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ON APPROXIMATION IMPROVEMENT FOR TRIGONOMETRIC SINGULAR INTEGRALS BY MEANS OF FINITE OSCILLATION KERNELS WITH SEPARATED ZEROS*

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Summary. The purpose of this paper is to present a method of constructing explicit examples of kernels of finite oscillation in order to break Korovkin's barrier on the optimal order of approximation by means of positive singular integrals. The main tool is the saturation limit of the corresponding kernel.

1. Let T_n denote the class of all even trigonometric polynomials of degree $n \in \mathsf{N} \subset \mathsf{N}_0 = \{0, 1, 2, \dots\}$ being normalized by $\int_{-\pi}^{\pi} p_n(t)dt = \pi$, thus

$$p_n(t) = \frac{1}{2} + \sum_{k=0}^{n} \varrho_{k,n}(p) \cos kt, \quad \varrho_{n,n}(p) \neq 0, \tag{1}$$

with row-finite convergence factors

$$\varrho_{k,n}(p) = \frac{1}{\pi} \int_{-\pi}^{\pi} p_n(t) \cos kt \, dt, \quad 1 \leq k \leq n; \tag{2}$$

$$\varrho_{0,n}(p) \equiv 1, \quad n \in \mathsf{N}, \quad \varrho_{k,n}(p) = 0, \quad k \geq n+1, \tag{3}$$

$$\varrho_{-k,n}(p) = \varrho_{k,n}(p), \quad k \in \mathsf{N}_0.$$

Moreover, for $m \in \mathsf{N}_0$, the class $\mathfrak{S}_{2m}$ consists of all $p_n \in T_n$ having exactly m changes of sign (zeros of odd multiplicity) in the interval $(0, \pi)$, m being independent of n; $p_n \in \mathfrak{S}_{2m}$ is said to be a kernel of finite oscillation of degree $2m$; see e. g. [9], [8], [7].

If $C_{2\pi}$ is the space of all real-valued functions continuous on the whole axis with period 2π, endowed with the usual maximum norm, then the singular integral of convolution type as defined for $n \in \mathsf{N}$ by

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(4) $$I_n(p;f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)p_n(t)dt, \quad f\in C_{2\pi},\ n\to\infty,$$

with kernel $p_n\in\mathfrak{S}_{2m}$ belongs to the Korovkin class S_{2m} of linear polynomial operators.

For integrals of class S_0, thus with positive kernels $p_n(t)\geq 0$, a well-known theorem of P. P. Korovkin states that

(5) $$\lim_{n\to\infty} n^2\| I_n(p;\cos u;x)-\cos x\|\neq 0;$$

i. e., the optimal order of appoximation for $I_n(p;f;x)$ with $p_n\in\mathfrak{S}_0$ cannot exceed* $O(n^{-2})$. This phenomenon sets up Korovkin's barrier.

On the other hand, a fundamental tool in saturation theory is the limit (if it exists)

(6) $$\lim_{n\to\infty} n^\tau(1-\varrho_{k,n}(p))=\psi(k), \quad \tau>0,$$

with $\psi(0)=0$ and $\psi(k)\neq 0$ for $k\neq 0$. For positive singular integrals it is known (see [1]) that the parameter τ of the saturation order is bounded with $\tau\leqq 2$, thus with maximal value $\tau=2$ just as in (5). Moreover, this sharpens (5) since

(7) $$|1-\varrho_{k,n}(p)|=\| I_n(p;\cos ku;x)-\cos kx\|, \quad 1\leqq k\leqq n.$$

If $I_n(p;f;x)\in S_{2m}$, it has been shown [2] that in (6) $\tau\leqq 2m+2$; in terms of (7) this, more exactly, means that for at least one k with $1\leqq k\leqq m+1$ the function $\cos kx$ cannot be approximated with an order better than $O(n^{-2m-2})$.

