^{(n,h)},}$$

provided that the denominators are different from zero.

It is easily verified that if n and j are natural numbers such that $n > j$, then we have

$$(1) \quad u^{(n,i)} = u^{(j,n)}(u-jh)^{(n-j,h)}.$$

3. We now wish to point out that in the same manner as the Bernstein polynomial $B_m f$, defined by

$$(2) \quad (B_m f)(x, y) = \sum_{0 \leq i+j \leq m} \frac{m!}{i!j!(m-i-j)!} x^i y^j (1-x-y)^{m-i-j} f\left(\frac{i}{m}, \frac{j}{m}\right)$$

originates in the identity

$$1 = (x+y+1-x-y)^m = \sum_{0 \leq i+j \leq m} \frac{m!}{i!j!(m-i-j)!} x^i y^j (1-x-y)^{m-i-j},$$

so if we consider the Vandermonde formula

$$(u+v+w)^{(m,h)} = \sum_{\substack{i,j,k=0 \\ (i+j+k=m)}}^m \frac{m!}{i!j!k!} u^{(i,h)} v^{(j,h)} w^{(k,h)}$$

and set $u=x$, $v=y$, $w=1-x-y$, $h=-\alpha$, we obtain the identity

$$1^{(m,-\alpha)} = \sum_{0 \leq i+j \leq m} \frac{m!}{i!j!(m-i-j)!} x^{(i,-\alpha)} y^{(j,-\alpha)} (1-x-y)^{(m-i-j,-\alpha)}$$

which gives rise to the polynomial $P_m^{[\alpha]} f$ defined by

$$(3) \quad (P_m^{[\alpha]} f)(x, y) = 1^{(-m,-\alpha)} \sum_{0 \leq i+j \leq m} \frac{m!}{i!j!(m-i-j)!} x^{(i,-\alpha)} y^{(j,-\alpha)} \\ \times (1-x-y)^{(m-i-j,-\alpha)} f\left(\frac{i}{m}, \frac{j}{m}\right).$$

Throughout this paper we assume that the real parameter α may depend only on the natural number m so that

$$1^{(m,-\alpha)} = (1+\alpha)(1+2\alpha) \dots (1+\overline{m-1}\alpha) \neq 0.$$

It is obvious that at (3) we have a polynomial of degree m with respect to x and y , which can be written in the form

$$(3') \quad (P_m^{[\alpha]} f)(x, y) = \sum_{i=0}^m \sum_{j=0}^{m-i} \tau_{m,i,j}^{[\alpha]}(x, y) f\left(\frac{i}{m}, \frac{j}{m}\right),$$

where

$$(3'') \quad \omega_{m,i,j}^{[a]}(x,y) = \binom{m}{i} \binom{m-i}{j} \frac{x^{(i,-a)} y^{(j,-a)} (1-x-y)^{(m-i-j,-a)}}{1^{(m,-a)}}.$$

4. As we have shown in [11], if $\alpha > 0$, $x > 0$, $y > 0$ and $x+y < 1$, then we can represent the operator $P_m^{[a]}$ by means of the Bernstein operator B_m in the following form

$$(P_m^{[a]}f)(x,y) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{y}{\alpha}, \frac{1-x-y}{\alpha}\right)} \int_D \int u^{\frac{x}{\alpha}-1} v^{\frac{y}{\alpha}-1} (1-u-v)^{\frac{1-x-y}{\alpha}-1} (B_m f)(u,v) du dv,$$

where by $B(a,b,c)$ one denotes the Dirichlet double integral, that is

$$B(a,b,c) = \int_D \int u^{a-1} v^{b-1} (1-u-v)^{c-1} du dv.$$

5. In accordance with our previous investigation [10] of the operator $P_m^{[a]}$ we should notice that one can arrive at this operator by making use of the Markov-Pólya bivariate probability distribution.

