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INVERSE RESULTS VIA SMOOTHING

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Summary. In 1972, Berens and Lorentz offered an elementary proof of the inverse theorem for Bernstein polynomials in case $0<\alpha<1$. Recently this was extended to all values $0<\alpha<2$ (cf. M. Becker, *Aequationes Math.*, in print), the main point being an appropriate use of integral means. The purpose of the present paper will be to examine this elementary procedure in connection with further classical problems, e. g. with (global) inverse theorems for a general class of linear approximation processes in Banach spaces, for best approximation by algebraic polynomials, and for a variant concerning Bernstein polynomials. It follows that this "elementary method" provides an interesting alternative to the classical Bernstein argument via telescoping sums.

1. Introduction. In 1972, Berens-Lorentz [8] offered an elementary proof of the inverse theorem for Bernstein polynomials in case $0<\alpha<1$. Recently this was extended to all values of $\alpha \in (0, 2)$ in [1], the main point being an appropriate use of integral means. In fact, this approach to (global) inverse results may be applied in other and more general situations. Details were worked out in [5] for trigonometric convolution operators as well as for a new proof of the Bernstein-Zygmund theorem concerning the best approximation by trigonometric polynomials, in [2] for the Szász-Mirakjan and Baskakov and in [3] for the Favard operators in polynomial weight spaces, respectively. For further comments and references, however, let us refer to [1-3].

The purpose of the present paper will be to test this elementary procedure in connection with further classical problems. Whereas Sec. 2 is concerned with some preliminary results, Sec. 3 treats inverse approximation theorems for a general class of approximation processes in Banach spaces. Section 4 is devoted to the inverse theorem for best approximation by algebraic polynomials. Finally, Sec. 5 deals with a variant concerning Bernstein polynomials. Let us emphasize that the purpose of this paper is not primarily to establish new results. In fact, particularly those of Sec. 4 are classical. The point will be to show that the elementary method as suggested in [1, 8] may be used in various situations, thus providing an interesting alternative to the classical Bernstein argument via telescoping sums. As for the Bernstein argument, the following procedure is restricted to nonsaturated rates of convergence. For the characterization of saturation classes via smoothing one may consult [7] (cf. [10, p. 502]).

2. Preliminaries. The proofs to be employed in this paper mainly rest upon the following lemma which slightly extends that given in [5] and [8], respectively.

Lemma 2.1. *Let Ω be monotonely increasing on* $[0, d]$, $d\leq 1$. *If for some* $0<\alpha<r$, $\lambda>0$ *one has for all* h, $t\in[0, d]$, $h<t<\sqrt{\lambda h}$ *that*

$$\Omega(h)\leq M[t^{\alpha}+(h/t)^{r}(t^{\alpha}+\Omega(t))], \tag{2.1}$$

then $\Omega(t)=O(t^{\alpha})$, $(t\to 0+)$.

Proof. Choosing $A>1$, $c\leq d$ such that $3M\leq A^{r-\alpha}$, $c<\lambda/A$, define for $m\in\mathbb{N}$, the set of natural numbers, $M_1:=\max\{1, c^{-\alpha}\Omega(c), 3MA^{\alpha}\}$, $h_m:=cA^{1-m}$. Then one has $\Omega(h_m)\leq M_1 h_m^{\alpha}$ via induction. Indeed, $\Omega(h_1)=\Omega(c)\leq M_1h_1^{\alpha}$. Since $A>1$, $c<\lambda A^{m-2}$ for all $m\in\mathbb{N}$, one has

$$h_{m+1}<h_m=cA^{1-m}<\sqrt{c\lambda}\,A^{-m/2}=\sqrt{\lambda h_{m+1}}.$$

Thus (2.1) for $h=h_m$, $t=h_{m-1}$ delivers

$$\Omega(h_m)\leq M[h_{m-1}^{\alpha}+(h_m/h_{m-1})^{r}(h_{m-1}^{\alpha}+\Omega(h_{m-1}))]$$

$$\leq M[A^{\alpha}h_m^{\alpha}+A^{-r}\{A^{\alpha}h_m^{\alpha}+M_1h_{m-1}^{\alpha}\}]\leq[MA^{\alpha}+MA^{\alpha-r}+MA^{\alpha-r}M_1]h_m^{\alpha}\leq M_1h_m^{\alpha}.$$

Let $t\in(0, c)$ be fixed and $m\in\mathbb{N}$ be such that $h_m\leq t<h_{m-1}$. Then the monotonicity of Ω yields

$$\Omega(t)\leq\Omega(h_{m-1})\leq M_1h_{m-1}^{\alpha}=M_1A^{\alpha}h_m^{\alpha}\leq M_1A^{\alpha}t^{\alpha}.$$

This completes the proof.

The next lemma contains some technical information about the functions

$$\varphi(x)=x(1-x);\quad x\in[0,1], \tag{2.2}$$

$$\psi(x)=1-x^2;\quad x\in[-1, 1]. \tag{2.3}$$

Lemma 2.2. *For* $0\leq\alpha\leq 1$, $0\leq\beta<2$ *one has*

$$0\leq\int_x^t\int_x^s\varphi(u)^{-\alpha}\,du\,ds\leq\varphi(x)^{-\alpha}(t-x)^2;\qquad x\in(0, 1),\ t\in[0, 1], \tag{2.4}$$

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\varphi(x+s+t)^{-\beta}\,ds\,dt\leq[\max\{\varphi(x\pm h), \varphi(x)\}]^{-\beta}M_{\beta}h^2 \tag{2.5}$$

$$\textit{for all } h\in(0, 1/8],\ x\in[h, 1-h],$$

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\psi(x+s+t)^{-\beta}\,ds\,dt\leq[\max\{\psi(x\pm h), \psi(x)\}]^{-\beta}M_{\beta}h^2 \tag{2.6}$$

$$\textit{for all } h\in(0, 1/4],\ |x|\leq 1-h.$$

Proof. For $\alpha=1$ there holds

$$\int_x^t\int_x^s(u(1-u))^{-1}\,du\,ds=t\log(t/x)+(1-t)\log((1-t)/(1-x))$$

$$\leq t(t/x-1)+(1-t)((1-t)/(1-x)-1)=(t-x)^2/x(1-x)$$

since $\log y\leq y-1$ for $y\geq 0$. Of course, the non-negativity of the integral results from the non-negativity of the integrand. This proves (2.4) for $\alpha=1$, the case $\alpha<1$ following by Hölder's inequality

$$\int_x^t \int_x^s \varphi(u)^{-\alpha}\, du\, ds \leq \{\int_x^t \int_x^s \varphi(u)^{-1}\, du\, ds\}^\alpha \{\int_x^t ds\}^{2(1-\alpha)} \leq \varphi(x)^{-\alpha}(t-x)^2.$$

