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ON THE ONE-SIDED TRIGONOMETRICAL AND SPLINE APPROXIMATION OF FUNCTIONS

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Summary. Jackson's type theorems are obtained for the best one-sided approximation in L_p, $1\leqq p\leqq\infty$, of the function f, bounded and 2π-periodical, by means of splines and trigonometrical polynomials. These theorems are obtained using the modulus

$$\tau(f;\delta)_{L_p}=\left(\int_0^{2\pi}(\omega(f,x;\delta))^p dx\right)^{1/p},$$

where $\omega(f,x;\delta)=\sup|f(t)-f(t')|$, $|t-x|\leqq\delta/2$, $|t'-x|\leqq\delta/2$.

The purpose of this paper is to suggest Jackson's type theorems for the best one-sided trigonometrical and spline approximation of functions.

Let T_n be the set of all trigonometrical polynomials of n-th order and let S_{k,Σ_n} be the set of all splines of k-th degree with knots in the points $\Sigma_n=\{a=x_0<x_1<\ldots<x_n=b\}$, i. e. $s\in S_{k,\Sigma_n}$ if $s\in C^{k-1}[a,b]$ and in each interval $[x_{i-1},x_i]$, $i=1,\ldots,n$, s is algebraical polynomials of k-th degree.

The best one-sided approximation of the function f bounded in the interval $[a,b]$ in L_p by means of splines of S_{k,Σ_n} is given by

$$\widetilde{E}_{k,\Sigma_n}(f)_{L_p}=\inf\|S-s\|_{L_p[a,b]},\ S,\ s\in S_{k,\Sigma_n},\ s(x)\leqq f(x)\leqq S(x),\ x\in[a,b],$$

and the best one-sided approximation of the bounded 2π-periodical function f in L_p by means of trigonometrical polynomials of n-th order is given by $\widetilde{E}^T_n(f)_{L_p}=\inf\|P-Q\|_{L_p(0,2\pi)}$, $P,\ Q\in T_n$, $Q(x)\leqq f(x)\leqq P(x)$ for all x.

There exist many works on one-sided approximation of functions. The first are the publications of G. Freud [1] and T. Ganelius [2]. The result of Ganelius is the following:

$$\widetilde{E}^T_n(f)_{L_1}\leqq c(k)n^{-k-1}V_0^{2\pi}f^{(k)},$$

where $V_0^{2\pi}f^{(k)}$ denotes the variation of the k-th derivative $f^{(k)}$ of the function f and $c(k)$ is a constant depending only on k (Ganelius gives its exact value).

V. F. Babenko and A. A. Ligun in [3] obtain the following result: $\sup\{\widetilde{E}_n^T(f)_{L_p}: f \in W^r L_p(0, 2\pi)\} = 0(n^{-r})$, $1 \leq p \leq \infty$, $r = 1, 2, \ldots$, where $W^r L_p(0, 2\pi)$ denotes the class of all 2π-periodical functions f with absolutely continuous $r-1$-th derivative for which $\|f^{(r)}\|_{L_p(0,2\pi)} \leq 1$.

A. Meir and A. Sharma [4] have considered one-sided L_1 approximation by means of splines of first and third degree. G. Freud and V. A. Popov [5] generalized their results for splines of arbitrary degree: $\widetilde{E}_{k,\Sigma_n}(f)_{L_1} \leq c_1(k) \Delta_n^{k+1} V_0^1 f^{(k)}$, where $\Delta_n = \max|x_i - x_{i-1}|$, $i = 1, \ldots, n$ and the constant $c_1(k)$ depends only on k. In [3] Babenko and Ligun obtain that $\sup\{\widetilde{E}_{k,\Sigma_n}(f)_{L_p}: f \in W^2 L_p(0, 2\pi)\} = 0(n^{-r})$, $1 \leq p \leq \infty$, $r = 1, 2, \ldots$, $\Sigma_n = \{x_i = i\pi/n,\ i = 0, \ldots, 2n\}$.

We shall give here Jackson's type theorems for $\widetilde{E}_n^T(f)_{L_p}$ and $\widetilde{E}_{k,\Sigma_n}(f)_{L_p}$, using the following modulus:

$$\tau(f;\delta)_{L_p[a,b]} = \left(\int_a^b (\omega(f, x; \delta))^p dx\right)^{1/p}, \quad 1 \leq p \leq \infty,$$

where $\omega(f, x; \delta) = \sup|f(t) - f(t')|$, $|t - x| \leq \delta/2$, $|t' - x| \leq \delta/2$, $t, t' \in [a, b]$.

For the history of this modulus see the paper of V. A. Popov in this volume.

The modulus $\tau(f;\delta)_{L_p}$ has the following properties:

1) $\tau(f;\delta)_{L_p} \leq \tau(f;\delta')_{L_p}$ for $\delta \leq \delta'$;

2) $\omega(f;\ \delta)_{L_p} \leq \tau(f;\ \delta)_{L_p}$, where $\omega(f;\delta)_{L_p[a,b]} = \sup\{(\int_a^{b-h}|f(x+h) - f(x)|^p dx)^{1/p} : 0 < h \leq \delta\}$;

3) $\tau(f+g;\delta)_{L_p} \leq \tau(f;\delta)_{L_p} + \tau(g;\delta)_{L_p}$;

4) $\tau(f;\lambda\delta)_{L_p} \leq (2\lambda+1)\tau(f;\delta)_{L_p}$;

5) $\tau(f;\delta)_{L_1} \leq 8\delta V_a^b f$;

6) $\tau(f;\delta)_{L_p} \leq \delta\|f'\|_{L_p}$;

7) $\tau(f;\ n^{-1})_{L_1} \leq 3\varkappa(f; 2n)/n$, where $\varkappa(f;\ 2n) = \sup_{\Sigma_n} \Sigma_{i=1}^n |f(x_i) - f(x_{i-1})|$ $\Sigma_n = \{a \leq x_1 \leq \ldots \leq x_n \leq b\}$ is the modulus of variation of the function f, (see [7], [8]).

