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CONSTRUCTION OF BEST CONTINUOUS APPROXIMATIONS TO A DISCONTINUOUS FUNCTION

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Summary. A singularity that arises naturally in relative and simultaneous approximation results in discontinuous errors. We indicate here how this discontinuity affects the alternation properties of best approximations which are necessary for efficient computation schemes à-la-Remes. This is a preliminary report of theoretical results and an algorithm, devised jointly with Irène Grimard, for dealing with such a singularity. It is based on the exchange algorithm of Remes and on recent characterizations of various types of extremal behaviour, obtained jointly with G. D. Taylor and O. Shisha. In addition, we claim that the convergence rates of our algorithm are "best possible expected" in the sense that they coincide with the rates corresponding to continuous errors. The rather involved details of the characterization and computational results will appear in [1, 2] and elsewhere. It is interesting to note that E. Ya. Remes has himself dealt with discontinuous errors (1957), a fact that was communicated to us by Professor Bl. Sendov in Blagoevgrad in the summer of 1977.

The problem. Let f be a function to be approximated, with the properties:

$$f\in C[-1,1];\ f(0)=0;\ f(x)\neq 0 \text{ for } x\in[-1,1]\setminus\{0\};$$

$$\operatorname{Sup}\{|x^{k-1}/f(x)|:|x|\leqq 1,\ x\neq 0\}=\infty \text{ and } \sup\{|x^k/f(x)|:|x|\leqq 1,\ x\neq 0\}<\infty$$

for some $k\in Z^+$. Let Π_n denote the class of real polynomials of degree less than or equal to n. We seek a preferred $p\in\Pi_n$ which minimizes the expression $\|\mathcal{R}_p\|$, where $\mathcal{R}_p(x)$ is defined by

$$\mathcal{R}_p(x)=\begin{cases}\max\{|e_p(x)|,\ E_p(x)|\}, & x\neq 0\\ \max\{|\overline{\lim}_{x\to 0}E_p(x)|,\ |\underline{\lim}_{x\to 0}E_p(x)|\}, & x=0\end{cases}$$

and $e_p(x)$ and $E_p(x)$ denote the expressions $e_p(x)=f(x)-x^k p(x)$ and $E_p(x)=1-x^k p(x)/f(x)$ and $\|\cdot\|$ is the supremum norm on $[-1,1]$. Such a minimizing polynomial p will be referred to as a best simultaneous approximation to f. The term simultaneous approximation is used because $\mathcal{R}_p(x)$ is the maximum of the absolute and relative Chebyshev errors of the approximation p. Questions of unicity are also answered.

Let the symbols μ and M be defined by $\mu=\underline{\lim}_{x\to 0}x^k/f(x)$ and

$M=\overline{\lim}_{x\to 0} x^k/f(x)$. Given a simultaneous approximation $p\in\Pi_n$, x_i will be called a critical point if and only if one of the following holds:

$$x_i\neq 0 \text{ and } \mathcal{R}_p(x_i)=\|\mathcal{R}_p\|,$$

$$x_i=0 \text{ and } \overline{\lim}_{x\to 0} E_p(x)=\|\mathcal{R}_p\| \text{ or } \underline{\lim}_{x\to 0} E_p(x)=-\|\mathcal{R}_p\|.$$

We shall distinguish two kinds of critical points, the s-extrema and the s-determining points. Each non-zero critical point will be called an s-extremum, i. e. there will be no s-determining points different from zero. In order to classify the different kinds of zero critical points as s-extrema or s-determining points we need the values of the sign functions $s(x)$, $\sigma(x)$ and $S(x)$, all of which are defined on the set of critical points. When $x_i\neq 0$ define $s(x_i)=\operatorname{sgn}(x_i^k)$, $\sigma(x_i)=\operatorname{sgn}(e(x_i))$ and $S(x_i)=1$. When $x_i=0$ define $s(x_i)=(-1)^k$ and $\sigma(x_i)=1$. The exact definition of the sign function $S(0)$ is too long and will be omitted in this report. However, the important classification which results from the definition of $S(0)$ is given in Table 1.