Concerning (2.5), consider $h \leq x \leq 2h \leq 1/4$ so that $\max\{\varphi(x+h),\ \varphi(x),\ \varphi(x-h)\} = \varphi(x+h)$. Then

$$J_{h,x} = \int\int_{-h/2}^{h/2} (x+s+t)^{-\beta}(1-x-s-t)^{-\beta}\, ds\, dt$$

$$\leq (1-x-h)^{-\beta} \int\int_{-h/2}^{h/2} (x+s+t)^{-\beta}\, ds\, dt.$$

Since $x \leq 2h$, one has for $\beta \neq 1$

$$J_{h,x} \leq (1-x-h)^{-\beta} . (\mid 1-\beta \mid (2-\beta))^{-1} \mid (x+h)^{2-\beta} - 2x^{2-\beta} + (x-h)^{2-\beta} \mid$$
$$\leq \varphi(x+h)^{-\beta} M_\beta[(x+h)^2 + 2x^{2-\beta}(x+h)^\beta + (x-h)^{2-\beta}(x+h)^\beta] \leq M_\beta\, \varphi(x+h)^{-\beta} h^2.$$

Here and in the following, M denotes a constant which may have different values at each occurrence. For $\beta = 1$ one has

$$J_{h,x} \leq (1-x-h)^{-1} \int\int_{-h/2}^{h/2} (h+s+t)^{-1}\, ds\, dt = (1-x-h)^{-1}[2h \log 2h - 2h \log h]$$

$$= 6h^2 \log 2/3h\, (1-x-h) \leq 6h^2 \log 2/((x+h)(1-x-h)) = Mh^2/\varphi(x+h).$$

Thus (2.5) is valid for $x \in [h, 2h]$ and by a symmetric argument for $x \in [1-2h, 1-h]$ as well. For $x \in [2h, 1-2h]$ one has

$$J_{h,x} \leq (x-h)^{-\beta}(1-x-h)^{-\beta} h^2 = \varphi(x)^{-\beta} h^2 [(1+h/(x-h))(1+h/(1-x-h))]^\beta$$
$$\leq \varphi(x)^{-\beta} 4^\beta h^2,$$

and analogously $h^2(x-h)^{-\beta}(1-x-h)^{-\beta} \leq 3^\beta h^2 \varphi(x \pm h)^{-\beta}$, so that (2.5) follows. The estimate (2.6) may be derived analogously (or via the transformation $x \to (x+1)/2$ giving $\varphi(x) \to \psi(x)/4$).

3. Approximation Processes in Banach Spaces. Let X be a Banach space with norm $\| \cdot \|_X$ and $Y \subset X$ a subspace with seminorm $| \cdot |_Y$. Structural properties of an element $f \in X$ to be approximated will be measured in terms of the K-functional which is given for $t \geq 0$ by

(3.1) $\quad K(t, f;\ X, Y) = K(t, f) := \inf\{\|f-g\|_X + t\,|g|_Y : g \in Y\}.$

Let $[X]$ be the space of bounded linear operators of X into itself. Let $\varrho > 0$ be a parameter tending to infinity, and suppose that $\varphi(\varrho) > 0$ is a function, monotonely increasing to infinity such that there exists a sequence $\{\varrho_k\}_{k=1}^\infty$, monotonely increasing to infinity with

(3.2) $\quad \sup_{k \in \mathbb{N}} \varphi(\varrho_{k+1})/\varphi(\varrho_k) = L < \infty.$

We shall be concerned with families $\{T_\varrho\}_{\varrho>0} \subset [X]$ of operators which satisfy a Bernstein-type inequality, thus $T_\varrho f \in Y$ for each $f \in X$, $\varrho > 0$ and

(3.3) $\quad | T_\varrho f |_Y \leq M \varphi(\varrho) \|f\|_X;\quad f \in X, \varrho > 0,$

as well as a certain invariance relation, namely

(3.4) $\quad | T_\varrho g |_Y \leq M | g |_Y;\quad g \in Y, \varrho > 0.$

Theorem 3.1. *Let* $\{T_\varrho\}_{\varrho>0}\subset[X]$ *satisfy* (3.2–4). *If* $f\in X$ *is such that for some* $0<\alpha<1$

$$(3.5)\qquad \| T_\varrho f-f\|_X=O(\varphi(\varrho)^{-\alpha}),\ (\varrho\to\infty),$$

then one has

$$(3.6)\qquad K(t, f;\ X,\ Y)=O(t^\alpha),\ (t\to 0+).$$

Proof. Let $g\in Y$ be arbitrary. Since $T_\varrho(X)\subset Y$, it follows by definition (3.1) as well as by (3.3-5) that

$$K(t, f)\leq \| f-T_\varrho f\|_X+t|T_\varrho f|_Y\leq M\varphi(\varrho)^{-\alpha}+t[|T_\varrho(f-g)|_Y+|T_\varrho g|_Y]\leq M\varphi(\varrho)^{-\alpha}$$
$$+Mt[\varphi(\varrho)\|f-g\|_X+|g|_Y].$$

Since $g\in Y$ is arbitrary, we conclude for any $t>0$, $\varrho>0$: $K(t, f)\leq M[\varphi(\varrho)^{-\alpha}+t\varphi(\varrho)K(\varphi(\varrho)^{-1}, f)]$. For any $\delta>0$ choose k such that (cf. (3.2))

$$\varphi(\varrho_{k+1})^{-1}\leq\delta<\varphi(\varrho_k)^{-1}\leq L\varphi(\varrho_{k+1})^{-1}.$$

Then this implies that for any $t,\ \delta>0: K(t, f)\leq M[\delta^\alpha+(t/\delta)K(\delta, f)]$ so that the assertion (3.6) follows by Lemma 2.1.

