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ON SOME APPLICATIONS OF GRAPH-THEORY TO ANALYSIS

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Summary. After recalling some simple notions and two theorems from the theory of graphs various applications are indicated to approximation theory, to potential theory and conformal mapping.

This lecture will deal with applications of graph theory to a) potential theory, b) conformal mapping, c) certain function spaces, b) calculus of probability.

From graph theory for the purposes of this talk we shall need besides the fundamental elements* "edge", "vertex", "degree of a vertex" only the notions of a subgraph and of a complete subgraph. The latter is a subgraph in which all pairs of vertices are connected by an edge in the original graph. The basis of almost all results mentioned in this talk is the following theorem** which I found in 1940 but its manysided applicability did not realize before 1968.

Theorem 1. Let $3 \leqq k \leqq n$ and h be defined by $n \equiv h \bmod (k-1)$ $0 \leqq h \leqq k-2$. Then in a graph G_n with n vertices which does not contain complete subgraph with k vertices, there are at most

$$(1) \qquad \frac{k-2}{2(k-1)}(n^2-h^2)+\binom{h}{2}$$

edges. Equality can be attained at the — up to isomorphism unique — graph $G^*_{n,k}$ in which the vertices can be distributed into $(k-1)$ disjoint classes, h of them containing each $[(n-h)/(k-1)]+1$ vertices, the other classes each $(n-h)/(k-1)$ vertices so that any two vertices from different classes are connected with an edge and no two from the same class.

* Throughout this talk we use the word "graph" in the sense that no multiple edges or loops are permitted, though extensions of the graph theorems to such graphs could be of significance in the applications.

** "Egy gráfelméleti extremumproblémáról" (In Hungarian with German abstract), *Matematikai és Fizikai Lapok* (1941), 436—452. An English translation of the proof is in the Appendix of my paper "On the theory of graphs" in *Colloquium Math.* 3 (1954), 19—30.

Thinking on the definition of the Newtonian potential one comes at once to the conclusion that — at least in the case of uniform mass distribution — everything depends after all on the distribution of distances. Having a point set A in R^m(m-dimensional Euclidean space) can one say anything reasonably general on this distribution? If yes, on what properties of A this distribution depends (which must be of course more easily treatable than the distribution itself)? We found that the answer to the first question is "yes". Let us fix any property H of A which satisfies only the following very mild requirements.

I. Property H should imply that A can be shifted into a fixed ball of R^m("boundedness").

II. With A each finite subset of A has property H.

III. Having for fixed l and $\nu=1, 2, \ldots$ finite point sets$(P_{1\nu}, P_{2\nu}, \ldots, P_{l\nu})$ in R^m having property H and for each fixed $1 \leq j \leq l$ and $\nu \to \infty$ $P_{j\nu} \to P_j$ then the point set $(P_1, P_2, \ldots, P_l)$ satisfies too property H ("closedness").

Plenty of such properties H can be given. Belonging to a bounded and closed domain in R^m, or fixing the length of the maximal chord, or restricting A to a g-dimensional surface in $R^m(g<m)$ or restricting the projection of A to any $(m-1)$-dimensional hyperplanes to a fixed domain on it after proper translation are all admissible H-properties and also many others.

Can one hope that something reasonable can be asserted on the distribution of distances of a point set A on which we require only the fulfilledness of an above defined property H? Perhaps a bit unexpectedly the answer is yes and we can answer also to the second question, i. e. on which way the fulfilledness of property H can regulate the distance-distribution of A. We found that this regulation is done by the so-called packing constants

$$d_\nu = d_\nu(H), \quad \nu=2, 3, \ldots, \tag{2}$$

belonging to the property H. These are defined by

$$d_\nu = \max \min_{1 \leq i < j \leq \nu} \overline{P_i P_j} \tag{3}$$

where the max refers to all $(P_1, \ldots, P_\nu)$ systems which satisfy the pro perty H. In order to justify the name let property H be the restriction o the points to the surface U of the unit sphere in R^3 (property H^*). Taking any ν points $P_1, \ldots, P_\nu$ on U and putting around each as centers spherical caps with the common spherical radius

