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RATIONAL APPROXIMATION TO CONTINUOUS FUNCTIONS ON THE ENTIRE REAL LINE

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Summary. We consider functions of one complex variable which are analytic in a strip, or more generally, in an infinite domain, equally containing the entire real line in their inside. If such a function has finite limits as the real variable tends to $-\infty$ or $+\infty$, then we can construct rational functions which approximate this function on $(-\infty, +\infty)$. The error of approximation depends on the domain of analyticity and on the continuity module of a certain transformed function. Of course, when the limits mentioned above are different then some weight-functions appear in the error-estimation. In the proofs results and methods of A. A. Gončar, D. J. Newman and the author are used.

1. Introduction. In § 2 we prove a general theorem concerning the rational approximation of continuous functions on the entire real line which have finite limits at $\pm\infty$. This theorem reduces the problem to a finite interval on which several results are proven concerning the rational approximation. In § 3 we give some concrete utilization of this theorem. The order f approximation depends on the properties of certain transformed functions n the general case also the weight-function $|x|$ appears. The use of our theorem lies in the fact that when the limits in $-\infty$ and $+\infty$ are the same then the approximation proves to be uniform, as we already know from the theory of Chebyshev. In § 4 we extend our results to unbounded functions. In the proofs results and methods of G. Freud, A. A. Gončar, D. J. Newman and the author are used.

2. The general theorem. Our starting point will be the following general

Theorem 1. Let $f(x)$ be continuous on the entire real line and assume that the finite limits

$$f(-\infty)=\lim_{x\to-\infty} f(x), \quad f(\infty)=\lim_{x\to\infty} f(x) \tag{1}$$

exist. Then there exist rational functions $r_n(x)$ of degree at most n such that

$$|f(x)-r_n(x)|\leq \max_{j=0,1} R_m(f_j)+e^{-3\sqrt{k}}\Big[\max_{\substack{j=0,1\\ -1\leq x\leq 1}}|r'_{m,j}(x)|+\frac{1}{2}|x|\,|f(\infty)-f(-\infty)|\Big], \tag{2}$$

for $-\infty<x<+\infty$; $n\geq n_0$,

where m and k are arbitrary positive integers with $4m+2k+6\leqq n$ and

$$(3)\qquad f_j(x)=f\left((-1)^j\sqrt{\frac{1-x}{1+x}}\right),\quad j=0,1,$$

are continuous functions on $[-1,+1]$ and

$$(4)\qquad \max_{-1\leqq x\leqq 1}|f_j(x)-r_{m,j}(x)|\leqq R_m(f_j),\quad j=0,1,$$

with suitable rational functions $r_{m,j}(x)$, $j=0,1$, of degree at most m.

Remark. Of course, our theorem gives a reasonable estimate only if we know that

$$(5)\qquad \max_{-1\leqq x\leqq 1}|r'_{m,j}(x)|=O(e^{\sqrt{k}}),\quad j=0,1.$$

Proof. By (4) we have

$$(6)\qquad \left|f\left(\sqrt{\frac{1-x}{1+x}}\right)-r_{m,0}(x)\right|\leqq R_m(f_0),\quad -1<x\leqq 1,$$

and

$$(7)\qquad \left|f\left(-\sqrt{\frac{1-x}{1+x}}\right)-r_{m,1}(x)\right|\leqq R_m(f_1),\quad -1<x\leqq 1.$$

Here we may assume

$$(8)\qquad r_{m,j}(-1)=f((-1)^j\infty)\quad\text{and}\quad r_{m,j}(1)=f(0),\quad j=0,1$$

(this can be attained by adding a properly chosen linear function to $r_{m,j}(x)$; by this the degree of $r_{m,j}(x)$ increases at most with 1, and the order of magnitude of $\max\{|r'_{m,j}(x)|, -1\leqq x\leqq 1\}$, $j=0,1$, does not change).

