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EXISTENCE AND CHARACTERIZATION OF MONOSPLINES OF LEAST L_p DEVIATION

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Summary. We prove the existence of a monospline of least L_p-norm in the set of all monosplines of degree r with free knots $(x_k)_1^n$, $a \leqq x_1 < \cdots < x_n \leqq b$ and fixed multiplicities $(\nu_k)_1^n$, $1 \leqq \nu_k \leqq r$, $k=1, \ldots, n$ satisfying certain boundary conditions.

The problem is related to the existence of optimal quadrature formulae of fixed type in Sobolev spaces $W_p^r[a, b]$.

1. Introduction. This paper is devoted to the problem of existence of a monospline with pre-assigned multiplicities $\nu_1, \ldots, \nu_n$ of the knots, which has a minimal L_p norm $(1 \leqq p < \infty)$ in $[a, b]$ among all monosplines with free knots of the same multiplicities $(\nu_k)_1^n$. Our results extend those of M. Powell [1], S. Karlin [2, 3] and R. Barrar and H. Loeb [4]. These authors consider the special case $\nu_1 = \cdots = \nu_n = 1$. We study both the periodic and finite interval case with certain boundary conditions. The relation between monosplines and quadrature formulae makes it possible to state our main results in another form, as existence theorems for optimal quadrature formulae of fixed type in Sobolev spaces $W_q^r[a, b]$,

$$W_q^r[a, b] := \{ f : f \in C^{r-1}[a, b],\ f^{(r-1)} \text{ abs. cont., } f^{(r)} \in L_q[a, b]\}$$

for $1 < q \leqq \infty$. Here as everywhere in this paper q is the conjugate number to p, i. e., $1/q + 1/p = 1$.

A monospline of degree r with knots $(x_k)_1^n$ of multiplicities $(\nu_k)_1^n$ respectively is a function of the form

$$M(t) = \frac{t^r}{r!} + \sum_{i=0}^{r-1} c_i t^i + \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} c_{k\lambda} (x_k - t)_+^{r-\lambda-1}. \tag{1.1}$$

We shall write

$$\mathbf{x} = \begin{pmatrix} x_1, \ldots, x_n \\ \nu_1, \ldots, \nu_n \end{pmatrix} \tag{1.2}$$

to denote that $\mathbf{x}$ is a system of n distinct knots $x_1 < \cdots < x_n$ in (a, b) of multiplicities $\nu_1, \ldots, \nu_n$, respectively. Given the multiplicities $(\nu_k)_1^n$ satisfying the inequalities

(1.3) $$1 \leq \nu_k \leq r, \quad k=1, \ldots, n,$$

we set

$$\Omega(\nu_1, \ldots, \nu_n) := \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} x_1, \ldots, x_n \\ \nu_1, \ldots, \nu_n \end{pmatrix} \right\}.$$

Let $\mathfrak{M}(\mathbf{x}; \mathfrak{B})$ denote the collection of monosplines $M(t)$ or the form (1.1) with fixed knots $\mathbf{x}$ satisfying the boundary conditions $\mathfrak{B}$. In this paper we consider boundary conditions $\mathfrak{B}$ of the kind:

(i) Periodic boundary conditions (PBC)

$$M^{(j)}(a) = M^{(j)}(b), \quad j=0, \ldots, r-1.$$

(ii) Zero boundary conditions (ZBC)

$$M^{(j)}(a) = 0, \quad j=0, \ldots, r-\nu-1, \quad 0 \leq \nu \leq r,$$

$$M^{(j)}(b) = 0, \quad j=0, \ldots, r-\mu-1, \quad 0 \leq \mu \leq r.$$

Denote by $M(\mathbf{x}; t)$ the monospline with knots $\mathbf{x}$ which has a minimal L_p norm in $[a, b]$ among all monosplines $M(t) \in \mathfrak{M}(\mathbf{x}; \mathfrak{B})$. The knots $\mathbf{x} \in \Omega(\nu_1, \ldots, \nu_n)$ are said to be optimal of the type $(\nu, \ldots, \nu_n; \mathfrak{B})$ if

$$\| M(\mathbf{x}; \cdot) \|_p = \inf \{ \| M(\mathbf{z}; \cdot) \|_p : \mathbf{z} \in \Omega(\nu_1, \ldots, \nu_n) \}.$$

Here, as usual, $\|f\|_p = \{\int_a^b |f(t)|^p dt\}^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_p = \operatorname{supvrai}_{a \leq t \leq b} |f(t)|$ for $p = \infty$.