In what follows we wish to give a general method for constructing kernels of finite oscillation $p_n\in\mathfrak{S}_2$, i. e., with two simple symmetric zeros, such that the order of approximation of the corresponding singular integral (4) is, in fact, increased to the optimal $O(n^{-4})$, thus to build up operators of class $S_2^{(\tau)}$ with $2\leq\tau\leq 4$.

Finally, we recall that a kernel of degree $(n+1)$ of $\mathfrak{S}_2$ with zeros at $\pm\alpha$ has the unique representation (see [2], [3])

(8) $$p^*_{n+1}(t)=\frac{\cos t-\cos\alpha}{\varrho_{1,n}(p)-\cos\alpha}p_n(t)=\frac{1}{2}+\sum_{k=1}^{n+1}\varrho^*_{k,n+1}(p)\cos kt, \quad p_n(t)\geqq 0,$$

with the (positive) factor kernel $p_n\in\mathfrak{S}_0\cap T_n$ and associated convergence factors

$$\varrho^*_{k,n+1}(p)=\frac{\varrho_{k-1,n}(p)-2\varrho_{k,n}(p)\cos\alpha+\varrho_{k+1,n}(p)}{2(\varrho_{1,n}(p)-\cos\alpha)}, \quad 1\leq k\leq n+1,$$

where $\varrho_{k,n}(p)$ are the convergence factors (2), (3) of $p_n(t)$. In view of the crucial importance of (6) we shall consider the asymptotic expansion of the characteristic difference in the appropriate form

(9) $$1-\varrho^*_{k,n+1}=\frac{(1-\varrho_{k-1,n})+(1-\varrho_{k+1,n})-2(1-\varrho_{1,n})-2(1-\varrho_{k,n})\cos\alpha}{2[(1-\cos\alpha)-(1-\varrho_{1,n})]}.$$

* All O, o-relations occurring are valid for $n\to\infty$.

2. First of all we have to select a suitable subclass of factor kernels such that the zero $\alpha=\alpha(p;n)$ which is still a free parameter may be chosen so as to increase τ of (6) to 4 once the expansion of (9) is established.

For this purpose we introduce the class $\mathfrak{S}_0^{(\tau,\mu)}$ of those kernels $p_n \in \mathfrak{S}_0$ satisfying the special asymptotic expansion

$$(10) \qquad 1-\varrho_{k,n}(p)=\sum_{j=1}^{\mu}(-1)^{j+1}\psi_j(k)n^{-\tau j}+o(n^{-\tau\mu}), \quad n\in \mathrm{N},\ 0<\tau\leq 2,$$

for at least one fixed $\mu=1,2,3,\ldots$ and all $k\in \mathrm{N}_0$; then μ is called the index of p_n. (From this definition it is clear that kernels of $\mathfrak{S}_0$ having limit (6) are, in particular, of class $\mathfrak{S}_0^{(\tau,1)}$ with $\psi(k)=\psi_1(k)$.) Of special interest will be the values $\tau=1$ and $\tau=2$. From (3) it follows that for $1\leq j\leq\mu$

$$(11) \qquad \psi_j(0)=0,\ \psi_j(-k)=\psi_j(k),\quad k=1,2,3,\ldots$$

With increasing μ expansion (10) becomes a more and more restrictive condition upon those p_n which are to generate the oscillation kernels (8). For our purpose we shall need $p_n\in\mathfrak{S}_0^{(\tau,3)}$, thus $\mu\geq 3$.

A necessary condition upon α for approximation improvement is given by (see [2])

$$(12) \qquad \lim_{n\to\infty}\alpha(p;n)=0.$$

Following (12) we set up the specific finite expression

$$(13) \qquad \alpha(p;n)=\gamma n^{-\tau/2},\ \gamma=\gamma(p)=\mathrm{const},$$

a so-called separated zero (implicit zeros given by $1-\cos\alpha$ are, in contrast, used in [10]). A zero of type (13) makes sense since in this case the Taylor-series expansion of $(1-\cos\alpha)$ occurring in (9) contains the same powers of n as the expansion (10).