A two-dimensional discrete random variable (X,Y) is said to have a Markov-Pólya distribution if its probability function: $P(X=i, Y=j)$, where $i=0, 1, \dots, m$ and $j=0, 1, \dots, m-i$, is given by

$$(4) \quad P(X=i, Y=j) = \frac{m^{(i+j)} p^{(i,-a)} q^{(j,-a)} r^{(m-i-j,-a)}}{i! j! 1^{(m,-a)}},$$

where p, q, r and a are any real numbers such that $p > 0, q > 0, r > 0$, $p+q+r=1$, $-ma \leq \min\{p, q, r\}$.

One can arrive at this formula when p, q and r are rational numbers by considering an urn model. This can be briefly explained as follows. An urn contains A white, B black and C red balls. One draws one ball at random and then it is replaced and moreover a balls of color drawn are added. This procedure is repeated m times. If we assume that the random vector (X,Y) takes on the values (i,j) , where $i=0, 1, \dots, m$ and $j=0, 1, \dots, m-i$, if during m trials one obtains exactly i times a white ball, j times a black ball and $m-i-j$ times a red ball, then

$$P(X=i, Y=j) = \frac{m^{(i+j)} A^{(i,-a)} B^{(j,-a)} C^{(m-i-j,-a)}}{i! j! (A+B+C)^{(m,-a)}}.$$

Now introducing the notations: $A+B+C=N$, $A/N=p$, $B/N=q$, $C/N=r$, $a/N=\alpha$, then the foregoing probability is given by a formula of the form (4).

If f is a real-valued function of two variables such that the mean value of $f(X/m, Y/m)$ exists for $m=1, 2, \dots$, then it is given just by $(P_m^{[a]}f)(p, q)$.

It should be observed that the integer a of balls added at each trial may be chosen negative, when one can give the following interpretation:

after each drawing we do not replace the ball drawn and moreover we shall eliminate from the urn $-(a+1)$ balls of the same colour as the ball just drawn.

In order that we can perform all the m trials when a is a negative integer we should assume that

$$(5) \quad A+ma \geq 0, \quad B+ma \geq 0, \quad C+ma \geq 0,$$

from which we find the restrictions: $p+ma \geq 0$, $q+ma \geq 0$, $r+ma = 1-p-q+ma \geq 0$. It is clear that these inequalities are satisfied if $a \geq 0$, but when a is negative we should assume that $-ma \leq 1/3$. In fact when $-ma = 1/3$ the foregoing inequalities are satisfied only for $p=q=1/3$.

Consequently if we assume that $a = \alpha_m$ is a negative number depending on m so that $-m\alpha_m \leq \varepsilon$, where $0 < \varepsilon < 1/3$, then we can define the operator $P_m^{[a]}$ by formulas (3) or (3') and it is of positive type on the triangle D_ε whose vertices are the points

$$(6) \quad (\varepsilon, \varepsilon), \quad (1-\varepsilon, \varepsilon), \quad (\varepsilon, 1-\varepsilon).$$

It is obvious that when $a = \alpha_m \geq 0$ then $P_m^{[a]}$ is of positive type over $D_0 = D$.

6. Now we state and prove

Theorem 1. The polynomial defined by (3')–(3'') can be represented by the following expansion

$$(7) \quad (P_m^{[a]}f)(x, y) = \sum_{r=0}^m \sum_{s=0}^{m-r} \binom{m}{r} \binom{m-r}{s} \frac{x^{(r,-a)} y^{(s,-a)}}{1^{(r+s,-a)}} \Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0),$$

where we have

$$\Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0) = \sum_{\nu=0}^r \sum_{\mu=0}^s (-1)^{\nu+\mu} \binom{r}{\nu} \binom{s}{\mu} f\left(\frac{r-\nu}{m}, \frac{s-\mu}{m}\right),$$

i. e., the finite difference of order (r, s) with the steps $h=k=1/m$ and the starting point $(0, 0)$ of the function f .