Put $(1-x^2)/(1+x^2)$ instead of x in (6) and (7). Then we have

$$(9)\qquad \left|f(x)-r_{m,0}\left(\frac{1-x^2}{1+x^2}\right)\right|\leqq R_m(f_0),\quad 0\leqq x<\infty,$$

and

$$(10)\qquad \left|f(x)-r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right|\leqq R_m(f_1),\quad -\infty<x\leqq 0.$$

Let

$$T_m(x)=\begin{cases}\frac{1}{2}\left[r_{m,0}\left(\frac{1-x^2}{1+x^2}\right)+r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right]+\frac{|x|}{2x}\left[r_{m,0}\left(\frac{1-x^2}{1+x^2}\right)-r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right] & \text{if } x\neq 0,\\ f(0) & \text{if } x=0.\end{cases}$$

This is a continuous function on $(-\infty,+\infty)$, and

$$T_m(x)=\begin{cases} r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) & \text{if } 0\leqq x<\infty,\\ r_{m,1}\left(\frac{1-x^2}{1+x^2}\right) & \text{if } -\infty<x\leqq 0.\end{cases}$$

Thus, by (9) and (10) we obtain

$$(11)\qquad |f(x)-T_m(x)|\leqq R_m(f_j),\quad -\infty<x<+\infty;\quad j=0,1.$$

In order to approximate $T_m(x)$ by rational functions, we use the result of D. J. Newman [1], improved by A. P. Bulanov [2], according to which there exist rational functions $Q_k(x)$ of degree at most k such that

$$(12) \qquad \max_{-1\leq x\leq 1} \| x| - Q_k(x)| \leq e^{-3\sqrt{k}}$$

for k large enough. Let

$$r_n(x) = \frac{1}{2}\left[r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) + r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right] + \frac{1+x^2}{4x} Q_k\left(\frac{2x}{1+x^2}\right)\left[r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right].$$

This is a rational function of degree at most $4m+2k+6\leq n$, and because of $2|x|/(1+x^2)\leq 1$, $-\infty<x<\infty$, and (12), we get

$$(13) \quad |T_m(x) - r_n(x)| \leq \frac{1+x^2}{4|x|} e^{-3\sqrt{k}} \left| r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right|, \quad -\infty<x<\infty.$$

We now distinguish two cases.

Case 1: $|x|\leq 1$. Then by (8)

$$\frac{1+x^2}{4|x|}\left| r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}\left(\frac{1-x^2}{1+x^2}\right)\right| \leq \frac{|x|}{2}\left[\left|\frac{r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,0}(1)}{\frac{1-x^2}{1+x^2}-1}\right|\right.$$

$$\left. + \left|\frac{r_{m,1}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}(1)}{\frac{1-x^2}{1+x^2}-1}\right|\right] \leq \max_{\substack{0\leq x\leq 1\\ j=0,\,1}} |r'_{m,j}(x)|.$$

Case 2: $|x|\geq 1$. Then by (8)

$$\frac{1+x^2}{4|x|}\left| r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}\left(\frac{1+x^2}{1+x^2}\right)\right| \leq \frac{1}{2|x|}\left[\left|\frac{r_{m,0}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,0}(-1)}{\frac{1-x^2}{1+x^2}-(-1)}\right|\right.$$

$$\left. + \left|\frac{r_{m,1}\left(\frac{1-x^2}{1+x^2}\right) - r_{m,1}(-1)}{\frac{1-x^2}{1+x^2}-(-1)}\right|\right] + \frac{1+x^2}{4|x|}|f(\infty) - f(-\infty)|$$

$$\leq \max_{\substack{-1\leq x\leq 0\\ j=0,\,1}} |r'_{m,j}(x)| + \frac{|x|}{2}|f(\infty) - f(-\infty)|.$$

Collecting our estimates we have from (13)

$$|T_m(x) - r_n(x)| \leq \left[\max_{\substack{-1\leq x\leq 1\\ j=0,\,1}} |r'_{m,j}(x)| + \frac{|x|}{2}|f(\infty) - f(-\infty)|\right] e^{-3\sqrt{k}},$$

$$-\infty<x<+\infty.$$

This together with (11) gives our Theorem 1.

3. Some applications. The first concrete realization of our general theorem will be given by Theorem 2. In order to formulate it, we give the following definition. Let $\psi(x)$ be a real-valued function with the following properties:

(i) $\psi(x)$ is continuous for all $-\infty<x<\infty$,
(ii) $\psi(x)=\psi(-x)>0$ for all $-\infty<x<\infty$;
(iii) $\psi(0)\leqq 1$;
(iv) $0\leqq d=d(\psi) \lim_{x\to\pm\infty} \frac{\psi(x)}{|x|}<\infty$.