The result of Theorem 3.1 is already contained in [7, p. 33] where it was proved via telescoping sums. Using that argument, one additionally has to assume the commutativity of the operators via

$$(3.7)\qquad T_{2^{-k}\varrho}-T_{2^{-(k+1)}\varrho}=T_{2^{-k}\varrho}[I-T_{2^{-(k+1)}\varrho}]-T_{2^{-(k+1)}\varrho}[I-T_{2^{-k}\varrho}].$$

In order to avoid (3.7), thus to extend telescoping arguments to noncommutative operators, Kuptsov [13] used the following modification of (3.3)

$$(3.3^*)\qquad |T_\varrho f-T_\sigma f|_Y\leq M\varphi(\varrho)\|T_\varrho f-T_\sigma f\|_X$$

to be valid for all $f\in X$ and $\sigma<\varrho$. From the point of view of applications, however, it seems to be impossible to verify (3.3*) for the standard examples.

In many applications (cf. [7]) one has

$$(3.8)\qquad Y:=D(B),\quad |g|_Y:=\|Bg\|_X,$$

where $D(B)\subset X$ is the domain of some closed linear operator B with range in X. In this situation, (3.4) reads

$$(3.4^*)\qquad \|BT_\varrho g\|_X\leq M\|Bg\|_X;\quad g\in D(B),\ \varrho>0,$$

which is trivial in case the operators T_ϱ and B commute and $\{T_\varrho\}$ is uniformly bounded.

The proof of Theorem 3.1 in particular shows that one may strengthen the Bernstein-type inequality (3.3) considerably. Indeed (see also [15]),

Corollary 3.2. *Let* $\{T_\varrho\}\subset[X]$ *satisfy* (3.3-4). *Then for any* $f\in X$, $\varrho>0$

$$(3.9)\qquad |T_\varrho f|_Y\leq M\varphi(\varrho)K(\varphi(\varrho)^{-1}, f; X, Y).$$

Proof. Again one has for any $g\in Y$

$$|T_\varrho f|_Y\leq|T_\varrho(f-g)|_Y+|T_\varrho g|_Y\leq M[\varphi(\varrho)\|f-g\|_X+|g|_Y],$$

which already implies (3.9).

Let us observe that, strengthening Bernstein inequalities of type (3.3) to those of type (3.9), may be equivalently expressed via a corresponding strengthening of Jackson-type inequalities (cf. [12]).

As an immediate consequence of Theorem 3.1, Cor. 3.2 we note

Corollary 3.3. *Let* $\{T_\varrho\}\subset[X]$ *satisfy* (3.2-4). *If* $f\in X$ *is such that* (3.5) *holds for some* $0<\alpha<1$, *then one has the Zamansky-type estimate*

$$\| T_\varrho f\|_Y = O(\varphi(\varrho)^{1-\alpha}), \quad (\varrho\to\infty). \tag{3.10}$$

Let us conclude these general considerations with the observation that in the particular case of semigroups of operators one may give a very elementary proof of (3.10), even for the limiting case $\alpha=1$ (see also [15]). To this end, let $\{T(t)\}_{t\geq0}\subset[X]$ be a uniformly bounded semigroup of operators of class (C_0), i. e.

$$T(t_1+t_2)=T(t_1)\,T(t_2), \quad T(0)=I, \text{ the identity}, \tag{3.11}$$

$$\| T(t)f\|_X\leqq M\|f\|_X, \quad \lim_{t\to0+}\| T(t)f-f\|_X=0.$$

Let A be the infinitesimal generator of the semigroup, i. e. $Af=\lim\limits_{t\to0+}t^{-1}[T(t)f-f]$, the domain consisting of all elements $f\in X$ for which the limit exists. Then A is a closed linear operator which is densely defined. Let for $f\in X$, $t>0$

$$S_t f=t^{-1}\int_0^t T(u)f\,du. \tag{3.12}$$

It follows that $S_t(X)\subset D(A)$ for each $t>0$ and

$$AS_tf=t^{-1}[T(t)f-f], \| S_t f-f\|_X\leqq\sup_{0\leqq u\leqq t}\| T(u)f-f\|_X. \tag{3.13}$$

For the basic facts of semigroup theory see [9].

Theorem 3. 4. *Let* $\{T(t)\}$ *be a uniformly bounded semigroup of operators of class* (C_0) *which is holomorphic, thus* $T(t)(X)\subset D(A)$ *and*

$$\| AT(t)f\|_X\leqq Mt^{-1}\|f\|_X; \quad f\in X,\ t>0. \tag{3.14}$$

If $f\in X$ *is such that for some monotonely increasing* φ

$$\| T(t)f-f\|_X=O(\varphi(t)), \quad (t\to0+), \tag{3.15}$$

then one has the Zamansky-type result

$$\| AT(t)f\|_X=O(t^{-1}\varphi(t)), \quad (t\to0+). \tag{3.16}$$

Proof. Using (3.13) one has $\| AT(t)f\|_X\leqq\| AT(t)[f-S_tf]\|_X+\| AT(t)S_tf\|_X\leqq Mt^{-1}\|f-S_tf\|_X+M\| AS_tf\|_X\leqq Mt^{-1}\sup\limits_{0\leqq u\leqq t}\| T(u)f-f\|_X$, which already completes the proof.

The assertion of Theorem 3.4 was essentially given in [6. p. 18] (cf. [9, p. 113]) where it was shown via telescoping sums. Let us consider the particular case $\varphi(t)=t^\alpha$.

Corollary 3. 5. *Under the assumptions of Theorem 3. 4 one has for any* $0<\alpha\leqq1$

$$\| T(t)f-f\|_X=O(t^\alpha)\Longrightarrow\| AT(t)f\|_X=O(t^{\alpha-1}).$$

For $\alpha=1$ this improves results as given in [6], [9] (see also [15]).