Proof. Let us consider the Newton-Biermann interpolating polynomial (see [4] and [6]) corresponding to the function f and the nodes $(i/m, j/m)$, where $i=0, 1, \dots, m$ and $j=0, 1, \dots, m-i$:

$$(N_m f)(t, \tau) = \sum_{r=0}^m \sum_{s=0}^{m-r} \frac{(mt)^{(r)} (m\tau)^{(s)}}{r!s!} \Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0).$$

Performing the changes of variables $mt = u$, $m\tau = v$ we obtain

$$(N_m f)\left(\frac{u}{m}, \frac{v}{m}\right) = \sum_{r=0}^m \sum_{s=0}^{m-r} \frac{u^{(r)} v^{(s)}}{r!s!} \Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0).$$

From this we readily arrive at the equation

$$\sum_{0 \leq i+j \leq m} \omega_{m,i,j}^{[a]}(x, y) (N_m f)\left(\frac{i}{m}, \frac{j}{m}\right) = \sum_{0 \leq r+s \leq m} \frac{M_{(r,s)}}{r!s!} \Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0),$$

where

$$M_{(r,s)} = \sum_{0 \leq i+j \leq m} i^{(r)} j^{(s)} \omega_{m,i,j}^{[a]}(x, y).$$

According to the interpolating properties of the polynomial $N_m f$ we have $(N_m f)(i/m, j/m) = f(i/m, j/m)$ and we conclude that

$$(8) \quad (P_m^{[a]} f)(x, y) = \sum_{0 \leq r+s \leq m} \frac{M_{(r,s)}}{r!s!} \frac{1}{m^r} \frac{1}{m^s} f(0, 0).$$

We will now proceed to find a convenient formula for $M_{(r,s)}$. Because of the fact that $k^{(n)} = 0$ if $k = 0, 1, \dots, n-1$, we can write

$$(9) \quad 1^{(m,-a)} M_{(r,s)} = \sum_{i=r}^m \sum_{j=s}^{m-i} i^{(r)} j^{(s)} \binom{m}{i} \binom{m-i}{j} x^{(i,-a)} y^{(j,-a)} (1-x-y)^{(m-i-j,-a)}.$$

Referring to the formula

$$(10) \quad k^{(q)} \binom{n}{k} = n^{(q)} \binom{n-q}{k-q}$$

we have

$$(11) \quad \begin{aligned} S &= \sum_{j=s}^{m-i} j^{(s)} \binom{m-i}{j} y^{(j,-a)} (1-x-y)^{(m-i-j,-a)} \\ &= (m-i)^{(s)} \sum_{j=s}^{m-i} \binom{m-i-s}{j-s} y^{(j,-a)} (1-x-y)^{(m-i-j,-a)}. \end{aligned}$$

If we set $j-s=l$ and take into account formula (1) we obtain

$$\begin{aligned} S &= (m-i)^{(s)} \sum_{l=0}^{m-i-s} \binom{m-i-s}{l} y^{(s+l,-a)} (1-x-y)^{(m-i-s-l,-a)} \\ &= (m-i)^{(s)} y^{(s,-a)} \sum_{l=0}^{m-i-s} \binom{m-i-s}{l} (y+s\alpha)^{(l,-a)} (1-x-y)^{(m-i-s-l,-a)}. \end{aligned}$$

Now if we refer to Vandermonde's formula

$$(a+b)^{(n,h)} = \sum_{k=0}^n \binom{n}{k} a^{(k,h)} b^{(n-k,h)}$$

we find that

$$(12) \quad S = (m-i)^{(s)} y^{(s,-a)} (1-x+s\alpha)^{(m-i-s,-a)}.$$

Consequently, by (9)–(12) we can write

$$1^{(m,-a)} M_{(r,s)} = y^{(s)} \sum_{i=r}^m i^{(r)} \binom{m}{i} (m-i)^{(s)} x^{(i,-a)} (1-x+s\alpha)^{(m-i-s,-a)}$$

$$= m^{(r)} y^{(s)} \sum_{i=r}^m \binom{m-r}{i-r} (m-i)^{(s)} x^{(i,-\alpha)} (1-x+sa)^{(m-i-s,-\alpha)}$$

