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TWO NOTES ABOUT THE CONVERGENCE OF THE APPROXIMATE SOLUTIONS OF FREDHOLM'S INTEGRAL EQUATION OF SECOND KIND

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Summary. The paper contains some estimations for the convergence of the approximate solution for Fredholm's integral equation of second kind.

1. A. I. Husseinov and G. M. Gassanov [1] estimated the approximation of integrable functions with linear positive operators in the space L by means of the modulus of non-monotonicity of the function. V. A. Popov and G. M. Gassanov [2] improved this result. Both results may be used for evaluation of the error in the method of collocation for Fredholm's equation.

We shall need some characteristics which are useful for improving the results mentioned above:

a) the modulus of non-monotonicity $\mu(f;\delta)$ of the function f, introduced by B. Sendov [3], given by

$$\mu(f;\delta)=2^{-1}\sup\{\sup\{|f(x)-f(x_1)|+|f(x)-f(x_2)|$$
$$-|f(x_2)-f(x_1)|\,|\,x\in[x_1,x_2]\}\,|\,|x_2-x_1|\leqq\delta\},$$

b) the modulus of variation $\varkappa(f;n)$ (see V. A. Popov and J. Szabados, [4] and Z. A. Chanturia [5]) for functions, bounded in the interval $[0, 1]$, defined by

$$\varkappa(f;n)=\sup\{\Sigma_{i=1}^{n}|f(x_i)-f(x_{i-1})|\,|\,0\leqq x_0\leqq x_1\leqq\cdots\leqq x_n\leqq 1\}.$$

We shall also use the inequalities

$$w(f;n^{-1})_L\leqq 2n^{-1}\varkappa(f;2n)\leqq Cn^{-1}\Sigma_{k=1}^{n}\mu(k^{-1}), \tag{1}$$

where $w(f;n^{-1})_L[0,1]$ is the integral modulus of continuity of f, given by $w(f;n^{-1})_L[0,1]=\sup\{\int_0^{1-h}|f(x+h)-f(x)|\,dx\,|0<h<n^{-1}\}$, C is an absolute constant (the left-hand side of (1) was obtained by V. A. Popov and J. Szabados [4], the right-hand one by P. Petrušev [6]).

Denote by $B_\mu[a, b]$ ($B_\mu[0, 2\pi]$) the class of locally monotone functions in the interval $[a, b]$ (2π periodically local monotone functions in the interval $[0, 2\pi]$), defined in the following manner: $f \in B_\mu[a, b]$ ($f \in B_\mu[0, 2\pi]$) if

a) $\sup\{|f(x)| \mid x \in [a, b]\} \leq B$ ($\sup\{|f(x)| \mid x \in [0, 2\pi]\} \leq B$,

b) the modulus of non-monotonicity $\mu(f; \delta)$ of f satisfies the condition $\mu(f; \delta) \leq \mu(\delta)$, where $\mu(\delta)$ is a given monotone non-decreasing function, $\mu(\delta) \to 0$ when $\delta \to 0$.

We are interested in the following two cases — approximation by positive integral operators and approximation by summation formulas.

First we treat the case of positive integral operators. Assume that $f \in B_\mu[0, 2\pi]$ and

$$(Pf)(x) = \int_{-\pi}^{\pi} K(t) f(x+t)\, dt, \tag{2}$$

where $K(t) \geq 0$, $\int_{-\pi}^{\pi} K(t) dt = 1$, K is an even 2π periodical kernel. For the distance between f and Pf in the space L we have:

$$\|f - Pf\|_{L[0, 2\pi]} = \int_0^{2\pi} |f(x) - (Pf)(x)|\, dx$$

$$\leq \int_{-\delta}^{\delta} K(t) \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx dt + \int_{\delta}^{\pi} K(t) \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx dt$$

$$+ \int_{-\pi}^{\delta} K(t) \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx dt \leq w(f; \delta)_L + 2\int_{\delta}^{\pi} K(t) w(f; t)_L\, dt.$$