The extremal monospline in the above problem will be referred to as a monospline of least L_p deviation of the type $(\nu_1, \ldots, \nu_n; \mathfrak{B})$. Note that the set $\{M(\mathbf{z}; \cdot) : \mathbf{z} \in \Omega(\nu_1, \ldots, \nu_n)\}$ is open and nonlinear. So the existence of an extremal element does not follow by classical arguments of compactness. It is conceivable that in minimizing $\| M(\mathbf{z}; \cdot) \|_p$ some of the knots coalesce or stream to the endpoints a, b. We show in this paper that it is not actually the case. For every system of multiplicities $(\nu_k)_1^n$ satisfying (1.3) the optimal knots of the type $(\nu_1, \ldots, \nu_n; \mathfrak{B})$ really exist.

In 1960 R. Johnson [5] proved the existence of optimal knots of the type $(1, \ldots, 1)$ without boundary conditions for $p = \infty$. M. Powell [1] obtained the same result for $p = 2$. In [2] S. Karlin announced existence theorems for $1 \leq p \leq \infty$ and $\nu_1 = \cdots = \nu_n = 1$. Recently Karlin [3] gave a proof of his result. Another independent proof was given by R. Barrar and H. Loeb [4]. N. Richter-Dyn [6] used an interesting approach to prove the existence of a monospline of least L_2 deviation of the type $(1, \ldots, 1)$. V. Motornii [7] and A. Ligun [8] showed that the equidistant knots are optimal of the type $(1, \ldots, 1; \mathfrak{B})$ for $p = 1$ and $p = \infty$, respectively, assuming $\mathfrak{B} = \text{PBC}$. In order to prove the same statement for $1 < p < \infty$ A. Žensykbaev [9, 10] uses the existence of optimal knots but he does not prove it. All cited results concerning the existence of a monospline of least L_p deviation for $1 \leq p < \infty$ and $\mathfrak{B}$ being of the kind PBC or ZBC follow from our main theorem. We give also characterization of the extremal monosplines.

2. Quadrature Formulae and Monosplines.

The proof of the main result of our work relies upon the connection between quadrature formulae and monosplines. Below we briefly describe this connection.

It will be more convenient to rewrite the monospline (1.1) in the following form

$$(2.1)\qquad M(t)=(-1)^r\left\{\frac{(b-t)^r}{r!}-\sum_{j=0}^{r-1}B_j\frac{(b-t)^{r-j-1}}{(r-j-1)!}-\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}\frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}\right\}.$$

Observe that the coefficients B_j and $a_{k\lambda}$ of $M(t)$ can be determined by the formulas

$$(2.2)\qquad \begin{gathered} B_j=(-1)^j M^{(r-j-1)}(b),\ j=0,\dots,r-1,\\ a_{k\lambda}=(-1)^\lambda\,[M^{(r-\lambda-1)}(x_k-0)-M^{(r-\lambda-1)}(x_k+0)],\\ k=1,\dots,n,\ \lambda=0,\dots,\nu_k-1. \end{gathered}$$

Suppose that $f\in W_q^r[a,b]$ $(1<q\leq\infty)$. Integration by parts produces the identity

$$(2.3)\ (-1)^r\int_a^b M(t)f^{(r)}(t)dt=\int_a^b f(t)dt-\sum_{j=0}^{r-1}A_jf^{(j)}(a)-\sum_{j=0}^{r-1}B_jf^{(j)}(b)-\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}f^{(\lambda)}(x_k),$$

where

$$(2.4)\qquad A_j=(-1)^{j+1}M^{(r-j-1)}(a),\ j=0,\dots,r-1.$$

This expression suggests a quadrature formula of the type

$$(2.5)\qquad I(f):=\int_a^b f(t)dt\approx Q(f):=\sum_{j=0}^{r-1}A_jf^{(j)}(a)+\sum_{j=0}^{r-1}B_jf^{(j)}(b)+\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}f^{(\lambda)}(x_k),$$

where (A_j), (B_j) and $(a_{k\lambda})$ are free real parameters. We shall use the notation $Q(\mathfrak{B};f)$ for the term $Q(f)$ of (2.5) to mark that the coefficients A_j, B_j are stipulated to fulfil the constraints

$$(2.6)\qquad \left.\begin{aligned} &A_j=0,\ j=\nu,\nu+1,\dots,r-1\\ &B_j=0,\ j=\mu,\mu+1,\dots,r-1\end{aligned}\right\}\text{ if }\mathfrak{B}=\text{ZBC}$$
$$A_j=-B_j,\ j=0,\dots,r-1\quad\text{if }\mathfrak{B}=\text{PBC}.$$