4. Best Approximation by Algebraic Polynominals. In this section we are concerned with the (algebraic) polynominal $p_n^*(f)$ of best approximation of $f\in C[-1, 1]$, the space of continuous functions on $[-1, 1]$ with the usual sup-norm, thus

$$\|f-p_n^*(f)\|_{C[-1,\ 1]}=\inf_{p_n\in \mathfrak{P}_n}\|f-p_n\|_{C[-1,\ 1]},$$

P_n being the set of algebraic polynomials of degree n. To formulate the result, for some $0<h\leq 1$ (cf. (2.3)) let

$$\Delta_h^2 f(x)=f(x+h)-2f(x)+f(x-h);\quad |x|\leq 1-h,$$

$$\omega_2(f,\ \delta)=\sup_{0<h\leq\delta}\ \sup_{|x|\leq 1-h}|\Delta_h^2 f(x)|,$$

$$\Delta_n(x)=\max\{\sqrt{1-x^2}/n,\ n^{-2}\}=n^{-1}\max\{\sqrt{\psi(x)},\ n^{-1}\}.$$

Then there holds the following classical result which is connected with many names including those of S. M. Nikolskii, A. F. Timan, V. K. Dzadyk, S. A. Teljakovskii, I. E. Gopengauz (see [14, 18] for the details).

Theorem 4.1. *Let* $\alpha\in(0, 2)$, $f\in C[-1, 1]$ *satisfy*

$$|p_n^*(f)(x)-f(x)|\leq M[\Delta_n(x)]^\alpha;\quad n\in\mathbb{N},\ x\in[-1, 1], \tag{4.1}$$

then $f\in \mathrm{Lip}_2\,\alpha=\{f:f\in C[-1, 1],\ \omega_2(f,\ \delta)\quad O(\delta^\alpha),\ \delta\to 0+\}$.

Proof. In view of Lemma 2.1 it suffices to show that

$$\omega_2(f,\ h)\leq M[\delta^\alpha+(h/\delta)^2\{\delta^\alpha+\omega_2(f,\ \delta)\}]. \tag{4.2}$$

To this end define for $n\in\mathbb{N}$, $h\in(0, 1]$, $x\in[-1+h,\ 1-h]:\delta_{n,x\ h}=\max\{\Delta_n(x+h),\ \Delta_n(x),\ \Delta_n(x-h)\}$. From (4.1) there follows

$$(4.3)\quad |\Delta_h^2 f(x)|\leq|f(x+h)-p_n^*(f)(x+h)|+2|p_n^*(f)(x)-f(x)|+|f(x-h)$$
$$-p_n^*(f)(x-h)|+|\Delta_h^2[p_n^*(f)](x)|\leq 4M(\delta_{n,x,h})^\alpha+\int_{-h/2}^{h/2}\!\!\int|[p_n^*(f)]''(x+s+t)|\,ds\,dt$$

Thus we have to estimate $[p_n^*(f)]''$ using an appropriate Bernstein-type inequality. This is given by the following lemma, the proof of which is postponed for a moment.

Lemma 4.2. *Let* $f\in C[-1, 1]$ *satisfy* (4.1). *Then for* $n\geq 2$ *and.* $y\in[x-h,\ x+h]\subset[-1, 1]$ *one has*

$$|[p_n^*(f)]''(y)|\leq M[\Delta_n(y)]^{-2}[(\delta_{n,x,h})^\alpha+\omega_2(f,\ \delta_{n,x,h})]. \tag{4.4}$$

Using Lemma 4.2, one has by (2.6) for all $|x|\leq 1-h$

$$\int_{-h/2}^{h/2}\!\!\int|[p_n^*(f)]''(x+s+t)|\,ds\,dt\leq M[(\delta_{n,x,h})^\alpha+\omega_2(f,\delta_{n,x,h})]\int_{-h/2}^{h/2}\!\!\int[\Delta_n(x+s+t)]^{-2}\,ds\,dt$$

$$=M[(\delta_{n,x,h})^\alpha+\omega_2(f,\delta_{n,x,h})]\,n^2\int_{-h/2}^{h/2}\!\!\int\frac{ds\ dt}{\max\{\psi(x+s+t),\ n^{-2}\}}$$

$$\leq M[(\delta_{n,x,h})^\alpha+\omega_2(f,\ \delta_{n,x,h})]\,h^2/(\delta_{n,x,h})^2.$$

Therefore we conclude from (4.3) for all $n \in \mathbb{N}$, $h \in (0, 1]$, $|x| \leq 1-h$: $|\Delta_h^2 f(x)| \leq 4M(\delta_{n,x,h})^\alpha + M(h/\delta_{n,x,h})^2 [(\delta_{n,x,h})^\alpha + \omega_2(f, \delta_{n,x,h})]$. For fixed $h, \delta \in (0, 1]$, $x \in [-1+h, 1-h]$ choose n such that $\delta_{n,x,h} \leq \delta < \delta_{n-1,x,h} \leq 4\delta_{n,x,h}$. Then $|\Delta_h^2 f(x)| \leq M[\delta^\alpha + (h/\delta)^2 \{\delta^\alpha + \omega_2(f, \delta)\}]$, proving (4.2), and hence Theorem 4.1.

Before proving Lemma 4.2, let us point out that the usual Bernstein-Markov inequality for algebraic polynomials

$$(4.5) \qquad |p_n''(x)| \leq M[\Delta_n(x)]^{-2} \|p_n\|$$

for $p_n \in P_n$, $x \in [-1, 1]$ (cf. [18, p. 227]) is too weak for our purposes. Instead, we use the following stronger result (cf. [14, p. 71], [18, p. 219, 224]).