Given a simultaneous approximation p, whenever both $\overline{\lim}_{x\to 0}E_p(x)=\|\mathcal{R}_p\|$ and $\underline{\lim}_{x\to 0} E_p(x)=-\|\mathcal{R}_p\|$ hold, the statement "$S(0)$ is consistent" will mean that the rather complicated definitions of $S(0)$ by cases (depending on the sign of $p(0)$, the integer k, the signs of μ and M and the directions from which the lim sup and inf are attained) all yield the same value, i. e. all give -1, 0 or 1.

When 0 is a critical point, it is said to be an s-extremum if $S(0)\neq 0$ and $S(0)$ is consistent. (Note that if $S(0)\neq 0$ and only one of $\overline{\lim}_{x\to 0} E_p(x)=\|\mathcal{R}_p\|$ and $\underline{\lim}_{x\to 0} E_p(x)=-\|\mathcal{R}_p\|$ holds, then $S(0)$ is automatically consistent and 0 will be an s-extremum.) It is said to be an s determining point if it is a critical point and not an s-extremum, i. e. if $S(0)=0$ or $S(0)$ is inconsistent. The results of these considerations for $S(0)$ are summarized in Table 1.

Theorem 1. *Let f satisfy the conditions stated in the beginning of this report. Then $p\in\Pi_n$ is the best simultaneous approximation to f if and only if at least one of the following holds:*

(1) *0 is an s-determining point (in which case $\|\mathcal{R}_p\|=1$ if $\mu M\leq 0$ and $\|\mathcal{R}_p\|=|(M-\mu)/(M+\mu)|$ if $\mu M>0$).*

(2) *there exists a sequence $\{x_i\}$ of $n+2$ s-extrema satisfying: $-1\leq x_1<x_2<\dots<x_{n+2}\leq 1$ and $s(x_{i+1})\sigma(x_{i+1})S(x_{i+1})=-s(x_i)\sigma(x_i)S(x_i)$, $i=1,\dots,n+1$.*

Theorem 2. *Let $B(f)$ denote the set of best simultaneous approximations from Π_n.*

Case I. *$B(f)$ is the set of best simultaneous approximations characterized by the alternation condition (2) of Theorem 1. In this case $B(f)$ contains a single element.*

Case II. *$B(f)$ is the set of best simultaneous approximations characterized by (1) of Theorem 1.*

If $\mu M\leq 0$, uniqueness of best simultaneous approximations fails, in general, and

$$B(f)=\{p\in\Pi_n : p(x)\equiv 0 \text{ or } \operatorname{sgn} p(x)=\operatorname{sgn}(x^k/f(x)) \text{ and}$$

$$|p(x)|\leq|2f(x)/x^k| \text{ throughout } [-1,1]\sim\{0\}\}.$$

Table I

Extremum and Sign Classification Table

μ, M	$p(0)$	$\overline{\lim}_{x\to 0} E_p(x) = \lVert R_p \rVert$	$p(0)$	$\underline{\lim}_{x\to 0} E_p(x) = -\lVert \mathcal{R}_p \rVert$
$0<\mu<M$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$
	$=0$		$=0$	$(-\lVert \mathcal{R}_p \rVert = 1)$
	<0		<0	cannot occur $(-\lVert \mathcal{R}_p \rVert > 1)$
$\mu<M<0$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$	>0	$(-\lVert \mathcal{R}_p \rVert > 1)$
	$=0$		$=0$	cannot occur $(-\lVert \mathcal{R}_p \rVert = 1)$
	<0		<0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$
$\mu<0<M$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$
	$=0$	s-determining point $(\lVert \mathcal{R}_p \rVert = 1)$	$=0$	cannot occur $(-\lVert \mathcal{R}_p \rVert = 1)$
	<0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$	<0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$
$\mu=0<M$	>0	s-determining point $(\lVert \mathcal{R}_p \rVert = 1)$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$ provided $\overline{\lim}_{x\to 0} \neq \lVert R_p \rVert$
	$=0$		$=0$	cannot occur $(-\lVert \mathcal{R}_p \rVert = 1)$
	<0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$	<0	
$\mu<0=M$	>0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=-1$	>0	cannot occur $(-\lVert \mathcal{R}_p \rVert = 1)$
	$=0$	s-determining point $(\lVert \mathcal{R}_p \rVert = 1)$	$=0$	
	<0		<0	s-extremum Sgn $\delta = s(0)\sigma(0)S(0)=1$ provided $\underline{\lim}_{x\to 0} \neq \lVert \mathcal{R}_p \rVert$
$\mu=0=M$	>0	s-determining point $(\lVert \mathcal{R}_p \rVert = 1)$	>0	cannot occur $(-\lVert \mathcal{R}_p \rVert = 1)$
	$=0$		$=0$	
	<0		<0	