Using the second differences $\Delta^2\psi_j(k)=\psi_j(k-1)-2\psi_j(k)+\psi_j(k+1)$ we may rewrite (9) in the form

$$(14) \qquad 1-\varrho^*_{k,n+1}(p)=Z/N$$

with

$$(15) \qquad Z=\{\Delta^2\psi_1(k)-2\psi_1(1)\}n^{-\tau}-\{\Delta^2\psi_2(k)-2\psi_2(1)-\gamma^2\psi_1(k)\}n^{-2\tau}$$
$$+\{\Delta^2\psi_3(k)-2\psi_3(1)-\gamma^2\psi_2(k)-\frac{1}{12}\gamma^4\psi_1(k)\}n^{-3\tau}+o(n^{-3\tau}),$$
$$N=\{\gamma^2-2\psi_1(1)\}n^{-\tau}+O(n^{-2\tau}).$$

From this representation we immediately conclude

Lemma 1. In order to construct an oscillation kernel (8) of class $\mathfrak{S}_2\cap T_{n+1}$ with zero (13) and generated by a factor kernel $p_n\in\mathfrak{S}_0^{(\tau,3)}\cap T_n$ such that $1-\varrho^*_{k,n+1}(p)=O(n^{-2\tau})$, $0<\tau\leq 2$, in expansion (9), the conditions

$$(16) \qquad \Delta^2\psi_1(k)-2\psi_1(1)=0,$$

$$(17) \qquad \Delta^2\psi_2(k)-2\psi_2(1)-\gamma^2\psi_1(k)=0$$

are necessary; if, in addition,

(18) $$\gamma^2-2\psi_1(1)\neq 0$$

is satisfied, then these three conditions are also sufficient.

The first equation (16) is independent of γ, thus of $\alpha(p;n)$, giving a further restriction on $p_n(t)$, thus on $\psi_1(k)$ of (10). On the other hand, since γ is independent of k equation (17) yields another restriction, namely on the structure of $\psi_2(k)$. Finally, (18) has to be verified with γ of (17) whereas the factor of $n^{-3\tau}$ in (15) must not vanish simultaneously.

The general solution of (16), an ordinary second order linear inhomogeneous difference equation with constant coefficients, is given by (see e. g. [6, p. 121 ff, 146]) $\psi_1(k)=(A+Bk)+\psi_1(1)k^2$, $A, B=\text{const}$; setting $k=0, 1$ this gives $A=B=0$ in view of (11), thus

(19) $$\psi_1(k)=\psi_1(1)k^2,\ \psi_1(1)=\text{const}>0.$$

The fact that $\psi_1(1)$ is indeed positive follows from (cf. [10], [4])

$$0<\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(2\sin\frac{t}{2}\right)^2 p_n(t)dt=1-\varrho_{1,n}(p)=\psi_1(1)n^{-\tau}+O(n^{-2\tau}).$$

This decisive condition (19) upon the structure of $\psi_1(k)$ restricts the class of factor kernels to those which a priori satisfy Korovkin's equivalence relation

$$\lim_{n\to\infty}\frac{1-\varrho_{k,n}(p)}{1-\varrho_{1,n}(p)}=k^2 \iff \lim_{n\to\infty}\frac{1-\varrho_{2,n}(p)}{1-\varrho_{1,n}(p)}=4.$$

(For this and further characterizations of these kernels see, in particular, [1], [4], [10].)

Inserting the result of (19) the further difference equation (17) reads

(20) $$\Delta^2\psi_2(k)=\psi_1(1)\gamma^2k^2+2\psi_2(1),$$

the right-hand side now depending upon k, too. The solution of (20), observing the auxiliary conditions (11), is given as (see [6])

(21) $$\psi_2(k)=\{\psi_2(1)-\frac{1}{12}\psi_1(1)\gamma^2\}k^2+\frac{1}{12}\psi_1(1)\gamma^2k^4;$$

this reveals that $\psi_2(k)$ is an even algebraic polynomial in k of degree exactly 4 without constant term.