L e m m a 4.3. *Let Ω be a modulus of continuity, and let r be an arbitrary integer. If $p_n \in P_n$ satiesfies $|p_n(x)| \leq [\Delta_n(x)]^r \Omega(\Delta_n(x))$; $|x| \leq 1$, then with some constant M only depending upon r*

$$|p_n'(x)| \leq M[\Delta_n(x)]^{r-1} \Omega(\Delta_n(x)); \ |x| \leq 1.$$

In fact, all we need in this section is that

$$(4.6) \qquad |p_n(x)| \leq \Omega(\Delta_n(x)); \ |x| \leq 1$$

implies

$$(4.7) \qquad |p_n''(x)| \leq M[\Delta_n(x)]^{-2} \Omega(\Delta_n(x)); \ |x| \leq 1.$$

P r o o f of L e m m a 4.2. For the Steklov means

$$(4.8) \qquad f_\delta(x) := \delta^{-2} \iint_{-\delta/2}^{\delta/2} f(x+s+t)\, ds dt; \ \delta > 0, \ |x| \leq 1$$

it is a well-known fact that for all $|x| \leq 1$ (cf. [10, p. 38])

$$(4.9) \qquad |f_\delta(x) - f(x)| \leq (1/2)\delta^{-2} \iint_{-\delta/2}^{\delta/2} |\Delta_{s+t}^2 f(x)|\, ds\, dt \leq (5/2)\omega_2(f, \delta),$$

$$|f_\delta''(x)| = \delta^{-2} |\Delta_\delta^2 f(x)| \leq 5\delta^{-2} \omega_2(f, \delta),$$

using a suitable, but standard extension of f from $[-1, 1]$ to some larger interval not effecting the modulus of continuity (apart from a constant factor 5, say, cf. [18, p. 121 f]). In addition we need a regularization process of polynomial type, i. e., let $\{J_n\}_{n=1}^\infty$ be a sequence of linear operators on $C[-1, 1]$ such that for each $f \in C[-1, 1]$

$$(4.10) \qquad J_n f \in P_n, \ \|J_n f\| \leq M\|f\|,$$

$$(4.11) \qquad |J_n f(x) - f(x)| \leq M\omega_2(f, \Delta_n(x)); \ |x| \leq 1.$$

For example, one may take processes constructed in [11, p. 146 ff] or [14, p. 66] via trigonometric convolution operators. Let us first show that

$$(4.12) \qquad |(J_n f)''(x)| \leq M[\Delta_n(x)]^{-2} \omega_2(f, \Delta_n(x)); \ |x| \leq 1.$$

P r o o f of (4.12). By (4.5), (4.9-10) one has

$$|(J_n f)''(x)| \leq |(J_n[f - f_\delta])''(x)| + |(J_n f_\delta)''(x) - f_\delta''(x)| + |f_\delta''(x)|$$

$$\leq M[\Delta_n(x)]^{-2} \|f - f_\delta\| + I_1 + \|f_\delta''\| \leq M\omega_2(f, \delta)([\Delta_n(x)]^{-2} + \delta^{-2}) + I_1,$$

say. In view of (4.11) and $\omega_2(f_\delta, t) \leqq \omega_2(f, t)$, a theorem of Teljakovskii [17], which gives an analog to Lemma 4. 3 for the difference $p_n - f$, might be used to deduce

$$(4.13) \qquad I_1 = |(J_n f_\delta)''(x) - f_\delta''(x)| \leqq M[\Delta_n(x)]^{-2}\omega_2(f, \Delta_n(x)); \ |x| \leqq 1,$$

so that (4.12) would immediately follow upon setting $\delta = \Delta_n(x)$. However, since Teljakovskii uses telescoping sums in his proof, we establish (4.13) by elementary methods using the Steklov means once again. One has by, (4.9), (4.11)

$$(4.14) \quad I_1 \leqq |(J_n f_\delta - [J_n f_\delta]_\delta)''(x)| + |([J_n f_\delta]_\delta)''(x) - ([f_\delta]_\delta)''(x)| + |([f_\delta]_\delta)''(x) - f_\delta''(x)|$$

$$\leqq I_2 + \delta^{-2}|\Delta_\delta^2[J_n f_\delta - f_\delta](x)| + \delta^{-2}|\Delta_\delta^2[f_\delta - f](x)|$$

$$\leqq I_2 + 4M\delta^{-2} \max_{y \in \{x-\delta, x, x+\delta\}} \omega_2(f, \Delta_n(y)) + 4\delta^{-2}\|f_\delta - f\|.$$

To estimate I_2 we use (4.6-7). There holds by (4.9-11)

$$(4.15) \quad |(J_n f_\delta - [J_n f_\delta]_\delta)(x)| = (1/2)\delta^{-4}\Big|\iiiint_{-\delta/2}^{\delta/2} \Delta_{s+t}^2 [J_n(f)(\cdot + u + v))](x)\, du\, dv\, ds\, dt\Big|$$

$$\leqq (1/2)\delta^{-4} \iiiint_{-\delta/2}^{\delta/2} |\Delta_{s+t}^2\{[J_n(f(\cdot + u + v))](x) - f(x+u+v)\}|\, du\, dv\, ds\, dt$$

$$+ (1/2)\delta^{-4} \iiiint_{-\delta/2}^{\delta/2} |\Delta_{s+t}^2 f(x+u+v)|\, du\, dv\, ds\, dt$$

$$\leqq (5/2) M \max\{\omega_2(f, \Delta_n(y)) + (5/2)\omega_2(f, \delta) : y \in [x-\delta, x+\delta]\}.$$

Next we prove that for $j_n(x, \delta) = \max\{\omega_2(f, \Delta_n(y)) : y \in [x-\delta, x+\delta]\}$; $\delta > 0$ one has

$$(4.16) \qquad j_n(x, \Delta_n(x)) \leqq M\omega_2(f, \Delta_n(x)).$$

Proof of (4.16). For $n \geqq 2$, $|x| \leqq \Delta_n(x) = \max\{\sqrt{1-x^2}/n, n^{-2}\} \leqq 1/2$ there follows $j_n(x, \Delta_n(x)) = \omega_2(f, 1/n) \leqq (1 + (1-x^2)^{-1/2})^2\,\omega_2(f, \sqrt{1-x^2}/n)$

$$\leqq (1 + (3/4)^{-1/2})^2\,\omega_2(f, \Delta_n(x)).$$

Next suppose $x > \Delta_n(x)$. Since the case $j_n(x, \Delta_n(x)) = \omega_2(f, 1/n^2)$ is trivial we have to consider $j_n(x, \Delta_n(x)) = \omega_2(f, \sqrt{1-(x-\Delta_n(x))^2}/n)$. If $\Delta_n(x) = n^{-2}$ i. e., $1 - x^2 \leqq n^{-2}$, then

$$1 - (x - \Delta_n(x))^2 = 1 - x^2 + 2x n^{-2} - n^{-4} \leqq n^{-2} + 2n^{-2} = 3n^{-2},$$

hence

$$j_n(x, \Delta_n(x)) \leqq \omega_2(f, \sqrt{3}n^{-2}) \leqq (1+\sqrt{3})^2\,\omega_2(f, \Delta_n(x)).$$

If $\Delta_n(x) = \sqrt{1-x^2}/n$, i. e., $\sqrt{1-x^2} > 1/n$, then $(1 - (x - \Delta_n(x))^2)/(1-x^2) < 1 + 2n^{-1}(1-x^2)^{-1/2} < 3$, hence $j_n(x, \Delta_n(x)) < (1+\sqrt{3})^2\,\omega_2(f, \sqrt{1-x^2}/n)$. The case $x < -\Delta_n(x)$ follows by an analogous argument so that the proof of (4.16) is complete.