Now suppose that $M\in\mathfrak{M}(\mathbf{x};\mathfrak{B})$. Then (2.3) induces a quadrature formula of the type

$$(2.7)\qquad I(f)\approx Q(\mathfrak{B};f),$$

which is evidently exact for all $f\in\pi_{r-1}$. (Here, as elsewhere in this paper, π_m denotes the class of algebraic polynomials of degree not greater than m.) Moreover,

$$I(f)-Q(\mathfrak{B};f)=(-1)^r\int_a^b M(t)f^{(r)}(t)dt$$

for every $f\in W_q^r[a,b]$. Conversely, let (2.7) be given with arbitrary fixed coefficients $(\tilde{A}_j)$, $(\tilde{B}_j)$, $(\tilde{a}_{k\lambda})$ satisfying the relation

$$(2.8) \qquad I(f)=Q(\mathfrak{B};f) \text{ for } f\in\pi_{r-1}.$$

Denote by $\widetilde{M}$ the monospline (2.1) with the same coefficients $\widetilde{B}_j$ and $\widetilde{a}_{k\lambda}$. It is not difficult to verify that the requirement (2.8) is equivalent to the next one $\widetilde{A}_j=(-1)^{j+1}\widetilde{M}^{(r-j-1)}(a)$ for $j=0,\dots,r-1$. Thus, on the basis of (2.3), the remainder term of the studied quadrature formula is $(-1)^r\int_a^b\widetilde{M}(t)f^{(r)}(t)dt$. This is the one-to-one correspondence between monosplines and quadrature formulae that we wished to describe.

Let the nodes x be fixed. The quadrature formula $I(f)\approx Q^*(\mathfrak{B};f)$ is said to be best for the class $W_q^r[a,b]$ if

$$R_q(\mathbf{x}):=\inf\{\sup\{|I(f)-Q(\mathfrak{B};f)|:f\in W_q^r[a,b],\ \|f^{(r)}\|_q\le 1\}:A_j,B_j,a_{k\lambda}\}$$
$$=\sup\{|I(f)-Q^*(\mathfrak{B};f)|:f\in W_q^r[a,b],\ \|f^{(r)}\|_q\le 1\}.$$

The quantity $R_q(\mathbf{x})$ is the error of the best quadrature formula in $W_q^r[a,b]$. It is easy to see that $R_q(\mathbf{x})$ is bounded for every $\mathbf{x}\in\Omega(\nu_1,\dots,\nu_n)$ if

$$(2.9) \qquad \begin{aligned} \nu+\mu+\nu_1+\dots+\nu_n&\ge r \quad \text{in the case} \quad \mathfrak{B}=\mathrm{ZBC},\\ \nu_1+\dots+\nu_n&\ge 1 \quad \text{in the case} \quad \mathfrak{B}=\mathrm{PBC}. \end{aligned}$$

In what follows we assume that (2.9) is postulated. Then the best quadrature formula must be exact for all polynomials $f\in\pi_{r-1}$. Indeed, if $I(f)-Q^*(\mathfrak{B};f)=\varepsilon>0$ for some $f\in\pi_{r-1}$ then $I(cf)-Q^*(\mathfrak{B};cf)=c\varepsilon$ for every number c. This contradicts the estimation $|I(cf)-Q^*(\mathfrak{B};cf)|\le R_q(\mathbf{x})<\infty$ for sufficiently large c. Thus, searching for the best quadrature formula we can restrict ourselves only to formulae which are exact for the class π_{r-1}. It is clear form the described one-to-one correspondence between quadrature formulae and monosplines that

$$(2.10) \quad R_q(\mathbf{x})=\inf\{\sup\{|\int_a^b M(t)f^{(r)}(t)dt|:f\in W_q^r[a,b],\ \|f^{(r)}\|_q\le 1\}:M\in\mathfrak{M}(\mathbf{x};\mathfrak{B})\}$$
$$=\min\{\|M\|_p:M\in\mathfrak{M}(\mathbf{x};\mathfrak{B})\}=\|M(\mathbf{x};\cdot)\|_p.$$