In the particular instance

(22) $$\gamma^2=12\frac{\psi_2(1)}{\psi_1(1)},$$

$\psi_2(k)$ consists only of the maximal power k^4 with a positive coefficient, thus is a positive function just as (19). In general, γ^2 is determined by letting $k=2$, say, in (21). This yields

(23) $$\gamma^2=\frac{\psi_2(2)-4\psi_2(1)}{\psi_1(1)},$$

and, instead of (21), moreover

$$(24)\qquad \psi_2(k)=\frac{1}{12}[\psi_2(2)-4\psi_2(1)]k^4-\frac{1}{12}[\psi_2(2)-16\psi_2(1)]k^2.$$

In connection with Lemma 1 these results may be summarized as

Lemma 2. The functions $\psi_j(k)$, $j=1, 2$, of (10) must necessarily be even algebraic polynomials in k of degree 2_j, namely

$$(25)\qquad \psi_1(k)=c_{11}k^2,$$

$$(26)\qquad \psi_2(k)=c_{12}k^2+c_{22}k^4$$

with positive maximal coefficients c_{11}, c_{22}.

With the notations (25), (26) we have for (23) — the same is true for (22) — and (14), respectively,

$$(27)\qquad \gamma^2=12\frac{c_{22}}{c_{11}},$$

$$(28)\qquad \lim_{n\to\infty} n^{2\tau}(1-\varrho^*_{k,n+1}(p))=\psi^*(k),\quad 0<\tau\leqq 2,$$

$$(29)\qquad \psi^*(k)=\frac{\left\{[\Delta^2\psi_3(k)-2\psi_3(1)]-\frac{12}{c_{11}}[(c_{11}+c_{22})c_{22}k^2+c_{22}^2k^4]\right\}n^{-3\tau}+o(n^{-3\tau})}{\frac{2}{c_{11}}(6c_{22}-c_{11}^2)n^{-\tau}+O(n^{-2\tau})}$$

provided $6c_{22}\neq c_{11}^2$ which is condition (18).

This gives rise to several remarks*. Whereas the zeros (13) are completely determined by (23) or (27), thus already by the first two functions $\psi_1(k)$, $\psi_2(k)$ of (10) which, in turn, must be of the special structure as given by Lemma 2, the additional third function $\psi_3(k)$ of (10) is needed only after $\psi^*(k)$ of (29) is calculated. On the other hand, it is just the power $n^{-3\tau}$ in the relevant expansion corresponding to $\psi_3(k)$ that involves the improved saturation order $O(n^{-2\tau})$ of (28). This is the reason why we started off with $\mu\geqq 3$ in the basic expansion (10) in order to build up optimal kernels of class $\mathfrak{S}_2$.

In order to formulate the final theorem in a form more convenient for the applications let us assume** that $\psi_3(k)$ is also of type (25), (26). Thus we define the functions of (10) consistently as

$$(30)\qquad \psi_j(k)=\sum_{i=1}^{j} c_{ij}k^{2i},\quad j=1, 2, 3;\ c_{jj}>0,$$

with triangular matrix c_{ij} of real numbers (for the fact that indeed $c_{jj}>0$ see [10]); furthermore we consider more specific functions

$$(31)\qquad \psi_j(k)=c_{jj}k^{2j},$$

i. e., if $\psi_j(k)$ only have maximal coefficients as is often the case.

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* In this connection compare the article of E. Görlich [5] in these Proceedings (p. 187—191) where structural properties of functions $\psi(k)$ appearing in general saturation limits are discussed; see furthermore [10].

** This is justified by the examples. If this assumption must necessarily be true in general remains open; see again [5].