The inequalities (1) allow to replace $w(f; \delta)_L$ with $Cn^{-1} \Sigma_{k=1}^{n} \mu(k^{-1})$. Assuming $\mu(\delta) = 0(\delta^\alpha)$, $0 < \alpha \leq 1$ one may obtain

$$w(f; n^{-1})_L \leq C \Sigma_{k=1}^{n} \mu(k^{-1}) n^{-1} = Cn^{-1} \Sigma_{k=1}^{n} K^{-\alpha} = \begin{cases} Cn^{-\alpha}, & \alpha < 1 \\ Cn^{-1} \operatorname{lm} n, & \alpha = 1 \end{cases}.$$

Therefore, the following theorem holds:

Theorem 1. *Assume that $f \in B_\mu[0, 2\pi]$, P is a positive linear operator of the type* (2) *and* $\mu(\delta) = 0(\delta^\alpha)$, $0 < \alpha < 1$. *Then for every $\delta > 0$ one has*

$$\|f - Pf\|_{L[0, 2\pi]} \leq C\delta^\alpha + 2C \int_{\delta}^{\pi} K(t)\, t^\alpha\, dt.$$

Especially when $K(t)$ is the kernel of Fejer $\Phi_n(t) = (\sin(nt/2)/\sin(t/2))^2/2n\pi$, choosing $\delta = n^{-1}$ we obtain $\|f - \Phi_n f\|_{L[0, 2\pi]} \leq C_1 n^{-\alpha}$, where C_1 is an absolute constant. The corresponding result in [2] is $\|f - \Phi_n f\|_{L} \leq CBn^{-\alpha/(1+\alpha)}$.

Approximation by summation formulas. Assume that $f \in B_\mu[a, b]$ and

$$(P_n f)(x) = \Sigma_{k=0}^{n} f(x_{kn})\, \psi_{kn}(x), \tag{3}$$

where $\psi_{kn}(x) \geq 0$, $\Sigma_{k=0}^{n} \psi_{kn}(x) = 1$, $x \in [a, b]$, $x_{kn} \in [a, b]$, $k = 0, 1, \ldots, n$. Denote $\varphi\delta = \sup\{\sum_{|x_{kn} - x| > \delta} \psi_{kn}(x) \mid x \in [a, b]\}$. Then for every $\delta > 0$ and $s = [(b-a)/\delta]$

$$\|f - p_n f\|_{L[a, b]} = \int_a^b |f(x) - \Sigma_{k=0}^{n} f(x_{kn}) \psi_{kn}(x)|\, dx = \Sigma_{i=1}^{s} \int_{x_i}^{x_{i+1}} |f(x)$$

$$- \Sigma_{k=0}^{n} f(x_{kn})\, \psi_{kn}(x)|\, dx \leq s^{-1} \Sigma_{i=1}^{s} |f(\xi_i) - \Sigma_{k=0}^{n} f(x_{kn})\, \psi_{kn}(\xi_i)|$$

$$\leq s^{-1} \Sigma_{i=1}^{s} \Big| \sum_{|\xi_i - x_{kn}| \leq \delta} \psi_{kn}(\xi_i)\, |f(\xi_i) - f(x_{kn})| \Big|$$

$$+ s^{-1} \Sigma_{i=1}^{s} \Big| \sum_{|\xi_i - x_{kn}| > \delta} \psi_{kn}(\xi_i)\, |f(\xi_i) - f(x_{kn})| \Big|.$$

Here $\xi_i \in [x_i, x_{i+1}]$, $x_i = a + i\delta$, $i = 1, 2, \ldots, s$. Obviously

$$(4) \qquad s^{-1}\Sigma_{i=1}^{s} \Big| \sum_{|\xi_i - x_{kn}| > \delta} \psi_{kn}(\xi_i) \, |f(\xi_i) - f(x_{kn})| \Big| \leq 2B\varphi\delta.$$

As for the other sum the following estimation holds

$$(5) \qquad s^{-1}\Sigma_{i=1}^{s} \Big| \sum_{|\xi_i - x_{kn}| \leq \delta} \psi_{kn}(\xi_i) \, |f(\xi_i) - f(x_{kn})| \Big| \leq 4\varkappa(f; 2s^{-1})s^{-1}.$$