Denote

$$R_q(\nu_1,\dots,\nu_n;\mathfrak{B})=\inf\{R_q(\mathbf{z}):\mathbf{z}\in\Omega(\nu_1,\dots,\nu_n)\}.$$

The best quadrature formula with nodes $\mathbf{x}\in\Omega(\nu_1,\dots,\nu_n)$ is said to be optimal of the type $(\nu_1,\dots,\nu_n;\mathfrak{B})$ if $R_q(\mathbf{x})=R_q(\nu_1,\dots,\nu_n;\mathfrak{B})$. It follows from (2.10) that

$$R_q(\nu_1,\dots,\nu_n;\mathfrak{B})=\inf\{\|M(\mathbf{z};\cdot)\|_p:\mathbf{z}\in\Omega(\nu_1,\dots,\nu_n)\}.$$

Moreover, the existence of an optimal quadrature formula of the type $(\nu_1,\dots,\nu_n;\mathfrak{B})$ in $W_q^r[a,b]$ is equivalent to the existence of a monospline of least L_p deviation of the type $(\nu_1,\dots,\nu_n;\mathfrak{B})$.

3. Characterization Results. To every fixed system of knots $\mathbf{x}\in\Omega(\nu_1,\dots,\nu_n)$ we compare a function

$$(3.1) \qquad \psi(\mathbf{x};t)=\sigma(\mathbf{x})|M(\mathbf{x};t)|^{p-1}\operatorname{sign}M(\mathbf{x};t),$$

where $\sigma(\mathbf{x})=(-1)^r(\int_a^b |M(\mathbf{x};t)|^p dt)^{-1/q}$. Recall that the existence of the extremal function $M(\mathbf{x};t)$ as a solution of the problem

$$(3.2)\qquad \min\{\|M\|_p : M\in\mathfrak{M}(\mathbf{x}:\mathfrak{B})\}$$

follows from well-known classical result (see [11], p. 17). Moreover, $M(\mathbf{x};t)$ is unique for fixed $\mathfrak{B}$ and p, $1<p<\infty$, since the L_p-norm is strictly convex for $1<p<\infty$. In the case $p=1$ the function sign $M(\mathbf{x};t)$ is the same (see [11], p. 92) for all extremal elements $M(\mathbf{x};t)$. Therefore, the equality (3.1) defines $\psi(\mathbf{x};t)$ uniquely. It is easily verified that $\|\psi(\mathbf{x};\cdot)\|_q=1$ and

$$(3.3)\qquad (-1)^r\int_a^b M(\mathbf{x};t)\,\psi(\mathbf{x};t)\,dt=\|M(\mathbf{x};\cdot)\|_p.$$

Employing the Hölder inequality, one can show that $\psi(\mathbf{x};t)$ is the unique function in the set $\{g:g\in L_q[a,b],\ \|g\|_q\leq 1\}$ for which the relation (3.3) holds. Therefore, the equality $I(f)-Q^*(\mathfrak{B};f)=R_q(\mathbf{x})$ and the constraint $\|f^{(r)}\|_q\leq 1$ implies $f^{(r)}(t)=\psi(\mathbf{x};t)$. Next we construct a special function for which the error of the best quadratute formula with fixed nodes is achieved.

With $\mathfrak{B}$ we associate the adjoint boundary conditions $\mathfrak{B}^*$,

$$\mathfrak{B}^*:\begin{cases} f^{(j)}(a)=0, & j=0,\dots,\nu-1\\ f^{(j)}(b)=0, & j=0,\dots,\mu-1\end{cases}\quad \text{for } \mathfrak{B}=\mathrm{ZBC}$$

$$\mathfrak{B}^*: f^{(j)}(a)=f^{(j)}(b),\ j=0,\dots,r-1 \text{ for } \mathfrak{B}=\mathrm{PBC}.$$

Denote, for convenience,

$$W_0(\mathbf{x};\mathfrak{B}^*)=\{f\in W_q^r[a,b]: f \text{ satisfies } \mathfrak{B}^*,\ \|f^{(r)}\|_q\leq 1,\ f^{(\lambda)}(x_k)=0,$$
$$k=1,\dots,n,\ \lambda=0,\dots,\nu_k-1\}.$$