Notation: $\mu = \underline{\lim}_{x\to 0} x^k/f(x)$; $M = \overline{\lim}_{x\to 0} x^k/f(x)$; δ = "corrector" added to $p(0)$.

Table II

Best Simultaneous Approximations to the Function

$$f(x)=\begin{cases}\sin(x), \in[-1,0), \\ 10\sin(x),\ x\in[0,1)\end{cases} \quad k=-1,\ \mu=-1,\ M=1.$$

n (degree of approximating class π_n)	Number of cycles	1 / x	x^2 / x^3	x^4 / x^5	x^6 / x^7	s-extrema	Contribution of s-extremum (by relative (R) or absolute (A) error)	$\lVert\mathcal{R}_{p*}\rVert$
2	4	2.805090 5.448630	1.966080			−1.0000 0.0000 0.4562 1.0000	R R A A	1.805090
3	5	2.448430 8.554200	3.513281 4.652772			−1.0000 −0.5748 0.0000 0 3737 1.0000	R R R A A	1.448430
4	8	2.121857 10.981482	11.635688 − 8.227796	− 9.218378		−1.0000 −0.4039 0.0000 0.2682 0.7906 1.0000	R R R A A A	1.121857
5	9	2.044914 12.190990	14.434221 −15.015920	−12.813135 6.528725		−1.0000 −0.8096 −0.3298 0.0000 0.2500 0.7696 1.0000	R R R R A A A	1.044914
6	14	1.845211 14.823568	29.948705 −25.517126	−61.611818 15.197881	34.573501	−0.9612 −0.6440 −0.2312 0.0000 0.1944 0.6022 0.8950 1.0000	R R R R A A A A	0.845211
7	11	1.826968 15.236118	31.975690 −30.157822	−69.434262 27.749269	41.021115 −8.975396	−1.0000 −0.8910 −0.6000 −0.2189 0.0000 0.1895 0.5946 0.8916 1.0000	R R R R R A A A A	0.826968

Table III

Best Simultaneous Approximations to the Function
$$f(x)=\begin{cases}\sin(x), & x\in[-1,0),\\ -10\sin(x), & x\in[0,1]\end{cases}\quad k=1,\ \mu=-1,\ M=1.$$

n (degree of approximating class π_n)	Number of cycles	1 / x	x^2 / x^3	x^4 / x^5	s-extrema	Contribution of s-extremum (by relative (R) or absolute (A) error)	$\lVert\mathcal{R}_{p_*}\rVert$
3	5	− 0.771339 −10.449957	− 4.646215 5.681463		−1.0000 −0.5751 0.0000 0.3740 1.0000	R R R A A	1.771339
4	8	− 0.371163 −13.421780	−14.591558 10.056168	11.284785	−1.0000 −0.4039 0.0000 0.2682 0.7906 1.0000	R R R A A A	1.371163
5	9	− 0.277122 −14.900052	−18.011947 18.352649	15.678341 −7.979457	−1.0000 −0.8096 −0.3298 0.0000 0.2500 0.7696 1.0000	R R R R A A A	1.277122

In addition, $\min_{p\in\Pi_n}\lVert\mathcal{R}_p\rVert=1$.

If $\mu M>0$, *uniqueness of best simultaneous approximations fails, in general, and*

$B(f)=\{p\in\Pi_n : p(0)=2/(M+\mu)$ *and* $x^{-k}2\mu f(x)/(M+\mu)\le p(x)\le x^{-k}2Mf(x)/(M+\mu)$ *throughout* $[-1,1]\sim\{0\}\}$.

In addition $\min_{p\in\Pi_n}\lVert\mathcal{R}_p\rVert=|(M-\mu)/(M+\mu)|$.