If we now set $i-r=k$ we obtain

$$1^{(m,-\alpha)} M_{(r,s)} = m^{(r)} y^{(s)} \sum_{k=0}^{m-r} (m-r-k)^{(s)} \binom{m-r}{k} x^{(r+k,-\alpha)} (1-x+sa)^{(m-i-s,-\alpha)}.$$

But by (1) and (10) we have

$$x^{(r+k,-\alpha)} = x^{(r,-\alpha)} (x+ra)^{(k,-\alpha)},$$

$$(m-r-k)^{(s)} \binom{m-r}{k} = (m-r-k)^{(s)} \binom{m-r}{m-r-k} = (m-r)^{(s)} \binom{m-r-s}{s},$$

so that

$$1^{(m,-\alpha)} M_{(r,s)}$$

$$= m^{(r)} (m-r)^{(s)} x^{(r,-\alpha)} y^{(s,-\alpha)} \sum_{k=0}^{m-r-s} \binom{m-r-s}{k} (x+ra)^{(k,-\alpha)} (1-x+sa)^{(m-r-s-k,-\alpha)}$$

$$= m^{(r)} (m-r)^{(s)} x^{(r,-\alpha)} y^{(s,-\alpha)} (1+r+sa)^{(m-r-s,-\alpha)}.$$

Hence we obtain finally

$$M_{(r,s)} = \frac{m^{(r)} (m-r)^{(s)} x^{(r,-\alpha)} y^{(s,-\alpha)}}{1^{(r+s,-\alpha)}}$$

and the insertion of this result into (8) leads us to the required formula (7).

For $\alpha=0$ (7) reduces to a formula established in [11] for the Bernstein polynomial (2).

By making use of the relation between divided differences and finite differences

$$\left[\begin{array}{c} a, a+h, \dots, a+rh \\ b, b+k, \dots, b+sk \end{array} ; f \right] = \frac{1}{r!s!h^r k^s} \Delta_{h,k}^{r,s} f(a, b)$$

we have

$$\Delta_{\frac{1}{m}, \frac{1}{m}}^{r,s} f(0, 0) = \frac{r!s!}{m^{r+s}} \left[\begin{array}{c} 0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{r}{m} \\ 0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{s}{m} \end{array} ; f \right].$$

Therefore, an immediate consequence of formula (7) is the following

$$(P_m^{[a]} f)(x, y) = \sum_{r=0}^m \sum_{s=0}^{m-r} \frac{1^{(r+s, m-1)} x^{(r,-\alpha)} y^{(s,-\alpha)}}{1^{(r+s,-\alpha)}} \left[\begin{array}{c} 0, \frac{1}{m}, \dots, \frac{r}{m} \\ 0, \frac{1}{m}, \dots, \frac{s}{m} \end{array} ; f \right].$$

7. In the next theorem we present the basic result of this paper.

Theorem 2. If $0 \leq \alpha = \alpha_m \rightarrow 0$ as $m \rightarrow \infty$ and f is continuous on the triangle D , then the sequence $(P_m^{[a]} f)$ converges to f uniformly on D .

If $0 > \alpha = \alpha_m$ and $-m\alpha_m \leq \varepsilon$, $m = 1, 2, \dots$, where $0 < \varepsilon < 1/3$, then if f is defined and bounded on D and continuous on the triangle D_ε having the vertices (6), then the sequence $(P_m^{[\alpha]})$ converges to f uniformly on D_ε .