Lemma 1. *Let $\mathfrak{B}$ be given and $1<q\leq\infty$. For every fixed system of nodes $\mathbf{x}\in\Omega(\nu_1,\dots,\nu_n)$ there exists a unique function $F(\mathbf{x};t)\in W_0(\mathbf{x};\mathfrak{B}^*)$ such that $I(F(\mathbf{x};\cdot))=R_q(\mathbf{x})$.*

Proof. Assume first that $\mathfrak{B}=\mathrm{ZBC}$. In view of (2.6) every monospline $M\in\mathfrak{M}(\mathbf{x};\mathfrak{B})$ must have the form

$$M(t)=(-1)^r\Big\{\frac{(b-t)^r}{r!}-\sum_{j=0}^{\mu-1}B_j\frac{(b-t)^{r-j-1}}{(r-j-1)!}-\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}\frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}\Big\}$$

with some real coefficients (B_j) and $(a_{k\lambda})$ satisfying the conditions $M^{(j)}(a)=0$, $j=0,\dots,r-\nu-1$. Because $M(\mathbf{x};t)$ is an extremal element in (3.2), we have

$$(3.4)\qquad \frac{\partial}{\partial a_{k\lambda}}\Big[\int_a^b |M(t)|^p dt+\sum_{i=0}^{r-\nu-1}\lambda_i M^{(i)}(a)\Big]_{M=M(\mathbf{x};\cdot)}=0$$

for $k=1,\dots,n$, $\lambda=0,\dots,\nu_k-1$ and

$$(3.5)\qquad \frac{\partial}{\partial B_j}\Big[\int_a^b |M(t)|^p dt+\sum_{i=0}^{r-\nu-1}\lambda_i M^{(i)}a)\Big]_{M=M(\mathbf{x};\cdot)}=0$$

for $j=0,\dots,\mu-1$, where (λ_i) are Lagrange multipliers. The computation of (3.4) gives

(3.6) $$(-1)^{r+1}p\int_a^b |M(\mathbf{x};t)|^{p-1}\operatorname{sgn} M(\mathbf{x};t)\{(x_k-t)_+^{r-\lambda-1}/(r-\lambda-1)!\}\,dt$$

$$+\sum_{i=0}^{r-\nu-1}(-1)^{r+i+1}\lambda_i(x_k-a)^{r-\lambda-i-1}/(r-\lambda-i-1)!=0,$$

where $1/m!=0$ for $m<0$ is stipulated. Now define

(3.7) $$F(\mathbf{x};t)=P(\mathbf{x};t)+\frac{(-1)^r}{(r-1)!}\int_a^b(t-\tau)_+^{r-1}\psi(\mathbf{x};\tau)\,d\tau,$$

where $$P(\mathbf{x};t)=p^{-1}\sigma(\mathbf{x})\sum_{i=0}^{r-\nu-1}(-1)^{r+i}\lambda_i(t-a)^{r-i-1}/(r-i-1)!.$$

Then (3.6) can be rewritten as

(3.8) $$F^{(\lambda)}(\mathbf{x};x_k)=0,\ k=1,\dots,n,\ \lambda=0,\dots,\nu_k-1.$$

In the same way (3.5) yields $F^{(j)}(\mathbf{x};b)=0$ for $j=0,\dots,\mu-1$. A direct computation of $F^{(j)}(\mathbf{x};a)$ shows that $F^{(j)}(\mathbf{x};a)=P^{(j)}(\mathbf{x};a)=0$ for $j=0,\dots,\nu-1$. Therefore $F(\mathbf{x};t)$ satisfies $\mathfrak{B}^*$. Further, using (3.7) and the definition of $\psi(\mathbf{x};t)$, we get $\|F^{(r)}(\mathbf{x};\cdot)\|_q=\|\psi(\mathbf{x};.)\|_q=1$ and consequently $F(\mathbf{x};\cdot)\in W_0(\mathbf{x};\mathfrak{B}^*)$. Finally, taking into account (3.8), (2.10) and (3.3) we obtain

$$I(F(\mathbf{x};\cdot))=I(F(\mathbf{x};\cdot))-Q^*(\mathfrak{B};F(\mathbf{x};\cdot))=(-1)^r\int_a^b M(\mathbf{x};t)\psi(x;t)dt=\|M(\mathbf{x};\cdot)\|_p=R_q(\mathbf{x}).$$

