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GEOMETRIC CONVERGENCE OF CHEBYSHEV RATIONAL APPROXIMATIONS ON $[0, \infty)$

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Summary. Let f be a continuous real-valued function on $[0, \infty)$,

$$\|f\|_r := \sup\{|f(x)| : 0 \leq x \leq r\} \text{ for } r > 0,\ \|f\| := \sup|\{f(x)| : x \geq 0\}.$$

The following problem is investigated:

For which functions f does there exist a number $q > 1$ and a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ such that $p_n \in \Pi_n$ and

$$\text{(1)} \qquad \overline{\lim_{n\to\infty}} \| 1/f - 1/p_n \|^{1/n} = 1/q.$$

Theorem: *If f is an entire function satisfying:*

(i) *f has only a finite number of zeros in $[0, \infty)$,*

(ii) *for every $s > 1$ there exist constants $K > 0$, $\theta > 0$ and $r_0 > 0$ such that* $M_f(r, s) \leq K(\|f\|_r)^{\theta}$ for all $r \geq r_0$,

(iii) *there exist real numbers $s > 1$, $\gamma > 0$ and $r_1 > 0$ such that* $\min\{f(x) \geq (\|f\|_r)^{\gamma} : x \geq r/(s+2)\}$ *for all $r \geq r_1$. Then for f the condition (1) holds.*

($M_f(r, s) := \max\{|f(z)| : z \in \mathcal{E}(r, s)\}$, $\mathcal{E}(r, s)$ denoting the closed ellipse in the complex plane with foci at $x=0$ and $x=r$, whose sum of both axes is rs).

Conditions (i) and (ii) are necessary for geometric convergence.

Let f be a continuous real-valued function on $[0, \infty)$ and define

$$\|f\|_r = \sup\{|f(x)| : r > 0,\ 0 \leq x \leq r\},\quad \|f\| = \sup\{|f(x)| : x \geq 0\}.$$

For each nonnegative integer n let Π_n denote the collection of all real polynomials of degree at most n. We investigate the following problem:

For which functions $f \in C[0, \infty)$ does there exist a number $q > 1$ and a sequence of polynomials $\{p_n\}_0^{\infty}$ such that $p_n \in \Pi_n$; $n = 0, 1, \ldots$ and

$$\text{(1)} \qquad \overline{\lim_{n\to\infty}} \| 1/f - 1/p_n \|^{1/n} = 1/q.$$

Such functions f are called functions with *geometric convergence*. The complete answer to this problem is not yet known, although many authors have investigated this problem in recent years. The first result on this topic was established in 1969 by W. Cody, G. Meinardus and R. Varga [4]. They proved, that e^x has geometric convergence. G. Meinardus and R. Varga [6] extended this result to functions of perfectly regular growth. In 1972 G. Meinardus, A. Reddy, G. Taylor and R. Varga [7] obtained a necessary as well as a sufficient condition for functions of geometric convergence. In order to formulate their results we need some definitions:

For given $r>0$ and $s>1$, let $\mathcal{E}(r, s)$ denote the closed ellipse in the complex plane with foci at $x=0$ and $x=r$ whose sum of both axes is the product rs. If f is an entire function, set

$$M_f(r, s)=\max\{|f(z)|:z\in\mathcal{E}(r, s)\}.$$

A necessary condition for geometric convergence is formulated in

Theorem 1 [7]: *If f has geometric convergence, then there exists an entire function $F(z)$ with $F(x)=f(x)$ for all $x\geqq 0$ and F is of finite order. In addition, for every $s>1$, there exist constants $K>0$, $\theta>0$ and $r_0>0$ such that*

$$M_f(r, s)\leqq K(\|f\|_r)^{\theta} \text{ for all } r\geqq r_0. \tag{2}$$

Conversely in [7] was proved

Theorem 2: *Let $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be an entire function with $a_\nu\geqq 0$ and $a_0>0$. If the growth condition (2) holds, then f has geometric convergence.*

It is this additional assumption of nonnegative Taylor coefficients that motivated much work in recent years to get weaker sufficient conditions. On the other hand, M. Henry and J. Roulier [5] obtained a new necessary condition for geometric convergence. Roughly speaking, the function f cannot oscillate too much. We present here a sufficient condition for geometric convergence containing all sufficient conditions known to us so far. Let f be an entire function having nonnegative zeros at precisely $\{x_i\}_{i=1}^{k}$, $0\leqq x_1<x_2<\cdots<x_k$, with respective orders $\beta_1, \beta_2, \ldots, \beta_k$ and assume that f satisfies the growth condition (2). For any nonnegative integer n we approximate $1/f$ with respect to $V_n=\{1/p:p\in\Pi_n\}$ in the Chebyshev sense, i. e. we want to minimize the error $\|1/f-1/p\|$ with respect to $p\in\Pi_n$.