Proof. It follows easily from (7) that for the functions e_{ij} , $0 \leq i+j \leq 2$, defined by $e_j(x, y) = x^i y^j$ for all (x, y) on D , we have

$$(13) \quad \begin{aligned} (P_m^{[\alpha]} e_{00})(x, y) &= 1, \quad (P_m^{[\alpha]} e_{10})(x, y) = x, \quad (P_m^{[\alpha]} e_{01})(x, y) = y, \\ (P_m^{[\alpha]} e_{20})(x, y) &= \frac{1}{1+\alpha} \left[\frac{x(1-x)}{m} + x(x+\alpha) \right], \quad (P_m^{[\alpha]} e_{11})(x, y) = \left(1 - \frac{1}{m} \right) \frac{xy}{1+\alpha}, \\ (P_m^{[\alpha]} e_{02})(x, y) &= \frac{1}{1+\alpha} \left[\frac{y(1-y)}{m} + y(y+\alpha) \right]. \end{aligned}$$

Therefore, under our assumptions we have

$$\lim_{m \rightarrow \infty} (P_m^{[\alpha]} e_{ij})(x, y) = e_{ij}(x, y), \quad 0 \leq i+j \leq 2,$$

uniformly on our basic regions.

Since $P_m^{[\alpha]}$ is a linear positive operator on D , respectively on D_ε , the assertions of our theorem now follow by virtue of the Bohman-Korovkin theorem corresponding to two variables (see, e. g., [1], [2] and [3]).

8. We now discuss the estimation of the order of approximation of the function f by means of the operator $P_m^{[\alpha]}$, in order to see the speed of convergence of $(P_m^{[\alpha]} f)$ to f . We shall make use of the modulus of continuity ω on D_ε , defined by

$$\omega(f; \delta) = \omega(\delta) = \sup |f(x'', y'') - f(x', y')|,$$

where $\delta > 0$, while (x', y') and (x'', y'') are points from D_ε so that

$$|x'' - x'| + |y'' - y'| \leq \delta.$$

Consider the space $C(D_\varepsilon)$ of functions continuous on D_ε , and the uniform norm, defined by

$$\|g\| = \|g\|_{D_\varepsilon} = \max \{ |g(x, y)| : (x, y) \in D_\varepsilon \}.$$

We shall now establish

Theorem 3. If $f \in C(D_\varepsilon)$ then we have

$$(14) \quad \|f - P_m^{[\alpha]} f\| \leq 2\omega \left(\sqrt{\frac{1+\alpha m}{m+\alpha m}} \right).$$

Proof. It should be clear that if $\alpha \geq 0$ then we take $\varepsilon = 0$, and if $\alpha < 0$ then we assume that α depends on m in such a way that $-m\alpha \leq \varepsilon$, where $0 < \varepsilon < 1/3$. In both cases the linear operator $P_m^{[\alpha]}$ is positive on D_ε , since $\omega_{m,i,j}^{[\alpha]}(x, y) \geq 0$ whenever $(x, y) \in D_\varepsilon$. Taking this into account and the first relation from (13) we can write

$$|f(x, y) - (P_m^{[\alpha]} f)(x, y)| \leq \sum_{0 \leq i+j \leq m} \omega_{m,i,j}^{[\alpha]}(x, y) \left| f(x, y) - f\left(\frac{i}{m}, \frac{j}{m}\right) \right|.$$

We shall use the following two properties of the modulus of continuity:

$$|f(x'', y'') - f(x', y')| \leq \omega(|x'' - x'| + |y'' - y'|), \quad \omega(\lambda\delta) \leq (1+\lambda)\omega(\delta), \quad \lambda > 0.$$

Since

$$\left| f(x, y) - f\left(\frac{i}{m}, \frac{j}{m}\right) \right| \leq \left(1 + \frac{1}{\delta} \left(\left| x - \frac{i}{m} \right| + \left| y - \frac{j}{m} \right| \right) \right) \omega(\delta),$$

it follows that

$$(15) \quad \left| f(x, y) - (P_m^{[a]}f)(x, y) \right| \leq [1 + \delta^{-1}(P_m^{[a]}|x-t|)(x, y) + \delta^{-1}(P_m^{[a]}|y-\tau|)(x, y)] \omega(\delta),$$

where in the second member the operator $P_m^{[a]}$ is to be performed first with respect to t and then with respect to τ , x and y being fixed.