It remains to prove the uniqueness of $F(\mathbf{x};\cdot)$. Suppose that there is another function $G(t)\in W_0(\mathbf{x};\mathfrak{B}^*)$ and such that $I(G)=R_q(\mathbf{x})$. Since $G^{(r)}(t)=\psi(\mathbf{x};t)$, we have $F^{(r)}(\mathbf{x};t)-G^{(r)}(t)=0$. Then the assumption $F(\mathbf{x};\cdot),\ G\in W_0(\mathbf{x};\mathfrak{B}^*)$ and (2.9) imply $F(\mathbf{x};t)=G(t)$. The lemma is proved for $\mathfrak{B}=\text{ZBC}$.

Now suppose that $\mathfrak{B}=\text{PBC}$. Then an arbitrary monospline $M\in\mathfrak{M}(\mathbf{x};\mathfrak{B})$ has the form (2.1) with coefficients satisfying the conditions $M^{(i)}(a)=M^{(i)}(b)$ for $i=0,\dots,r-1$. The necessary conditions

$$\frac{\partial}{\partial a_{k\lambda}}\Big[\int_a^b|M(t)|^p dt+\sum_{i=0}^{r-1}\lambda_i(M^{(i)}(a)-M^{(i)}(b))\Big]_{M=M(\mathbf{x};\cdot)}=0,$$

$$\frac{\partial}{\partial B_j}\Big[\int_a^b|M(t)|^p dt+\sum_{i=0}^{r-1}\lambda_i(M^{(i)}(a)-M^{(i)}(b))\Big]_{M=M(\mathbf{x};\cdot)}=0$$

for the extremality of $M(\mathbf{x};t)$ gives, as in the previous case,

(3.9) $$F^{(\lambda)}(\mathbf{x};x_k)=0,\ k=1,\dots,n,\ \lambda=0,\dots,\nu_k-1,$$
$$F^{(j)}(\mathbf{x};b)=p^{-1}\sigma(\mathbf{x})(-1)^j\lambda_{r-j-1},\ j=0,\dots,r-1,$$

where $F(\mathbf{x};t)$ is defined by (3.7) with

(3.10) $$P(\mathbf{x};t)=p^{-1}\sigma(\mathbf{x})\sum_{i=0}^{r-1}(-1)^{r+1+i}\lambda_i(t-a)^{r-i-1}/(r-i-1)!.$$

But $F^{(j)}(\mathbf{x}; a) = p^{(j)}(\mathbf{x}; a) = p^{-1}\sigma(\mathbf{x})(-1)^j \lambda_{r-j-1}$. Therefore, in view of (3.10), $F^{(j)}(\mathbf{x}; a) = F^{(r)}(\mathbf{x}; b)$ for $j = 0, \dots, r-1$ and consequently $F(\mathbf{x}; \cdot)$ satisfies $\mathfrak{B}^*$. The other assertions of the lemma are proved in the same way as in the case $\mathfrak{B} = \mathrm{ZBC}$. The proof is complete.

Lemma 2. *Let $1 \leq p < \infty$. Suppose that the knots (1.2) are optimal of the type $(\nu_1, \dots, \nu_n; \mathfrak{B})$. Then*

$$a_{k,\nu_k-1} F^{(\nu_k)}(\mathbf{x}; x_k) = 0 \tag{3.11}$$

for $\nu_k < r$ and

$$|M(\mathbf{x}; x_k - 0)| = |M(\mathbf{x}; x_k + 0)| \tag{3.12}$$

for $\nu_k = r$, where $(a_{k\lambda})$ are the coefficients of $M(\mathbf{x}; t)$.