Theorem. If for an oscillation kernel p_{n+1}^* of class $\mathfrak{S}_2$ the generating factor kernel satisfies both (10) with index $\mu \geq 3$ and (30) with $6c_{22} \neq c_{11}^2$ then the saturation limit (28) exists with

$$\psi^*(k) = 3\,\frac{5c_{11}c_{33} - 2c_{22}^2}{6c_{22} - c_{11}^2}\,(k^4 + k^2) + 6\,\frac{c_{11}c_{23} - c_{12}c_{22}}{6c_{22} - c_{11}^2}\,k^2; \tag{32}$$

if moreover $p_n \in \mathfrak{S}_0^{(\tau,3)}$ satisfies (31), then

$$\lim_{n\to\infty} n^{2\tau}(1 - \varrho_{k,n+1}^*(p)) = 3\,\frac{5c_{11}c_{33} - 2c_{22}^2}{6c_{22} - c_{11}^2}\,k^2(k^2 + 1). \tag{33}$$

In any case, the zeros are determined by (13) together with $\pm\gamma$ of (27). The corresponding singular integral $I_n(p^*; f; x)$ is a uniform approximation process with improved saturation order $O(n^{-2\tau})$, $0 < \tau \leq 2$, thus belongs to class $S_2^{(2\tau)}$.

To prove that $\lim_{n\to\infty} \| I_n(p^*; f; x) - f(x) \| = 0$, in view of the fact that $\lim_{n\to\infty} \varrho_{k,n+1}^*(p) = 1$, $k = 1, 2, 3, \ldots$, it only remains to show that the Lebesgue constants of p_{n+1}^* are uniformly bounded (cf. [1]). But it is easily seen rom (8) that

$$L_n(p^*) \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} |p_{n+1}^*(t)|\,dt = \frac{\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \left|(1-\cos\alpha) - \frac{1}{2}\left(2\sin\frac{t}{2}\right)^2\right| p_n(t)\,dt}{|(1-\cos\alpha) - (1-\varrho_{1,n})|}$$

$$\leq \frac{|1-\cos\alpha| + |1-\varrho_{1,n}|}{|(1-\cos\alpha) - (1-\varrho_{1,n})|} = O(1)$$

by construction; thus the norm of these operators is bounded per se.

In order to solve the complete saturation problem for these approximation processes (also for the spaces $L_{2\pi}^p$, $1 \leq p < \infty$) including characterizations of the saturation class, the general theory of [1] may be applied using the special functions $\psi^*(k)$ of (32) or (33), respectively. The same is true for establishing theorems of Voronovskaja-type, pointwise as well as in the norm.

3. Let us conclude with three representative examples of factor kernels. The singular integral of Fejér-Korovkin has kernel (of degree $n-2$)

$$K_{n-2}(t) = \frac{1}{n}\sin^2\frac{\pi}{n}\,\frac{\cos^2 n\dfrac{t}{2}}{\left(\cos t - \cos\dfrac{\pi}{n}\right)^2}, \quad n = 1, 2, 3, \ldots,$$

with convergence factors

$$\varrho_{k,n-2}(K) = \left(1 - \frac{k}{n}\right)\cos\frac{k\pi}{n} + \frac{1}{n}\cot\frac{\pi}{n}\sin\frac{k\pi}{n}.$$

The asymptotic expansion

$$1 - \varrho_{k,n-2}(K) = \frac{\pi^2}{2}k^2\frac{1}{n^2} - \frac{\pi^3}{3}(k^3 - k)\frac{1}{n^3} - \frac{\pi^4}{24}k^4\frac{1}{n^4} + O(n^{-5}) = \frac{\pi^2}{2}k^2\frac{1}{n^2} + O(n^{-3})$$

is easily shown to be valid. Thus, in view of (10) with $\mu=1$, the kernel $K_{n-2}(t)$ only belongs to $\mathfrak{S}_0^{(2,1)}$ and is not a generating factor kernel — though this kernel is, in some sense, the best possible positive kernel since it satisfies at least two intrinsic extremal properties for polynomials of class T_{n-2}; compare e. g. [4], [10]. It is this negative example that uncovers the special features of factor kernels which are to generate (optimal) kernels of finite oscillation (cf. [3]).