Setting $\delta = \Delta_n(x)$ in (4.15) gives $|(J_n f_\delta - [J_n f_\delta]_\delta)(x)| \leqq M\omega_2(f, \Delta_n(x))$, so ithat since (4.6) implies (4.7) one has

$$I_2 = |(J_n f_\delta - [J_n f_\delta]_\delta)''(x)| \leqq M[\Delta_n(x)]^{-2}\omega_2(f, \Delta_n(x)).$$

With $\delta = \Delta_n(x)$ in (4.14) this yields (4.13), and thus (4.12).

By (4.1), (4.10-11) one has $|p_n^*[f-J_nf](y)|\leq |p_n^*(f)(y)-f(y)|+|f(y)-J_nf(y)|\leq M[(\Delta_n(y))^\alpha+\omega_2(f,\Delta_n(y))]$. Since (4.6) yields (4.7) with modulus of continuity $\Omega(t)=M[t^\alpha+\omega_2(f,t)]$, it therefore follows that

(4.17) $$|(p_n^*[f-J_nf])''(y)|\leq M[\Delta_n(y)]^{-2}[(\Delta_n(y))^\alpha+\omega_2(f,\Delta_n(y))].$$

Now we are ready to prove (4.4). Using (4.12), (4.17) one has for $y\in[x-h,x+h]$

$$|[p_n^*(f)]''(y)|\leq|(p_n^*[f-J_nf])''(y)|+|(J_nf)''(y)|$$
$$\leq M[\Delta_n(y)]^{-2}[(\Delta_n(y))^\alpha+\omega_2(f,\Delta_n(y))]\leq M[\Delta_n(y)]^{-2}[(\delta_{n,x,h})^\alpha+\omega_2(f,\delta_{n,x,h})].$$

The last inequality follows by arguments similar to those used for (4.16). This proves Lemma 4.2.

At the end of this rather lengthy argument we have to point out that the present procedure is by no means shorter than the classical one using telescoping sums (cf. [14, p. 73 f]). It is the different arrangement of the details which may be of some interest. In particular, it may be possible to shorten the present argument considerably in case it is possible to develop a more clever way in dealing with regularization processes in connection with Lemma 4.2.

5. Approximation by Bernstein Polynomials. In this section we would like to discuss another variant of approximation by algebraic polynomials, namely by Bernstein polynomials

$$B_nf(x):=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)p_{k,n}(x),\quad p_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k},$$

where $x\in[0,1]$, $n\in\mathbb{N}$, $f\in C[0,1]$. With $\varphi(x)=x(1-x)$ (cf. (2.2)) one has

Theorem 5.1. *For $f\in C[0,1]$, $\alpha\in(0,2]$ the following statements are equivalent*

(5.1) $$|B_nf(x)-f(x)|\leq Mn^{-\alpha/2};\quad n\in\mathbb{N},\ x\in[0,1],$$

(5.2) $$\varphi(x)^{\alpha/2}|\Delta_h^2f(x)|\leq Mh^\alpha;\quad h\in(0,1/8],\ x\in[h,1-h].$$

In this theorem the pointwise structure as described by the endpoint weight $\varphi(x)$ has been moved into the Lipschitz condition (5.2) (cf. Theorem. 4.1). For the saturation case $\alpha=2$ this was shown in [4] (see also [14a]); the similar saturation conclusion $\Delta_h^2[\varphi f]=O(h^2)$ for modified Bernstein polynomials was obtained in [16]. On the other hand, in [1], [8] the equivalence of

(5.1*) $$|B_nf(x)-f(x)|\leq M[\varphi(x)/n]^{\alpha/2};\quad n\in\mathbb{N},\ x\in[0,1]$$

with $f\in \mathrm{Lip}_2\alpha$ was shown. Although most authors discuss (5.1*) which exhibits a pointwise approximation rate, the nonoptimal inverse part of Theorem 5.1, i. e. (5.1) $\Rightarrow$ (5.2) for $0<\alpha<2$, is indeed an immediate consequence of Theorem 5 in [8], which is proved via an intricate argument using intermediate space methods. In the following we would like to show how the elementary method may be used to derive Theorems 5.1. This was suggested to the authors by Prof. Berens, Erlangen, which is gratefully acknowledged.

Before proving the theorem, we need some estimates for the Bernstein polynomials. Define

$$\omega_{2,\alpha}(f, \delta)= \sup_{0<h\leq\delta} \sup_{x\in[h,1-h]} \varphi(x)^{\alpha/2} |\Delta_h^2 f(x)|.$$

Let us again note that the definition of the Steklov means (4. 8) for $x\in[0, 1]$, $\delta\leq 1/2$ requires a suitable extension of f from $[0, 1]$ to the real axis R, namely (cf. [18, p. 121f])

$$\bar{f}(x)=\begin{cases} f(x) \text{ for } x\in[0, 1], \\ -f(-x)+2f(0) \text{ for } x\in[-1, 0], \end{cases} \tag{5.3}$$

$$\bar{f}(x+2)=\bar{f}(x)+2[f(1)-f(0)];\ x\in\mathbb{R}.$$

With this definition one has for $x\in(0, 1)$ (cf. (4.9))

$$f_\delta(0)-f(0)=f_\delta(1)-f(1)=0, \tag{5.4}$$

$$|f_\delta(x)-f(x)|\leq(1/2)\delta^{-2}\iint_{-\delta/2}^{\delta/2}|\Delta_{s+t}^2\bar{f}(x)|\,ds\,dt\leq(9/2)\varphi(x)^{-\alpha/2}\,\omega_{2,\alpha}(f,\delta), \tag{5.5}$$

$$|f_\delta''(x)|=\delta^{-2}|\Delta_\delta^2\bar{f}(x)|\leq 9\delta^{-2}\varphi(x)^{-\alpha/2}\,\omega_{2,\alpha}(f,\delta).$$