$$\frac{1}{2} \min_{1 \leq i < j \leq \nu} \widehat{P_i P_j} = \arcsin \left(\frac{1}{2} \min_{1 \leq i < j \leq \nu} \overline{P_i P_j} \right)$$

they do not overlap and cover a certain part S of U. Maximizing $\min\limits_{1 \leq i < j \leq \nu} \overline{P_i P_j}$ the resulting $(P_1^*, \ldots, P_\nu^*)$ points realize the centers of the densest packing of U by ν congruent spherical caps.* We have obviously

$$d_2 \geq d_3 \geq \cdots \tag{4}$$

* These special packing constants d_ν (H^*) received attention of mathematicians since the dispute of Gregory and Newton (which boils down to the question whether $d_{13}(H^*)$ equals to 1 resp. <1) until recently as shown by the works of Fejes-Tóth, van der Waerden and Schütte, R. Robinson and others.

Equality sign can occur in various places; in distance-distribution — remarkably enough* — important role is played by the "critical indices" i_ν defined by $i_1=1$ and

(5) $$d_2=\cdots=d_{i_2}>d_{i_2+1}=\cdots=d_{i_3}>d_{i_3+1}=\cdots.$$

With all these we assert the

Theorem 2. If the point system $(P_1, P_2, \ldots, P_n)$ satisfies one of the above defined properties H with packing-constants d_ν then for $l\geqq 1$, $n>i_l$ the inequality

$$\overline{P_iP_j}\leqq d_{i_{l+1}}\,(=d_{i_l+1})$$

is satisfied for at least $\left(\frac{n^2}{2i_l}-\frac{n}{2}\right)$ (i, j)-pairs $(1\leqq i<j)\leqq n)$.

Theorem 2 is best possible for all of our H-properties, for all $l\geqq 1$ and for all $n\equiv 0 \bmod i_l$. The only known instance of the theorem is due to Paul Erdös. He gave in 1955 in *Elemente der Mathematik* in the Problemsektion the following problem. Having on the plane $3k$ points with the maximal distance 1 then at least $3k^2$ distances $\leqq\sqrt{2}/2$.

Are the packing constants more easily treatable than the distance-distribution itself? We can show in several instances that this is indeed the case. In the case when the points are restricted to the unit circle one can see at once that $d_\nu=2\sin(\pi/\nu)$, $\nu=2, 3, \ldots$, i. e. $i_\nu=\nu$ and hence Theorem 2 gives — with $l=5$ and a bit roughly expressed — that having n points on the unit circle at least one fifth part of all distances are $\leqq 1$ and this is best possible. Several packing constants were determined for the two and three dimensional cube domain resp. for its boundary by Prof. A. Méir. To mention also a general case, calling an arc L a "monotone arc" having the property that fixing any point P on it and moving another point Q along L off P the distance increases strictly monotonically one can see easily that in this case d_ν is furnished by the side length of the (uniquely determined) "quasiregular" ν-gon $P_1, P_2, \ldots, P_\nu$ with

(6) $$\overline{P_1P_2}=\overline{P_2P_3}=\cdots=\overline{P_{\nu-1}P_\nu},$$

P_1 and P_ν being the endpoints of L. Most probably all packing constants can be determined somewhat analogously for convex curves. Estimations for the packing constants can be often obtained from the remark that if A_1 and A_2 are sets with $A_1\subset A_2$ then

(7) $$d_\nu(A_1)\leqq d_\nu(A_2), \quad \nu=2, 3, \ldots$$

Now we can turn briefly to applications in potential theory and conformal mapping. Having a uniform mass distribution over the domain B in R^m its potential energy is given by the integral

$$\int\limits_{(B)}\int\limits_{(B)}\frac{dv_P dv_Q}{\overline{PQ}^{m-2}}$$

* In the literature of the packing constants d_ν (H^*) the strict decrease is very often investigated; e. g. another form of the Gregory-Newton dispute is whether $d_{12}(H^*)=d_{13}(H^*)$ or $d_{12}(H^*)>d_{13}(H^*)$. All these problems get by theorem 2 perhaps an additional interest.