In accordance with the Cauchy-Schwarz inequality and with the identities (13) we have

$$(16) \quad (P_m^{[a]}|x-t|)(x, y) \leq [(P_m^{[a]}(x-t)^2(x, y))]^{1/2} \\ = [x^2 - 2x(P_m^{[a]}t)(x, y) + (P_m^{[a]}t^2)(x, y)]^{1/2} = \left[\frac{1+am}{1+a} \frac{x(1-x)}{m} \right]^{1/2} \leq \frac{1}{2} \sqrt{\frac{1+am}{m+am}}.$$

Similarly it may be shown that

$$(16') \quad (P_m^{[a]}|y-t|)(x, y) \leq \frac{1}{2} \sqrt{\frac{1+am}{m+am}}.$$

By using these inequalities, (15) leads us to

$$\|f - P_m^{[a]}f\| \leq \left(1 + \frac{1}{\delta} \sqrt{\frac{1+am}{m+am}} \right) \omega(\delta).$$

Now if we set

$$\delta = \delta_m = \sqrt{\frac{1+am}{m+am}}$$

we obtain the desired inequality (14).

Specializing to the case $a=0$ one obtains the corresponding inequality for the Bernstein polynomials (2), established first in [5].

9. Our next result yields a way of describing an asymptotic estimate of remainder, of Voronovskaja type, of the approximation formula of the function f by $P_m^{[a]}f$.

Theorem 4. Let f be defined and bounded on D . If $0 \leq a = a_m \rightarrow 0$ and f has on $(x, y) \in D$ the differential $d^2f(x, y)$, then

$$(17) \quad (P_m^{[a]}f)(x, y) = f(x, y) + \frac{1+am}{1+a} \left[\frac{x(1-x)}{2m} f''_{xx}(x, y) - \frac{xy}{m} f''_{xy}(x, y) - \frac{y(1-y)}{2m} f''_{yy}(x, y) \right] + \frac{\eta_m^{[a]}}{m},$$

where $\eta_m^{[a]} \rightarrow 0$ as $m \rightarrow \infty$.

If $a = a_m < 0$ and $-ma_m \leq \varepsilon$, $m = 1, 2, \dots$, where $0 < \varepsilon < 1/3$, then if f has on $(x, y) \in D_\varepsilon$ the differential $d^2f(x, y)$, then formula (17) is also valid.

Proof. By virtue of Taylor's theorem we may write the expansion

$$\begin{aligned}
f(t, \tau) &= f(x, y) + (t-x)f'_x(x, y) + (\tau-y)f'_y(x, y) \\
&\quad + \frac{1}{2}(t-x)^2 f''_{xx}(x, y) + (t-x)(\tau-y)f''_{xy}(x, y) \\
&\quad + \frac{1}{2}(\tau-y)^2 f''_{yy}(x, y) + (t-x)^2 g_1(t, y) + (t-x)(\tau-y)g_2(t, \tau) + (\tau-y)^2 g_3(x, \tau),
\end{aligned}$$

where $(t, \tau) \in D$ and $g_1, g_2, g_3 \rightarrow 0$ at $t \rightarrow x$ and $\tau \rightarrow y$.

We see at once that if we set $t=i/m$, $\tau=j/m$, multiply by (3''), sum for i and j such that $0 \leq i+j \leq m$ and use (13) we obtain

$$\begin{aligned}
(P_m^{[\alpha]}f)(x, y) &= f(x, y) + \frac{1+\alpha m}{1+\alpha} \left[\frac{x(1-x)}{2m} f''_{xx}(x, y) - \frac{xy}{m} f''_{xy}(x, y) \right. \\
&\quad \left. + \frac{y(1-y)}{2m} f''_{yy}(x, y) \right] + \varrho_m^{[\alpha]}(x, y),
\end{aligned}$$

where

$$\begin{aligned}
\varrho_m^{[\alpha]}(x, y) &= \sum_{0 \leq i+j \leq m} \omega_{m,i,j}^{[\alpha]}(x, y) \left[\left(\frac{i}{m} - x \right)^2 g_1 \left(\frac{i}{m}, y \right) \right. \\
&\quad \left. + \left(\frac{i}{m} - x \right) \left(\frac{j}{m} - y \right) g_2 \left(\frac{i}{m}, \frac{j}{m} \right) + \left(\frac{j}{m} - y \right)^2 g_3 \left(x, \frac{j}{m} \right) \right].
\end{aligned}$$