The kernel of de La Vallée Poussin as given by

$$V_n(t)=\frac{(n!)^2}{2(2n)!}\left(2\cos\frac{t}{2}\right)^{2n},\quad \varrho_{k,n}(V)=\frac{(n!)^2}{(n-k)!\,(n+k)!},\quad n=0,1,2,\ldots,$$

admits the expansion (which may be continued infinitely in this way; see [10])

$$1-\varrho_{k,n}(V)=k^2\frac{1}{n}-\frac{1}{2!}(k^4+k^2)\frac{1}{n^2}+\frac{1}{3!}(k^6+4k^4+k^2)\frac{1}{n^3}+O(n^{-4});$$

this gives (at least) $\mu=3$ together with nonoptimal $\tau=1$ in (10), thus $V_n\in\mathfrak{S}_0^{(1,3)}$.

Therefore, with general functions $\psi_j(k)$ of (30), it follows from (27) and (13) that $\gamma^2(V)=6$, $\alpha(V;n)=\pm\sqrt{6/n}$; since $\gamma^2(V)\neq 2$ condition (18) is satisfied. The new saturation limit is due to (32), namely

$$\lim_{n\to\infty} n^2(1-\varrho^*_{k,n+1}(V))=\frac{1}{4}(2k^4+7k^2).$$

The associated kernel of $\mathfrak{S}_2$ is explicitly given by (8) as

$$V^*_{n+1}(t)=\frac{\cos t-\cos\sqrt{6/n}}{(n/n+1)-\cos\sqrt{6/n}}V_n(t),$$

defining a singular integral of class $S_2^{(2)}$.

For the first generalized Korovkin-kernel

$$(34)\qquad K^{(1)}_{n-2}(t)=\frac{8}{5n}\sin^6\frac{\pi}{n}\,\frac{\left(\cos t+2\cos\frac{\pi}{n}\right)^2\cos^2 n\frac{t}{2}}{\left(\cos t-\cos\frac{\pi}{n}\right)^2\left(\cos t-\cos\frac{3\pi}{n}\right)^2},\quad n=4,5,6,\ldots,$$

the asymptotic expansion of the corresponding convergence factors (see [3 p. 456]) leads to

$$1-\varrho_{k,n-2}(K^{(1)})=\frac{9}{10}\pi^2k^2n^{-2}-\frac{3}{8}\pi^4k^4n^{-4}+\frac{41}{400}\pi^6k^6n^{-6}+O(n^{-7})$$

the order of the remainder being exact; here the functions $\psi_j(k)$ are of type (31) with pure maximal powers. Thus, with $\mu=3$, $\tau=2$ in (10), i. e., $K^{(1)}_{n-2}\in\mathfrak{S}_0^{(2,3)}$, one has immediately

$$(35)\qquad \gamma^2(K^{(1)})=5\pi^2,\quad \alpha(K^{(1)};n)=\pm\sqrt{5}\frac{\pi}{n}$$

with (18) being satisfied. (33) leads to the saturation limit

$$(36)\qquad \lim_{n\to\infty} n^4(1-\varrho^*_{k,n-1}(K^{(1)}))=\frac{3}{8}\pi^4(k^4+k^2)$$

which attains the optimal order of approximation related to kernels of $\mathfrak{S}_2$. Thus, the new singular integral with kernel $K_{n-1}^{(1)}(t)$ belongs to $S_2^{(4)}$.

The explicit evalution of the kernel (34) together with its convergence factors is to be found in [3]. (35) as well as (36) have been derived there by a procedure which is a forerunner of the method developed in this note in order to obtain a first concrete example of a polynomial approximation process of class $S_2^{(4)}$.

The straightforward generalization of this method to $m\geqq 2$, which is clear from the above considerations, will be published elsewhere. For another quite different method which indeed starts off with the expansion (10) with $\mu\geqq 3$, too, but gives families of new oscillation kernels with nonseparated implicit zeros as well as for a more thorough examination of suitable factor kernels (including nonpolymial ones) together with an extensive list of references we must refer to [10].

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