In fact we have

$$\Sigma_{i=1}^{s} \Big| \sum_{|\xi_i - x_{kn}| \leq \delta} \psi_{kn}(\xi_i) \, |f(\xi_i) - f(x_{kn})| \Big| \leq \Sigma_{i=1}^{s} |f(\xi_i) - f(x_{k_i n})|,$$

where $x_{k_i n} \in [\xi_i - \delta, \xi_i + \delta]$. In order to obtain a set of points q_k, $k = 0, 1, 2, \ldots, K$, $K \leq 2s$, such that $a \leq q_0 \leq q_1 \leq \cdots \leq q_k \leq b$, we have to enumerate the points $\xi_i, x_{k_i n}$ in a proper way and to find out how many times the difference $|f(q_l) - f(q_{l+1})|$ may be counted. Consider the interval $[\xi_i, \xi_{i+1}]$. It may contain up to four points $x_{k_j n}$, i. e. for $j = i-1$, i, $i+1$, $i+2$ ($|\xi_{i+1} - \xi_i| \leq 2\delta$ and $|\xi_i - x_{k_i n}| \leq \delta$). Then we have

$$\begin{aligned} &|f(\xi_{i-1}) - f(x_{k_{i-1} n})| + |f(\xi_i) - f(x_{k_i n})| + |f(\xi_{i+1}) - f(x_{k_{i+1} n})| \\ &\quad + |f(\xi_{i+2}) - f(x_{k_{i+2} n})| \leq |f(\xi_{i-1}) - f(\xi_i)| + |f(\xi_i) - f(x_{k_{i-1} n})| \\ &\quad + |f(\xi_i) - f(x_{k_i n})| + |f(\xi_{i+1}) - f(x_{k_{i+1} n})| + |f(\xi_{i+1}) - f(x_{k_{i+2} n})| \\ &\quad + |f(\xi_{i+1}) - f(\xi_{i+2})| \leq |f(\xi_{i-1}) - f(\xi_i)| + 2|f(\xi_i) - f(x^*_{k_i n})| \\ &\quad + 2|f(x^*_{k_{i+1} n}) - f(\xi_{i+1})| + |f(\xi_{i+1}) - f(\xi_{i+2})|, \end{aligned}$$

where

$$x^*_{k_m n} = \begin{cases} x_{k_{m-1} n} & \text{if } |f(\xi_i) - f(x_{k_{i-1} n})| > |f(\xi_i) - f(x_{k_i n})|, \\ x_{k_m n} & \text{otherwise}; \qquad m = i, \, i+2. \end{cases}$$

If $x^*_{k_i n} < x^*_{k_{i+1} n}$, then

$$\begin{aligned} &2|f(\xi_i) - f(x^*_{k_i n})| + 2|f(\xi_{i+1}) - f(x^*_{k_{i+1} n})| \leq 2|f(\xi_i) - f(x^*_{k_i n})| \\ &\quad + |f(x^*_{k_i n}) - f(x^*_{k_{i+1} n})| + 2|f(\xi_{i+1}) - f(x^*_{k_{i+1} n})|. \end{aligned}$$

When $x^*_{k_i n} > x^*_{k_{i+1} n}$ we have

$$\begin{aligned} &2|f(\xi_i) - f(x^*_{k_i n})| + 2|f(\xi_{i+1}) - f(x^*_{k_{i+1} n})| \leq 2|f(\xi_i) - f(x^*_{k_{i+1} n})| \\ &\quad + 4|f(x^*_{k_{i+1} n}) - f(x^*_{k_i n})| + 2|f(x^*_{k_i n}) - f(\xi_{i+1})|. \end{aligned}$$

So the points q_k we are looking for become $q_{l-2} = \xi_{i-1}$, $q_{l-1} = \xi_i$,

$$q_l = x^*_{k_{i+1} n} (x^*_{k_i n}), \; q_{l+1} = x^*_{k_i n} (x^*_{k_{i+1} n}), \; q_{l+2} = \xi_{i+1}, \; q_{l+3} = \xi_{i+2}.$$

One might obtain, proceeding in the same manner, a similar result for the cases of less than four points in $[\xi_i, \xi_{i+1}]$ and will find that the number of repetitions for the distance $|f(q_l)-f(q_{l+1})|$ is not greater than four. Taking into account (4) and (5) we state the following

Theorem 2. *Assume that $f \in B_\mu[a, b]$ and P_n is an operator of type* (3). *Then for every $\delta>0$ it holds $\|f-P_n f\|_{L[a,b]} \leq 4\delta\varkappa(f; 2\delta)+2B\varphi\delta$.*

In the case of Fejer's kernel,

(6) $$\psi_{kn}(x)=\Phi_n(x)=(T_n(x)/n(x-x_{kn}))^2(1-xx_{kn}),$$

where x_{kn} are zeros of Chebishev's polynomial $T_n(x)=\cos(n \arccos x)$, knowing that $\varphi\delta=0(n^{-1}\delta^{-1})$ and considering $\mu(\delta)=0(\delta^\alpha)$, $0<\alpha<1$, we obtain $\|f-\Phi_n f\|_{L[-1,1]} \leq 0(n^{-\alpha/(1+\alpha)})$. The corresponding result in [2] is $0(n^{-\alpha/(1+\alpha+\alpha^2)})$.