One also has by (2.5), (5.5) for $x\in[1/n, 1-1/n]$

$$|\Delta_{1/n}^2 f_\delta(x)|\leq\iint_{-1/2n}^{1/2n}|f_\delta''(x+s+t)|\,ds\,dt\leq 9\delta^{-2}\,\omega_{2,\alpha}(f,\delta)\iint_{-1/2n}^{1/2n}\varphi(x+s+t)^{-\alpha/2}ds\,dt$$

$$\leq[\max\{\varphi(x+1/n),\ \varphi(x),\ \varphi(x-1/n)\}]^{-\alpha/2}\ M(n\delta)^{-2}\,\omega_{2,\alpha}(f,\delta). \tag{5.6}$$

Lemma 5.2. *For $0<\alpha\leq 2$, $x\in[0, 1]$ one has*

$$\sum_{k=1}^{n-1}\varphi(k/n)^{-\alpha/2}\,p_{k,n}(x)\leq 4\varphi(x)^{-\alpha/2};\ x\neq 0,\ 1, \tag{5.7}$$

$$A:=\sum_{k=1}^{n-1}|(k-nx)^2-(1-2x)\,k-nx^2|\,\varphi(k/n)^{-1}\,p_{k,n}(x)\leq 20\,n, \tag{5.8}$$

$$B:=\sum_{k=0}^{n}|(k-nx)^2-(1-2x)\,k-nx^2|\,p_{k,n}(x)\leq 2n\,\varphi(x). \tag{5.9}$$

Proof. Obviously there holds

$$\varphi(k,n)^{-1}p_{k,n}(x)\leq\frac{k+1}{k}\ \frac{n-k+1}{n-k}\binom{n+2}{k+1}x^k(1-x)^{n-k}\leq\frac{4}{x(1-x)}p_{k+1,n+2}(x).$$

This proves (5.7) for $\alpha=2$, whereas for $\alpha<2$ this follows by Hölder's inequality. In view of $(1-2x)\,k+nx^2\geq 0;\ 1\leq k\leq n$,

$$(k-nx)^2+(1-2x)\,k+nx^2\leq(k+1-(n+2)\,x)^2+2nx\,(1-x),$$

there follows

$$A\leq\frac{4}{x(1-x)}\sum_{k=1}^{n-1}[(k-nx)^2+(1-2x)\,k+nx^2]\,p_{k+1,n+2}(x)$$

$$\leq\frac{4(n+2)^2}{x(1-x)}\sum_{k=1}^{n-1}\left(\frac{k+1}{n+2}-x\right)^2p_{k+1,n+2}(x)+8n\sum_{k=1}^{n-1}p_{k+1,n+2}(x)$$

$$\leq 4\,(n+2)+8n\leq 20\,n,$$

which proves (5.8). For (5.9) see [8, p. 700].

Now one has the following Bernstein-type inequality for the Bernstein polynomials (cf. [1, (5)]).

Lemma 5.3. *For* $f \in C[0,1]$, $\alpha \in [0,2]$, $n \geq 3$, $\delta \in (0, 1/2)$, $x \in (0,1)$ *one has*

(5.10) $$|(B_n f)''(x)| \leq M\varphi(x)^{-\alpha/2}\omega_{2,\alpha}(f,\delta)[n/\varphi(x)+\delta^{-2}].$$

Proof. For $(B_n f)''$ there hold the representations (cf. [8, p. 705])

$$(B_n f)''(x) = n(n-1)\sum_{k=0}^{n-2} \Delta^2_{1/n} f((k+1)/n)\, p_{k,n-2}(x); \quad 0 \leq x \leq 1,$$

$$(B_n f)''(x) = \varphi(x)^{-2}\sum_{k=0}^{n}[(k-nx)^2-(1-2x)k-nx^2] f(k/n)\, p_{k,n}(x); \quad 0<x<1.$$

In view of (5.4-6) one obtains

$$|(B_n f)''(x)| \leq |(B_n[f-f_\delta])''(x)| + |(B_n f_\delta)''(x)|$$

$$\leq \varphi(x)^{-2}\sum_{k=1}^{n-1}|(k-nx)^2-(1-2x)k-nx^2|\,|f(\tfrac{k}{n})-f_\delta(\tfrac{k}{n})|\,p_{k,n}(x)$$

$$+n(n-1)\sum_{k=0}^{n-2}|\Delta^2_{1/n} f_\delta(\tfrac{k+1}{n})|\,p_{k,n-2}(x)$$

$$\leq \frac{9}{2}\varphi(x)^{-2}\omega_{2,\alpha}(f,\delta)\sum_{k=1}^{n-1}|(k-nx)^2-(1-2x)k-nx^2|\,\varphi(\tfrac{k}{n})^{-\alpha/2}\,p_{k,n}(x)$$

$$+M\delta^{-2}\omega_{2,\alpha}(f,\delta)\sum_{k=0}^{n-2}p_{k,n-2}(x)[\max\{\varphi(\tfrac{k+2}{n}),\varphi(\tfrac{k+1}{n}),\varphi(\tfrac{k}{n})\}]^{-\alpha/2}$$

$$\leq M\omega_{2,\alpha}(f,\delta)\,[\varphi(x)^{-2}A^{\alpha/2}B^{1-\alpha/2}+\delta^{-2}C^{\alpha/2}]$$

with

$$C=\sum_{k=0}^{n-2}p_{k,n-2}(x)/\max\{\varphi(\tfrac{k+2}{n}),\varphi(\tfrac{k+1}{n}),\varphi(\tfrac{k}{n})\}$$

$$\leq \varphi(1/n)^{-1}p_{0,n-2}(x)+\sum_{k=1}^{n-3}\varphi(k/n)^{-1}p_{k,n-2}(x)+\varphi(1/n)^{-1}p_{n-2,n-2}(x)$$