Now, since $g_1, g_2, g_3 \rightarrow 0$ when $t \rightarrow x$ and $\tau \rightarrow y$, it follows that given $\eta > 0$, we may choose $\delta > 0$ so that $|g_i| < \varepsilon$, $i=1, 2, 3$, whenever t and τ obey the conditions: $|t-x| \leq \delta$, $|\tau-y| \leq \delta$. By setting $t=i/m$ and $\tau=j/m$ we should have

$$(18) \quad \left| g_1 \left(\frac{i}{m}, y \right) \right| < \varepsilon, \quad \left| g_2 \left(\frac{i}{m}, \frac{j}{m} \right) \right| < \varepsilon, \quad \left| g_3 \left(x, \frac{j}{m} \right) \right| < \varepsilon,$$

$$\text{whenever } \left| \frac{i}{m} - x \right| \leq \delta \quad \text{and} \quad \left| \frac{j}{m} - y \right| \leq \delta.$$

Clearly

$$(19) \quad |\varrho_m^{[\alpha]}(x, y)| \leq \sum_{0 \leq i+j \leq m} G_{m,i,j}^{[\alpha]}(x, y),$$

where, for brevity, we used the notation

$$\begin{aligned}
G_{m,i,j}^{[\alpha]}(x, y) &= \omega_{m,i,j}^{[\alpha]}(x, y) \left[\left(\frac{i}{m} - x \right)^2 \left| g_1 \left(\frac{i}{m}, y \right) \right| \right. \\
&\quad \left. + \left| \left(\frac{i}{m} - x \right) \left(\frac{j}{m} - y \right) \right| \left| g_2 \left(\frac{i}{m}, \frac{j}{m} \right) \right| + \left(\frac{j}{m} - y \right)^2 \left| g_3 \left(x, \frac{j}{m} \right) \right| \right].
\end{aligned}$$

If we divide the set of indices (i, j) into four classes:

$$I_1: \left\{ (i, j) : 0 \leq i+j \leq m, \left| \frac{i}{m} - x \right| \leq \delta, \left| \frac{j}{m} - y \right| \leq \delta \right\},$$

$$I_2: \left\{ (i, j) : 0 \leq i+j \leq m, \left| \frac{i}{m} - x \right| \leq \delta, \left| \frac{j}{m} - y \right| \geq \delta \right\},$$

$$I_3: \left\{ (i, j) : 0 \leq i+j \leq m, \left| \frac{i}{m} - x \right| \geq \delta, \left| \frac{j}{m} - y \right| \leq \delta \right\},$$

$$I_4: \left\{ (i, j) : 0 \leq i+j \leq m, \left| \frac{i}{m} - x \right| \geq \delta, \left| \frac{j}{m} - y \right| \geq \delta \right\}$$

and split the sum from (19) in a corresponding way we have:

$$|e_m^{[\alpha]}(x, y)| \leq \sum_{\nu=1}^4 \sum_{(i,j) \in I_\nu} G_{m,i,j}^{[\alpha]}(x, y).$$

Further if we take into consideration the foregoing results and proceed as in [5] one sees that $e_m^{[\alpha]}(x, y) = \frac{1}{m} \eta_m^{[\alpha]}(x, y)$, where $\eta_m^{[\alpha]}(x, y) \rightarrow 0$ as $m \rightarrow \infty$ and that this convergence is uniform on D (resp. on D_ϵ) if the function f has all its second-order partial derivatives continuous on D (resp. on D_ϵ), under the assumptions made on the parameter α .

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