(supposing "Newton" law and $k \geqq 3$). We consider more general energy integrals of the form

$$(8) \qquad I(B, f) = \int\limits_{(B)} \int\limits_{(B)} f(\overline{PQ})\, dv_P dv_Q$$

where B is bounded closed and Jordan-measurable domain with positive m-dimensional measure $|B|$ and $f(x)$ is for $x > 0$ positive and monotone decreasing. Using properly Theorem 2 an usual passage to the limit gives the following

Theorem 3. If the domain B satisfies one of the above defined properties H with the packing constants d_ν then for the energy integral $I(B, f)$ in (8) the inequality

$$(9) \qquad \frac{1}{|B|^2} I(B, f) \geqq \sum_{\nu=2}^{\infty} \frac{f(d_\nu)}{(\nu-1)\nu}$$

holds.

For $f(x) \equiv 1$ we have certainly equality in (9). The inequality (9) gives of course not one lower bound for the energy integral but several, according to the choice of property H (the strongest being the one where property H is the sole restriction to B). Other points of (9) are of course the facts that on the right side the places d_ν are independent of f and the factors $1/(\nu-1)\nu$ are universal.* If B is a $g < m$ dimensional subspace of R^m then (9) holds with the understanding that $|B|$ means the m-dimensional (positive) Jordan measure and dv_P, dv_Q are g-dimensional volume elements.

It follows from (9) at once that under the conditions of Theorem 3 the inequality

$$(10) \qquad \sup_{P \in B} \int\limits_{(B)} f(\overline{PQ})\, dv_Q \geqq |B| \sum_{\nu=2}^{\infty} \frac{f(d_\nu)}{(\nu-1)\nu}$$

holds for the (generalised) potential generated by f (under uniform mass distribution). Let us compare it with the corresponding case of the important upper bound of H. Cartan

$$(11) \qquad \int\limits_{(B)} f(\overline{PQ})\, dv_Q \leqq |B| \int\limits_{0}^{\varrho} f(x)\, dh(x)$$

(f monotone decreasing, h monotone increasing with $h(0)=0$, $h(\infty)>1$, $h(\varrho)=1$); this holds in R^m except perhaps a set which can be covered by countably many balls with radii r_ν obeying the inequality

$$\sum_{\nu} h(r) \leqq 6.$$

This made plausible the conjecture that the generalized potential on the left of (11) can be estimated in B from below with exception of a "small" set perhaps. This conjecture was proved, using again graph theory, by Vera T. Sós. She proved:

* These facts remind one vaguely of the classical formulae of mechanical quadrature.

Theorem 4. Under the restrictons of Theorem 3 and $0<\lambda\leq 1/i_l$ ($l\geq 2$ and prescribed) the inequality

$$\int_{(B)} f(\overline{PQ})\, dv_Q \geq |B|\, \lambda f(d_{i_{l+1}})$$

holds in B with exception perhaps of a set with measure $\leq \lambda |B| (i_l - 1)$.

Next we turn to some applications in the theory of conformal mapping. Let B be again a domain in the plane with boundary $\overline{B}$ and $r(B)$ its outer mapping radius. This $r(B)$ is an important geometrical constant of the domain B and its connections with other "more geometrical" data of B (in form of equalities or inequalities) were the subject of several investigations. We can now exhibit a connection (in fact several) of $r(B)$ with packing constants belonging to the boundary $\overline{B}$ of B. More exactly let H be one of the above defined geometrical properties of $\overline{B}$ and $\overline{d}_\nu = \overline{d}_\nu(H)$ be the packing constants belonging to it. Then we assert (with normalisation $d_2=1$) the

Theorem 5. For $r(B)$ the inequality

$$(12)\qquad r(B) \leq \prod_{\nu=2}^{\infty} (\overline{d}_\nu)^{1/(\nu-1)\nu}$$

holds.