An estimation for the error of the approximate solution of a linear integral equation (solved by a collocation method) in the metric L, may be obtained in terms of the modulus of non-monotonicity of the kernel and of the free term of the equation. Consider Fredholm's integral equation of second kind

(7) $$Y(x)=\lambda \int_a^b K(x, t)\, y(t)\, dt+f(x),$$

where $K(x, t)$, $y(t)$ are given integrable functions in $a \leq x$, $t \leq b$, $a \leq t \leq b$ respectively. We are looking for approximate solutions of the kind $Y_n(x)=\sum_{k=0}^n C_k \psi_{kn}(x)$, where

$$\psi_{kn}(x) \geqq 0,\ k=0, 1, \ldots, n,\ \sum_{k=0}^n \psi_{kn}(x)=1,\ x \in [a, b],$$

$$\psi_{kn}(x_{in})=\begin{cases} 0, & k \neq i, \\ 1, & k=i. \end{cases}$$

Denote by $B_{x\mu}[a, b]$ the class of functions $F(x, t)$, defined in $a \leq x$, $t \leq b$ for which $\sup\{|F(x, t)|\ |x \in [a, b]\} \leq B$ and $\mu(F; \delta)_x \leq \mu(\delta)$ for every $t \in [a, b]$, where $\mu(F; \delta)_x$ is the modulus of non-monotonicity of $F(x, t)$ related to x, t is fixed, and $\mu(\delta)$ is a nondecreasing function, $\mu(\delta) \rightarrow 0$, when $\delta \rightarrow 0$. We introduce the following notations:

$$(P_n K)(x, t)=\sum_{k=0}^n K(x_{kn}, t)\, \psi_{kn}(x),$$

$$(P_n f)(x)=\sum_{k=0}^n f(x_{kn})\, \psi_{kn}(x),\ \beta_n=\| f-P_n f\|_{L[a,b[},$$

$$\varepsilon_n=\sup\{\| P_n K-K\|_{L[a,b]} \mid t \in [a, b]\},$$

where the last norm is related to x.

Having in mind that for every continuous function f in the interval $[a, b]$ $P_n f$ converges to f uniformly, the following assertion holds (V. A. Popov, G. M. Gassanov [2]): Assume that (7) has a unique solution $Y \in L[a, b]$ and $f \in B_{\mu 1}[a, b]$, $K \in B_{x_{\mu 2}}[a, b]$. Then for every appropriate large number n and approximate solutions y_n, obtained by the method of collocation, the following inequation holds:

$$\| Y-Y_n\|_{L[a,b]} \leqq (1+R\lambda)\{\beta_n+(b-a)B\,|\lambda|\,(1+R\lambda)\,\varepsilon_n/(1-|\lambda|\,(1+R\lambda)\,\varepsilon_n)\},$$

where $R\lambda=|\lambda| \sup\{\int_a^b |R(x, t; \lambda)|\, dx \mid t \in [a, b]\}$, $R(x, t; \lambda)$ is the resolvent of $K(x, t)$.

According to the results obtained above, this theorem has the following corollary: Let P_n be the Fejer's operator, defined by (6), and $\mu 1(\delta)=0(\delta^\alpha)$,

$\mu 2(\delta)=0(\delta^{\beta})$, $0<\alpha, \beta<1$. Then $\| Y-Y_n \|_{L[a,b]}=0(n^{-s})$, where $s=\min\{\alpha/(1+\alpha), \beta/(1+\beta)\}$.

2. In this part an estimation of the rate of convergence for the solutions of Fredholm's integral equation (7), by means of Hausdorff distance, is derived.