Proof. Since $\|M(\mathbf{x}; \cdot)\|_p = \min\{\|M(\mathbf{z}; \cdot)\|_p : \mathbf{z} \in \Omega(\nu_1, \dots, \nu_n)\}$ we have

$$\frac{\partial}{\partial x_k}\Big[\int_a^b |M(t)|^p dt + \sum_{i=0}^{r-\nu-1} \lambda_i M^{(i)}(a)\Big]_{M = M(\mathbf{x};\, \cdot)} = 0 \tag{3.13}$$

for $\mathfrak{B} = \mathrm{ZBC}$ and

$$\frac{\partial}{\partial x_k}\Big[\int_a^b |M(t)|^p dt + \sum_{i=0}^{r-1} \lambda_i (M^{(i)}(a) - M^{(i)}(b))\Big]_{M = M(\mathbf{x};\, \cdot)} = 0 \tag{3.14}$$

for $\mathfrak{B} = \mathrm{PBC}$, where (λ_i) are Lagrange multipliers. In order to compute the derivatives in (3.14) and (3.15) we divide the integral into two parts as follows $\int_a^b |M(t)|^p dt = \int_a^{x_k} |M(t)|^p dt + \int_{x_k}^b |M(t)|^p dt$. Performing the differentiation we get

$$p \sum_{\lambda=0}^{\min(\nu_k-1,\, r-2)} a_{k\lambda} \int_a^{x_k} |M(\mathbf{x}; t)|^{p-1} \operatorname{sgn} M(\mathbf{x}; t) \{(x_k - t)^{r-\lambda-2}/(r-\lambda-2)!\}$$

$$+ |M(\mathbf{x}; x_k - 0)|^p - |M(\mathbf{x}; x_k + 0)|^p + \sum_{\lambda=0}^{\nu_k - 1} a_{k\lambda} \sum_{i=0}^{\varrho} (-1)^{r+i+1}$$

$$\times \lambda_i (x_k - a)^{r-\lambda-i-2}/(r-\lambda-i-2)! = 0,$$

where $\varrho = r - \nu - 1$ if $\mathfrak{B} = \mathrm{ZBC}$ and $\varrho = r - 1$ if $\mathfrak{B} = \mathrm{PBC}$. Using the definitions (3.7) and (3.8) (respectively (3.11)) we rewrite the above relation as

$$\sum_{\lambda=0}^{\min(\nu_k-1,\, r-2)} a_{k\lambda} F^{(\lambda)}(\mathbf{x}; x_k) + |M(\mathbf{x}; x_k - 0)|^p - |(M(\mathbf{x}; x_k + 0)|^p = 0. \tag{3.15}$$

But $F^{(i)}(\mathbf{x}; x_k) = 0$ for $i = 0, \dots, \nu_k - 1$, according to Lemma 1. This together with (3.16) implies (3.13). To derive (3.12) we need only to observe that $M(\mathbf{x}; t)$ is continuous at x_k for $\nu_k < r$. The proof is complete.

In order to show some other properties of the extremal function $F(\mathbf{x}; \cdot)$ we need a bound for the zeros of monosplines. We begin by defining a zero of multiplicity m of $M(t)$ where m is allowed to be as large as $r+1$.

We say the monospline (2.1) has a zero of multiplicity m at a point $t \notin (x_1, \dots, x_n)$ provided

$$M(t) = M'(t) = \cdots = M^{(m-1)}(t) = 0, \quad M^{(m)}(t) \neq 0. \tag{3.16}$$

Let M_+ (M_-) be the monospline which agrees with $M(t)$ to the right (left) of x_k and has no knots in $(-\infty, x_k]$ ($[x_k, \infty)$). Suppose M_+ has a zero of multiplicity α (according to the convention (3.17)) at x_k and M_- has a zero of multiplicity β at x_k. Then M has a zero of multiplicity $m=\max(\alpha, \beta)$ provided the sign of $M_-^{(r-\nu_k)}(t)$ to the right of x_k is the same as the sign of $M_+^{(r-\nu_k)}(t)$ to the left of x_k. Otherwise M has a zero of multiplicity $\max(\alpha, \beta)+1$. The definition is chosen so that M changes sign if m is odd and does not change sign if m is even.

Denote the number of zeros of M in (a, b), counting multiplicities, by $Z(M; (a, b))$. Let $S^+(t_1, \dots, t_n)$ stand for the maximum number of sign changes in the sequence $t_1, \dots, t_n$ by allowing each zero to be replaced by $+1$ or -1. Let us also set $\sigma_k=1$ if ν_k is odd, and zero otherwise.