Denote by $D(\psi)$ an infinite domain of the complex plane defined by

$$D(\psi)=\{z=x+iy: |y|<\psi(x), -\infty<x<\infty\}.$$

Using the previous notations, we state

Theorem 2. Let $f(z)$ be analytic and bounded in every finite point of $D(\psi)$ and assume that the finite limits (1) exist. Then there exist rational functions $r_n(x)$ of degree at most n such that

(14) $$|f(x)-r_n(x)|=O[\max_{j=0,1} \omega(f_j, \varepsilon_m)]+e^{-2\sqrt{m}|x|}|f(\infty)-f(-\infty)|,$$
$$-\infty<x<\infty,\ 5m+6\leqq n,$$

where ω denotes the module of continuity of the corresponding function on $-1, +1]$; ε_m depends on $d(\psi)$ and

15) $$O(e^{-\sqrt{m}})\leqq \varepsilon_m \leqq o\left(\frac{1}{m}\right).$$

Proof. Consider the analytic continuation of the functions (3) where we take that branch of the square root for which $f_0(0)=f(1)$ and $f_1(0)=f(-1)$ holds. It is easy to see that the domain of analyticity $\bar{D}(\psi)$ of these functions is symmetric to the real line and is bounded by a closed continuous curve having intersection-points with the real line at -1 and $+1$. Thus we can utilize Theorem 4 of [3]:

16) $$R_m(f_j)=O(\omega(f_j, \varepsilon_m)),\quad j=0, 1,$$

where $\varepsilon_m=o\left(\frac{1}{m}\right)$, and for the rational functions $r_{m,j}(x)$ realizing (16)

(17) $$\max_{-1\leqq x\leqq 1} |r'_{m,j}(x)|=O(m^2 e^{\sqrt{m}}),\quad j=0, 1,$$

holds. Substituting (16) and (17) into (2) and putting $k=[m/2]$ we get Theorem 2, q. e. d.

To see the dependence of ε_m on $d(\psi)$, we examine in details the image $\bar{D}(\psi)$ of $D(\psi)$ by the mapping $u=[(1-z)/(1+z)]^{1/2}$. If $u=x+i\psi(x)$ $-\infty<x<\infty$, then by easy calculation we obtain

$$\operatorname{Re} z=\frac{1-[x^2+\psi(x)^2]^2}{[1+x^2-\psi(x)^2]^2+4x^2\psi(x)^2},$$

$$\operatorname{Im} z=-\frac{4x\psi(x)}{[1+x^2-\psi(x)^2]^2+4x^2\psi(x)^2}.$$

The order of magnitude of ε_m depends on the character of the analyticity-domain $\bar{D}(\varphi)$ at the point -1. By (iv) we have

$$\left|\frac{\operatorname{Im} z}{\operatorname{Re} z+1}\right|=\left|\frac{2x\psi(x)}{1+x^2-\psi(x)^2}\right|=O\left(\frac{\psi(x)}{|x|}\right) \quad \text{if} \quad x\to\pm\infty.$$

Thus the "peak" of $\bar{D}(\psi)$ at -1 depends on the behaviour of $\psi(x)$ for $x \to \pm\infty$. Consider some special cases. If $\psi(x)=\delta(1+|x|)$, $0<\delta<1$, then

$$\left|\frac{\operatorname{Im} z}{\operatorname{Re} z+1}\right| \to \frac{2\delta}{1-\delta^2}, \quad x \to \pm\infty,$$

and thus, using the localization theorem cited in [3, § 5], we get

$$\varepsilon_m = O\{\exp[-c(m^{1/3})]\}.$$

(Here the constant $c>0$ depends on δ. We could not use Theorem 3 from [3] because it seems difficult to estimate the derivative of the corresponding rational function.) If $\psi(x)=\delta(1+|x|)^a$, $0<\delta<1$, $-\infty<a<1$, then $\operatorname{Im} z = O(|\operatorname{Re} z+1|^{(3-a)/2})$, $x \to \pm\infty$, and thus (see [3], § 5, Case 1°)

$$\varepsilon_m = \log^\gamma m / m^{(3-a)/(1-a)} \ (\gamma > (3-a)/(1-a) \text{ arbitrary.})$$