The algorithm. This algorithm seeks out the unique best simultaneous approximation of the type characterized by $n+2$ s-extrema as described in (2) of Theorem 1, Section 2. For any given f, as defined in the beginning of this paper, there always does exist at least one best simultaneous approximation. However, for such an f, it is not the case that there always exists a best simultaneous approximation of the type characterized by the above-

Table IV

Best Simultaneous Approximations of Fixed Degree 3 to the Function
$$f(x)=\begin{cases}\Lambda(e^x-1), & x\in[-1,0),\ \Lambda\in[1/1.000001,\ 1/1.001],\ k=1,\ \mu=1,\ M=1/\Lambda\\ e^x-1, & x\in[0,1]\end{cases}$$

Λ („jump" parameter)	$\lvert(M-\mu)/(M+\mu)\rvert$	Number of cycles	1 / x	x^2 / x^3	s-extrema	Contribution of s-extremum (by relative (R) or absolute (A) error)	$\lVert\mathcal{R}_{p_*}\rVert$
1/1.000001	3×10^{-7}	6	0.998914 0.500540	0.175374 0.042335	—1.0000 —0.7377 —0.0628 0.7526 1.0000	R R R A A	0.001120
1/1.00001	5×10^{-6}	6	0.998906 0.500545	0.175379 0.042333	—1.0000 —0.7378 —0.0632 0.7527 1.0000	R R R A A	0.001119
	5×10^{-5}						
1/1.001	5×10^{-4}	4	0.998793 0.501335	0.175108 0.041839	—1.0000 —0.7146 0.0000 0.7476 1.0000	R R R A A	0.001207

Table V

Best Simultaneous Approximation to the Function
$$f(x)=\begin{cases}-2x, & x\in[-1,0),\ k=1,\quad \mu=-.5,\ M=1\\ x, & x\in[0,1].\end{cases}$$

n (degree of approximating class π_n)	Number of cycles	$p^*(x)$	s-extrema	Contribution of s-extremum (by relative (R) or absolute (A) error)	$\lVert\mathcal{R}_{p^*}\rVert$
0	1	—0.500000	—1.0000 0.0500	A R	1.50000_0

mentioned $n+2$ s-extrema (see Theorem 1). In case such an approximation involving $n+2$ s-extrema does not exist for a given f, this algorithm will necessarily break down. The breakdown will be signalled by the fact that at some cycle of the iteration the "exchange" or the "levelling" will be impossible.

This algorithm does not seek out the best simultaneous approximations characterized by the s-determining point 0 (see Theorem 1). However, if at some cycle the computed polynomial $p^m(x)$ is a best approximation of the s-determining point type, the algorithm recognizes $p^m(x)$ as such and stops.

Assume that the m-th cycle of the algorithm has been completed and we have a number $h^m>0$, a polynomial $p^m(x)\in\Pi_n$ and a sequence of $n+2$ points $\{x_i^m\}$ $(-1\leq x_1^m<x_2^m<\cdots<x_{n+2}^m\leq 1)$ satisfying $R_pm(x_i^m)=h^m$, if $x_i^m\neq 0$ and either $\overline{\lim}_{x\to 0}E_pm(x)=h^m$ or $\underline{\lim}_{x\to 0}E_pm(x)=-h^m$, if $x_i^m=0$ and also satisfying

$$s^m(x_{i+1}^m)\,\sigma^m(x_{i+1}^m)\,S^m(x_{i+1}^m)=-s^m(x_i^m)\,\sigma^m(x_i^m)\,S^m(x_i^m),\ i=1,2,\ldots,n+1.$$

If $h^m=\|\mathcal{R}_pm\|$ or if 0 is an s-determining point of $p^m(x)$ (see Table I), then the algorithm has generated the best simultaneous approximation $p^m(x)$ (within computational error); therefore terminate the algorithm.

The details involving the many cases of $S(0)$, judicious extensions of the sign functions s, σ and S, the feasibility of "levelling" and "monotonicity", the condition that the errors do not oscillate at zero, a refined version of the de la Vallé-Poussin inequality and other adaptations leading to a contraction map (which results in quadratic convergence) will be given elsewhere. A non-trivial aspect of this algorithm has been to show there are no "loops" as the iteration switches from one case to another (see Table I). Numerical examples are given in Tables II-V.

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