Besides the independent interest of this inequality (or rather inequalities according to the possible choices of property H) this can be used in order to obtain effective upper bounds for $r(B)$ in the (very numerous) cases when the first few packing constants are known. Then keeping only finitely many factors in (12) we obtain already upper bounds for $r(B)$. This can be combined with (7) too, and also with a proper choice of property H. Theorem 5 holds also in the important case when B is a given arc; then the remark in (6) can be useful.

A widely investigated subject in the function theory is the capacity of bounded and closed point sets in the plane (see e. g. R. Nevanlinna's book "Eindeutige Funktionen"); in particular criteria for the capacity being 0. In the case of such point sets A an analogous reasoning leads such criteria in terms of packing constants $d_\nu(H)$ belonging to an admissible property H of $\overline{A}$. This is the

Theorem 6. The divergence of the series

$$(13)\qquad \sum_\nu \frac{1}{\nu^2} \log \frac{1}{d_\nu}$$

implies that the capacity of A is 0.

As Paul Erdös remarked the theorem is best possible in the sense that for any fixed positive ε the divergence of the series

$$\sum_\nu \frac{1}{\nu^{2-\varepsilon}} \log \frac{1}{d_\nu(H)}$$

does not imply in general the vanishing of the capacity. One can construct easily point sets A for which the series (13) diverges indeed.*

* There are graph theoretical and other possibilities to extend the results to more general mass distributions, but they are not yet in definitive form.

It was ab ovo clear from the proof that Theorem 2 holds for metric spaces too. Prof. A. Méir observed that it can be applied successfully also to some function spaces. Namely the use of it permits to solve exactly for some spaces K of functions in $[0, 1]$ the problem what can be said on the "probability" of the inequality

$$(14) \qquad |f(x)-g(x)|\leqq\varepsilon,$$

f, g in K, $0\leqq x\leqq 1$ and $\varepsilon>0$. By this we mean the question that taking n different functions $f_\nu(x)$ from K arbitrarily where $n>n_0(\varepsilon, K)$ and denoting by D the number of pairs among them for which (14) holds, what can be said for

$$(15) \qquad \binom{n}{2}^{-1} D$$

independently of the f_ν's, if n goes to infinity. The class for which we know the solution is the important Lip_1 1-class; more exactly the class K_0 defined by

$$f(0)=0, \quad |f(x_1)-f(x_2)|\leqq|x_1-x_2|; \quad 0\leqq x_1, x_2\leqq 1.$$

A little stronger than necessary to the above aim we assert the

Theorem 7. For $l=1, 2, 3, \ldots$ and $n>2^{l-1}$ from each set of n different functions $f_\nu(x)$, $\nu=1, 2, \ldots$, from K_0 the inequality

$$(16) \qquad |f_\mu(x)-f_\nu(x)|\leqq\frac{2}{l}, \quad 0\leqq x\leqq 1; \quad 1\leqq\mu<\nu\leqq n,$$

holds for at least

$$(17) \qquad \frac{n^2}{2^l}-\frac{n}{2}$$

pairs.

The lower bound in (17) is best possible for each positive integer l and n's with $n\equiv 0 \bmod 2^l$ and $n>2^{l-1}$. From this theorem we got that for each natural l the probability of (16) in the above sense is at least $1/2^{l-1}$, and this is best possible.

The applied method seems to lead to — at least — nearly best possible results of above type in the case of other function classes, too. These investigations have points of contact with deep results of Kolmogorov, Tihomirov and Vituskin.