Especially for $a=0$ (i. e. when $f(z)$ is analytic in the strip $|y| \leq \delta$) we get $\varepsilon_m = (\log^\gamma m)/m^3$ ($\gamma>3$ arbitrary). If $\psi(x)=\exp(-|x|^a)$, $0<a<\infty$, then

$$\operatorname{Im} z = O\left(\frac{e^{-|\operatorname{Re} z+1|-a/2}}{|\operatorname{Re} z+1|}\right), \quad x \to \pm\infty,$$

and hence (see [3], § 5, Case 2°) $\varepsilon_m = (m \log^\gamma m)^{-1}$, ($\gamma < 2/a$ arbitrary).

If we drop the condition of analyticity then using the best approximating polynomials as $r_{m,j}(x)$ we get by the classical Jackson theorem that (14) holds with $\varepsilon_m = 1/m$.

While the order of ε_m depends on the analyticity-domain, the modules of continuity $\omega(f_i, \varepsilon_m)$ depend on the behaviour of $f(x)$ at $\pm\infty$. E. g. if

$$\sup_{-\infty<x<\infty} |f'(x)x^{2\varrho+1}| < \infty, \ 0<\varrho\leq 1,$$

then easy to verify that

$$\omega(f_j, \varepsilon_m) = O(\varepsilon_m^\varrho), \quad j=0, 1.$$

4. Unbounded functions. Finally, we prove that Theorems 1 and 2 also hold for a wider class of functions (with obvious modifications).

Theorem 3. Let $f(x)$ be a continuous function on the entire real line such that

$$|f(x)| \leq A + B|x|^{2s-1}, \quad -\infty<x<\infty; \quad A, B>0, \tag{18}$$

with a suitable positive integer s. Then there exist rational functions $r_n(x)$ of degree at most n such that

$$|f(x)-r_n(x)| \leq [\max_{j=0,1} R_m(f_j) + e^{r-3\sqrt{k}} \max_{\substack{-1\leq x\leq 1 \\ j=0,1}} |r'_{m,j}(x)|](A+Bx^{2s}), \tag{19}$$

$$-\infty<x<\infty,$$

where $4m+2k+2s+6 \leq n$ and

$$f_j(x) = \left[f\left((-1)^j\sqrt{\frac{1-x}{1+x}}\right)\right]\left[A+B\left(\frac{1-x}{1+x}\right)^s\right]^{-1}, \quad j=0, 1,$$

are continuous functions on $[-1, +1]$ and (4) holds with suitable rational functions $r_{m,j}(x)$, $j=0, 1$, of degree at most m.

Proof. Consider the function $F(x)=f(x)/(A+Bx^{2s})$. By (18), this satisfies the conditions of Theorem 1 ($A+Bx^{2s}$ has no roots on the real line) such that $\lim_{x\to\pm\infty} F(x)=0$. Thus there exist rational functions $\bar{r}_l(x)$ of degree at most l such that

$$|F(x)-\bar{r}_l(x)|\leqq \max_{j=0,1} R_m(f_j)+e^{-3\sqrt{k}} \max_{\substack{-1\leqq x\leqq 1\\ j=0,1}} |r'_{m,j}(x)|, \quad -\infty<x<+\infty,$$

with $4m+2k+6\leqq l$, i. e. by (20) we get (19) with $r_n(x)=(A+Bx^{2s})\bar{r}_l(x)$ q. e. d.

Using the previous notations we state

Theorem 4. Let $f(z)$ be analytic in $D(\psi)$ and assume that (18) holds for all $x\in D(\psi)$. Then there exist rational functions $r_n(x)$ of degree at most n such that

$$|f(x)-r_n(x)|=O[\max_{j=0,1} \omega(f_j, \varepsilon_m)](A+Bx^{2s}), \quad -\infty<x<\infty,$$

where $5m+2s+6\leqq n$ and $\varepsilon_m\leqq O(1/m)$.

The proof is the same as that of Theorems 2 and 3.

Our Theorems 1 and 3 can be applied to several problems of the theory of rational approximation if we can prove (5) for the derivative of the corresponding approximating rational functions.

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