Assume that the kernel $K(x, t)$ is a continuous function in the interval $a \leq x$, $t \leq b$ and λ is not an eigenvalue. The function $f(x)$ is supposed to be bounded and continuous almost everywhere. Consider also the integral equation

$$\lambda \int_a^b K(x, t) Y_n(t) dt + Y_n(x) = f_n(x). \tag{8}$$

In [7] V. A. Popov proved the following theorem: Let the Hausdorff distance $r(f_n, f)$ tend to zero when $n \to \infty$. Then for the solutions of (7) and (8) we have $r(y_n, y) \to 0$, $n \to \infty$.

Our purpose here is to obtain an estimation for the error of the approximate solution $y_n(x)$ given by

Theorem 3. *Assuming* $\mu(f; \delta)=0(\delta^{\alpha})$, $w(K; \delta)=0(\delta^{\beta})$, $0<\alpha, \beta \leq 1$, *then* $r(y_n, y)=0(\delta^{\gamma})$, $\gamma=\min\{\alpha/(1+\alpha), \beta\}$. *Here* $w(f; \delta)$ *is the modulus of continuity of* f, *defined by* $w(f; \delta)=\sup\{|f(x+h)-f(x)| \mid h<\delta\}$, $x \in [a, b]$.

Proof. The solutions of (7) and (8) are given by

$$y(x)=f(x)+\lambda \int_a^b R(x, t; \lambda) f(t)\, dt,$$

$$y_n(x)=f_n(x)+\lambda \int_a^b R(x, t; \lambda) f_n(t)\, dt.$$

According to the definition of Hausdorff distance we have

$$\begin{aligned} &\max\{|x-s|, |f(x)+\lambda \int_a^b R(x, t; \lambda) f(t)\, dt - f_n(s) \\ &\quad -\lambda \int_a^b R(s, t; \lambda) f_n(t) dt |\} \leq \max\{|x-s|, |f(x)-f_n(s)|\} \\ &\quad + |\lambda \int_a^b R(s, t; \lambda)(f(t)-f_n(t)) dt| + |\lambda \int_a^b f(t)(R(x, t; \lambda)-R(s, t; \lambda)) dt| \\ &\leq r(f, f_n)+|\lambda| \, \|R\| \int_a^b |f(t)-f_n(t)|\, dt + |\lambda| M (w_t(R; \delta) \| x-s| \leq \delta). \end{aligned} \tag{9}$$

But

$$\int_a^b |f(t)-f_n(t)|\, dt \leq 4Mrp+(b-a)\{\mu((b-a)/p)+r\}, \tag{10}$$

where $M=|f|$, $r=r(f(x), f_n(x))$, p is a natural number [8].

It is easy to show that $w(R; \delta) \leq C w(K; \delta)$. In fact we have $R(x, t; \lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_i(x, t)$, where $K_i(x, t)=\int_a^b K_{i-1}(x, s) K(s, t)\, ds$, $K_1(x, t)=K(x, t)$. Considering the difference $K_{n+1}(x+h, t)-k_{n+1}(x, t)=\int_a^b K(s, t)\{K_n(x+h, s) -K_n(x, s)\}\, ds$ we obtain

$$\begin{aligned} w(K_{n+1}; \delta) &= \max\{| K_{n+1}(x+h, t)-K_{n+1}(x, t) \| h \leq \delta\} \\ &= \max\{\int_a^b K(s, t)(K_n(x+h, s)-K_n(x, s))\, ds \mid h \leq \delta\} \\ &\leq \max|K_n(x+h, s)-K_n(x, s)| \, \|K\|(b-a) = w(K_n; \delta) \|K\| (b-a). \end{aligned}$$

Using the same technique one obtains $w(K_{n+1}; \delta) \leq \|K\|^n (b-a)^n w(K; \delta)$' which, when $\|K\|^n (b-a)^n |\lambda|^n<1$, gives

(11) $$w(R;\delta)\leqq Cw(K;\delta).$$

Applying (10) and (11) to (9) we have $r(y_n, y)\leqq r(f_n, f)+|\lambda|Mw(K;\delta)+|\lambda||R|\{4Mrp+(b-a)(\mu((b-a)/p)+r)\}$, where, after optimization by p, we obtain $r(y_n, y)=0(\delta^\gamma)$, $\gamma=\min\{\alpha/(1+\alpha), \beta\}$, which has to be shown.

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