Another sort of applications of Theorem 1 to function spaces was given recently by Gy. Katona. With the notation

$$E(n)=\begin{cases}\frac{n}{n}\left(\frac{n}{2}-1\right) & \text{if } n \text{ is even,}\\ \left(\frac{n-1}{2}\right)^2 & \text{if } n \text{ is odd}\end{cases}$$

he proved that having functions $f_\nu(x)$ with

$$(18) \qquad \int_a^b |f_\nu(x)|^2\,dx\geqq 1, \quad \nu=1, 2, \ldots, n,$$

implies for at least $E(n)$ pairs (μ, ν) with $1 \leqq \mu < \nu \leqq n$ the inequality

$$(19) \qquad \int_a^b |f_\mu(x) + f_\nu(x)|^2 \, dx \geqq 1$$

and this is again best possible. So (18) implies that (19) holds for roughly the half of all pairs (f_μ, f_ν); this could be stated again as a probability statement for the L_2-space.

All these lead directly to further applications to the calculus of probability. The possibility of such applications is clear already from Theorem 2. More exactly a form of it — suitably modified for our present aims — says that having a bounded and closed Jordan measurable domain B satisfying one of the above defined properties H having the packing constants $d_\nu = d_\nu(H)$ and critical indices i_ν then for each ϑ with $0 < \vartheta \leqq d_{i_2} = d_2$ the probability of $\overline{PQ} \leqq \vartheta$ (P, Q in B) with an obvious interpretation of the probability is $\geqq 1/i_l$ where l is determined by $d_{i_{l+1}} \leqq \vartheta < d_{i_l}$ and this is best possible for each of our H-properties and ϑ-values. It was plausible to ask for corresponding results on triangles. Here the graph theoretical method — owing to the incompleteness of the analogon of Theorem 1 — is less successful though it leads to some results which seem to be new and not obtainable with the same ease with other methods. So for instance if the above domain B is such that max area $\triangle PQR = 1$ (P, Q, R in B) then one can prove that the probability of area $\triangle PQR \leqq (\sqrt{5}-1)/2$ is at least $1/7$ (which is probably not best possible). But applications to calculus of probability of different sort can be given, too. As Gy. Katona proved recently — using again theorem I — if ξ and η are independent random variables with values from R^d and identical distributions then the inequality

$$\text{Prob}(|\xi + \eta| \geqq x) \geqq \frac{1}{2} \{\text{Prob}(|\xi| \geqq x)\}^2$$

holds; here $1/2$ is best possible.

The proofs of the above said theorems — together with further results — will appear in a sequence of joint papers with Paul Erdös, A. Méir and Vera T. Sós. In these papers — as in this talk — we lay the stress on exhibiting the manysided applicability with elegance and as clearly as possible and not to squeeze out the strongest possible results the method could give. A dangerous mathematical philosophy for the authors indeed in an age when only world records count; several theorems are nowadays considered as trivialities quoted without names (if quoted at all) the possibility of which nobody had the faintest idea in the time of the discovery.

After all these applications it is plausible to ask for the ultimate reasons of the power of the graph theoretical method. For this sake let l and n be arbitrary natural numbers and consider $ln+1$ elements each having exactly one of l different properties. Then the usual box-principle says that we have at least $n+1$ elements having the same property. Representing our elements as vertices of a graph and connecting two different vertex if and only if they possess the same property then the box-principle assured us the existence of a complete subgraph with $n+1$ vertices at least. This shows that every theorem, which assures in a graph from some property the existence of complete subgraph, can be considered as extensions of the box-principle. This part of the graph theory is not “the slum of topology” as was told 30 years ago; it has nothing to do with topology at all. It is rather a chapter of logic and a consequent reduction of the solution of all sorts of mathematical problems to appropriate graph theorems seems to us a fruitful new trend in mathematics (which reduction was done in the treatment of several individual problems of number theory and elementary geometry by Paul Erdös).

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