These properties (1—5) for $p=1$ are given in [6].

The following theorem holds:

Theorem 1. *Let f be a bounded 2π-periodical function. Then* $(1 \leq p \leq \infty)$:

$$\widetilde{E}_n^T(f)_{L_p} \leq c\tau(f;\ n^{-1})_{L_p}, \quad \widetilde{E}_{0,\Sigma_n}(f)_{L_p} \leq c'\tau(f;\ \Delta_n)_{L_p},$$

where c and c' are absolute constants.

We will sketch the proof of Theorem 1.

Let us denote $x_i = i\pi/n$, $i = 0, \ldots, 2n$, $y_i = (x_{i-1} + x_i)/2$, $i = 1, \ldots, 2n$, and let us define the functions S_n and s_n as follows:

$$S_n(x) = \begin{cases} \sup\{f(t): t \in [x_{i-1}, x_i]\} \quad \text{for} \quad x = y_i, \quad i = 1, \ldots, 2n; \\ \max\{S_n(y_{i+1}),\ S_n(y_i)\} \text{ for } x = x_i,\ i = 1, \ldots, 2n-1; \\ S(y_1) \text{ for } x = 0,\ S(y_{2n}) \text{ for } x = 2\pi, \text{ continuous and linear} \\ \text{in } [x_{i-1}, y_i] \text{ and } [y_i, x_i],\ i = 1, \ldots, 2n. \end{cases}$$

$$s_n(x)=\begin{cases}\inf\{f(t): t\in[x_{i-1}, x_i]\} \text{ for } x=y_i,\ i=1,\dots,2n;\\ \min\{s_n(y_{i+1}),\ s_n(y_i)\} \text{ for } x=x_i,\ i=1,\dots,2n-1;\\ s_n(y_1) \text{ for } x=0,\ s_n(y_{2n}) \text{ for } x=2\pi \text{ continuous and linear}\\ \text{in } [x_{i-1}, y_i] \text{ and } [y_i, x_i],\ i=1,\dots,2n.\end{cases}$$

Obviously we have

$$s_n(x)\leqq f(x)\leqq S_n(x),\quad x\in[0, 2\pi] \tag{5}$$

and the derivatives of S_n and s_n exist in each point in $[0, 2\pi]$ except the points x_i, $i=0,\dots,2n$, y_i, $i=1,\dots,2n$. Moreover, we have

$$|S_n'(x)|\leqq\omega(f, x; 8\pi/n)\, n/\pi,\ x\neq x_i, y_i,$$

$$|S_n'(x)|\leqq\omega(f, x; 8\pi/n)n/\pi,\quad x\neq x_i, y_i,$$

$$|S_n(x)-s_n(x)|\leqq\omega(f, x; 2\pi/n).$$

Consequently

$$\begin{aligned}&\|S_n'\|_{L_p}\leqq n/\pi\ \tau(f; 8\pi/n)_{L_p};\\ &\|s_n'\|_{L_p}\leqq n/\pi\ \tau(f; 8\pi/n)_{L_p};\ \|S_n-s_n\|_{L_p}\leqq\tau(f; 2\pi/n)_{L_p}.\end{aligned} \tag{6}$$

From (2) for $r=1$, (5) and (6) follows theorem 1 for the trigonometrical case.

For the case of approximation by means of spline function of zero order (with step functions) we consider the functions

$$\widetilde{S}_n(x)=\{\sup f(t): t\in[x_{i-1}, x_i]\},\quad x\in[x_{i-1}, x_i)$$

$$s_n(x)=\inf\{f(t): t\in[x_{i-1}, x_i]\},\quad x\in[x_{i-1}, x_i),\ i=1,\dots,2n.$$

Obviously $\|\widetilde{S}_n-\widetilde{s}_n\|_{L_p}\leqq\|\omega(f, x;\ 2\pi/n)\|_{L_p}=\tau(f; 2\pi/n)_{L_p}$. The following two lemmas permit to obtain estimates with derivatives of functions:

L e m m a 1. *Let f be 2π-periodical function, $f\in L_p(0,2\pi)$, $\int_0^{2\pi}f(x)dx=0$, and $T\in T_n$ be such that $T(x)\geqq f(x)$, $x\in[0, 2\pi]$ and $\|T-f\|_{L_p}=\eta$. Then there exists $R\in T_n$ such that $R(x)\geqq\int_0^x f(t)dt$, $x\in[0, 2\pi]$ and $\{\int_0^{2\pi}(R(x)-\int_0^x f(t)dt)^p dx\}^{1/p}\leqq c''\eta/n$.*

L e m m a 2. *Let the function f has a derivative $f'\in L[a, b]$. Then $\widetilde{E}_{k,\Sigma_n}(f)_{L_p}\leqq(k+1)\Delta_n\widetilde{E}_{k-1,\Sigma_n}(f')_{L_p}$.*

Using Theorem 1 and Lemmas 1 and 2 we obtain

T h e o r e m 2. *Let the 2π-periodical function f have a bounded derivative $f^{(k)}$. Then*

$$\widetilde{E}_n^T(f)_{L_p}\leqq c_1(k)n^{-k}\tau(f^{(k)}; n^{-1})_{L_p}$$

$$\widetilde{E}_{k,\Sigma_n}(f)_{L_p}\leqq c_2(k)\Delta_n^k\tau(f^{(k)}; \Delta_n)_{L_p},$$

where $c_1(k)$ and $c_2(k)$ are constants, depending only on k.

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