$$\leq \frac{p_{1,n-2}(x)+p_{n-3,n-2}(x)}{\varphi(1/n)\,(n-2)\,x(1-x)}+9\sum_{k=1}^{n-3}\varphi(\tfrac{k}{n-2})^{-1}p_{k,n-2}(x) \leq M\varphi(x)^{-1},$$

using (5.7) and $\varphi(1/n)\,(n-2)=(1-2/n)\,(1-1/n)\geq 2/9$ for $n\geq 3$. Hence by Lemma 5.2

$$|(B_n f)''(x)| \leq M\omega_{2,\alpha}(f,\delta)[\varphi(x)^{-2}n^{\alpha/2}(n\varphi(x))^{1-\alpha/2}+\delta^{-2}\varphi(x)^{-\alpha/2}]$$

$$\leq M\varphi(x)^{-\alpha/2}\omega_{2,\alpha}(f,\delta)\,[n/\varphi(x)+\delta^{-2}],$$

which yields the assertion.

Proof of Theorem 5.1. To prove the direct part let f satisfy (5.2). One has for $x\in(0,1)$ ($x\in\{0,1\}$ being trivial) $|B_n f(x)-f(x)|\leq|B_n[f-f_\delta](x)|+|B_n f_\delta(x)-f_\delta(x)|+|f_\delta(x)-f(x)|=:I_1+I_2+I_3$, say. By (5.4-5), (5.7) there follows

$$I_1+I_3\leq(9/2)\,\omega_{2,\alpha}(f,\delta)\{\sum_{k=1}^{n-1}\varphi(k/n)^{-\alpha/2}p_{k,n}(x)+\varphi(x)^{-\alpha/2}\}\leq M\varphi(x)^{-\alpha/2}\,\omega_{2,\alpha}(f,\delta).$$

In view of $f_\delta(t)-f_\delta(x)=(t-x)f_\delta'(x)+\int_x^t\int_x^s f_\delta''(u)\,du\,ds$ one has by (2.4), (5.5)

$$I_2\le\sum_{k=0}^{n}\int_x^{k/n}\int_x^{s}|f_\delta''(u)|\,du\,ds\,p_{k,n}(x)$$

$$\le 9\delta^{-2}\omega_{2,\alpha}(f,\ \delta)\sum_{k=0}^{n}\int_x^{k/n}\int_x^{s}\varphi(u)^{-\alpha/2}\,du\,ds\,p_{k,n}(x)$$

$$\le 9\delta^{-2}\omega_{2,\alpha}(f,\ \delta)\,\varphi(x)^{-\alpha/2}\sum_{k=0}^{n}(k/n-x)^2\,p_{k,n}(x)\le 9\omega_{2,\alpha}(f,\ \delta)\,\varphi(x)^{1-\alpha/2}/n\,\delta^2.$$

This gives (5.1) upon setting $\delta=\sqrt{\varphi(x)/n}$. Indeed, in view of (5.2)

$$|B_nf(x)-f(x)|\le M\varphi(x)^{-\alpha/2}\,\omega_{2,\alpha}(f,\ \delta)\,[1+\varphi(x)/n\,\delta^2]$$

$$\le M\varphi(x)^{-\alpha/2}\,\omega_{2,\alpha}(f,\sqrt{\varphi(x)/n})\le M\varphi(x)^{-\alpha/2}(\sqrt{\varphi(x)/n})^{\alpha}=Mn^{-\alpha/2}.$$

To consider the inverse part, let f satisfy (5.1). For $\alpha=2$ see [4, 14a]. To establish (5.2) for $0<\alpha<2$, in view of Lemma 2.1 it suffices to show that for $0<h\le 1/8$, $t<\sqrt{h/2}$

(5.11) $$\omega_{2,\alpha}(f,\ h)\le M[t^\alpha+(h/t)^2\,\omega_{2,\alpha}(f,\ t)].$$

To this end define for $n\ge 3$, $h\in(0,1/8]$, $x\in[h,1-h]$: $\delta_{n,x,n}=\max\{\sqrt{\varphi(x\pm h)/nh},\ \sqrt{\varphi(x)/n}\}$. Then by (5.1), Lemma 2.2, 5.3 one has

$$\varphi(x)^{\alpha/2}|\Delta_h^2 f(x)|\le 4\varphi(x)^{\alpha/2}\|B_nf-f\|+\varphi(x)^{\alpha/2}|\Delta_h^2(B_nf)(x)|$$

$$\le 4M(\varphi(x)/n)^{\alpha/2}+\varphi(x)^{\alpha/2}\iint_{-h/2}^{h/2}|(B_nf)''(x+s+t)|\,ds\,dt$$

$$\le 4M(\delta_{n,x,h})^\alpha+M\varphi(x)^{\alpha/2}\omega_{2,\alpha}(f,\ \delta)\,\Big[n\iint_{-h/2}^{h/2}\varphi(x+s+t)^{-(1+\alpha/2)}\,ds\,dt$$

$$+\delta^{-2}\iint_{-h/2}^{h/2}\varphi(x+s+t)^{-\alpha/2}\,ds\,dt\Big]\le 4M(\delta_{n,x,h})^\alpha$$

$$+M[\max\{\varphi(x),\ \varphi(x\pm h)\}]^{-\alpha/2}\varphi(x)^{\alpha/2}\,\omega_{2,\alpha}(f,\ \delta)\,[(\delta_{n,x,h})^{-2}h^2+h^2\delta^{-2}]$$

$$\le M[(\delta_{n,x,h})^\alpha+(h/\delta_{n,x,h})^2\,\omega_{2,\alpha}(f,\ \delta_{n,x,h})]$$

upon setting $\delta=\delta_{n,x,h}$. Note that $\delta_{3,x,h}\ge\sqrt{2h(1-2h)/3}\ge\sqrt{h/2}$ for $h\le 1/8$. For fixed $h\le 1/8$, $0<t<\sqrt{h/2}$, $x\in[h,\ 1-h]$ now choose n such that $\delta_{n,x,h}\le t<\delta_{n-1,x,h}\le 2\delta_{n,x,h}$. This gives (5.11), and thus the theorem.

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