For abbreviation we define $\beta:=\sum_{i=1}^{k}\beta_i$, $w(x):=\Pi_{i=1}^{k}(x-x_i)^{\beta_i}$. As in the case $1/f\in C[0, \infty]$ ([2, 3]) we get a similar characterization for the best approximation.

Theorem 3. *Let $n\geqq\max(1, 3\beta)$, $\lim_{x\to\infty} f(x)=\infty$. Then $v_0=1/q_0$ is the best approximation with respect to V_n iff:*

(a) *For $q_0\in\Pi_n-\Pi_{n-1}$, the error $1/f-1/q_0$ has $n+2-2\beta$ alternation points in $[0, \infty)$.*

(b) *For $q_0\in\Pi_{n-1}$, the error function has $n+1-2\beta$ alternation points in $[0, \infty)$, but the greatest alternation point is a maximum of $1/f-1/q_0$.*

We denote by $p\in\Pi_{2\beta-1}$ (resp. $\tilde{p}\in\Pi_{3\beta-1}$) the polynomial satisfying $p^{(j)}(x_i)=f^{(j)}(x_i)$ (resp. $\tilde{p}^{(j)}(x_i)=f^{(j)}(x_i)$) for $i=1, 2, \ldots, k$ and $j=0, 1, \ldots, 2\beta_i-1$ (resp. $3\beta_i-1$). Then $f-p=w^2.F$ and $f-\tilde{p}=w^3.\tilde{F}$ with entire functions $F, \tilde{F}$. We define

$$m(r)=\sup\{1/f(x):x\geqq r\},\ \varrho_n(f)=\inf\{\|1/f-1/p_n\|:p_n\in\Pi_n\},$$

and for $g \in C[0, r]$:

$$E_n(g, r) := \inf\{\|g - p_n\|_r : p_n \in \Pi_n\}.$$

The following lemma combines the minimal distance $\varrho_n(f)$ with certain polynomial approximations on finite intervals, using the above-mentioned characterization.

L e m m a: *For $s>1$ we choose r such that $f(r)=s^n$. Then there exist a real number $A>0$, an integer $n_0>0$ and real numbers $\alpha=\alpha(r)\geq r$ such that $\varrho_n(f)\leq m(r)+\max(m(r/(s+2)), A.\ E(r))$ holds for all $n\geq n_0$ with*

$$E(r) = \max\left(\left(\frac{s}{s+1}\right)^n, \alpha.\ E_{n-2\beta-1}\left(\frac{F}{x-\alpha}, \frac{r}{s+2}\right), E_{n-3\beta-1}(\widetilde{F}, r)\right).$$

This lemma and the well-known inequalities of S. B e r n s t e i n [1] are the keys for proving our main

T h e o r e m 4: *Let f be an entire function with the properties:*

(i) *f has only a finite number of zeros in $[0, \infty)$,*

(ii) *f satisfies the condition (2),*

(iii) *there exist real numbers $s>1, \gamma>0$ and $r_1>0$ such that $m(r/(s+2)) \leq (\|f\|_r)^{-\gamma}$ for all $r\geq r_1$.*

Then f has geometric convergence.

We remark that the conditions (i) and (ii) are necessary for geometric convergence. The additional condition (iii) is a certain kind of oscillation. Remember that M. H e n r y and J. R o u l i e r [5] have proved an oscillation condition being necessary for geometric convergence. Hence we think that the complete characterization of functions f with geometric convergence has to be found in oscillation properties of f.

From our main theorem one can deduce the

C o r o l l a r y: *If f satisfies* (i) *and* (ii), *and if there exists $r_1\geq 0$ such that $f'(x)\geq 0$ for all $x\geq r_1$, then f has geometric convergence.*

REFERFNCES

1. S. B e r n s t e i n. Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle. New York, 1970.
2. H.-P. B l a t t. Rationale Tschebyscheff-Approximation über unbeschränkten Intervallen. *Numer. Math.*, **27**, 1977, 179—190.
3. H.-P. B l a t t. Rationale Approximierbarkeit singulärer Funktionen über $[0,\infty)$. Approximation Theory, Bonn 1976. Berlin, 1976, 1—15.
4. W. J. C o d y, G. M e i n a r d u s, R. S. V a r g a. Chebyshev Rational Approximations to e^{-x} in $[0,\infty)$ and Applications to Heat-Conduction Problems. *J. Approx. Theory*, **2**, 1969, 50—65.
5. M. S. H e n r y, J. A. R o u l i e r. Geometric Convergence of Chebyshev Rational Approximations on $[0,\infty)$. June 1976.
6. G. M e i n a r d u s, R. S. V a r g a. Chebyshev Rational Approximations to Certain Entire Functions in $[0,\infty)$. *J. Approx. Theory*, **3**, 1970, 300—309.
7. G. M e i n a r d u s, A. R. R e d d y, G. D. T a y l o r, R. S. V a r g a. Converse Theorems and Extensions in Chebyshev Rational Approximation to Certain Entire Functions in $[0,\infty)$. *Trans. Amer. Math. Soc.*, **170**, 1972, 171—185

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