Lemma 3. *Let $M(t)$ be an arbitrary monospline of the form* (2.1) *with $a<x_1<\cdots<x_n<b$. Then*

$$(3.17)\qquad Z(M; (a, b))\leq r+\sum_{k=1}^{n}(\nu_k+\sigma_k)-S^+(M(a), \dots, (-1)^r M^{(r)}(a)) -S^+(M(b), \dots, M^{(r)}(b)).$$

Further,

$$(3.18)\qquad Z(M; (-\infty, \infty))\leq r+\sum_{k=1}^{n}(\nu_k+\sigma_k).$$

Moreover, when equality holds in (3.19), *the next two statements are valid*

$$(3.19)\qquad \text{If } \nu_k \text{ is odd then } a_{k\lambda}>0,\ \lambda=0, 2, \dots, \nu_k-1$$

$$(3.20)\qquad \begin{cases} M^{(i)}(t)>0, & i=0, \dots, r \text{ for } t>x_n, \\ (-1)^{r-i} M^{(i)}(t)>0, & i=0, \dots, r \text{ for } t<x_1. \end{cases}$$

The proof of this important result can be seen in [12].

Lemma 4. *Let $M(t)$ be an arbitrary monospline of the form* (2.1) *with $a<x_1<\cdots<x_n<b$ and let M satisfy PBC. Then*

$$(3.21)\qquad Z(M; (a, b))\leq \sum_{k=1}^{n}(\nu_k+\sigma_k).$$

Moreover, when equality holds in (3.22), *the next statement is valid.*

$$(3.22)\qquad \text{If } \nu_k \text{ is odd then } a_{k\lambda}>0,\ \lambda=0, 2, \dots, \nu_k-1.$$

Proof. The estimation (3.22) follows immediately from (3.18) and the simple fact that $S^+(t_0, \dots, t_r)+S^+(t_0, \dots, (-1)^r t_r)\geq r$. The second part of the lemma is proved as (3.20) in [12].

We say that a monospline M of the form (2.1) has a maximal number of zeros if

$$Z(M; (-\infty, \infty))=r+\sum_{k=1}^{n}(\nu_k+\sigma_k) \text{ for } \mathfrak{B}=\text{ZBC},$$

$$Z(M; (a, b))=\sum_{k=1}^{n}(\nu_k+\sigma_k) \text{ for } \mathfrak{B}=\text{PBC}.$$

Lemma 5. *Let the multiplicities* $(\nu_k)_1^n$ *satisfy the inequalities* $1 \leq \nu_k \leq r$, $k=1, \ldots, n$ *and the following condition (evenness condition):*

$$\nu_k \text{ is even if } \nu_k < r. \tag{3.23}$$

Then, for every system of knots $\mathbf{x} \in \Omega(\nu_1, \ldots, \nu_n)$ *the monospline* $M(\mathbf{x}; t)$ *has a maximal number of zeros.*

Proof. Assume first that $\mathfrak{B}=\text{ZBC}$. Let only the knots $x_{k_1}, \ldots, x_{k_m}$ have multiplicities equal to r. Then the function $F(\mathbf{x}; t)$ is continuous over the intervals (a, x_{k_1}), (x_{k_m}, b) and $(x_{k_i}, x_{k_{i+1}})$ for $i=1, \ldots, m-1$. Since, according to Lemma 1, $F(\mathbf{x}; t)$ vanishes at $\mathbf{x}$, Rolle's theorem yields

$$Z(M; (x_{k_i}, x_{k_{i+1}})) \geq r + \sum_{j=k_i+1}^{k_{i+1}-1} \nu_j. \tag{3.24}$$

Further, taking into account that $F(\mathbf{x}; t)$ satisfies $\mathfrak{B}^*$ and $M(\mathbf{x}; t)$ satisfies $\mathfrak{B}$, after a repeated application of Rolle's theorem we get

$$Z(M; [a, x_{k_1})) \geq \nu_1 + \cdots + \nu_{k_1-1} + r,$$
$$Z(M; (x_{k_m}, b]) \geq \nu_{k_m+1} + \cdots + \nu_n + r.$$

Therefore the monospline which coincides with $M(\mathbf{x}; t)$ over one of the subintervals (a, x_{k_1}), (x_{k_m}, b), $(x_{k_i}, x_{k_{i+1}})$, $i=1, \ldots, m-1$ and has no knots out of this subinterval, has a maximum number of zeros. Then the property (3.21) gives

$$\begin{cases} \operatorname{sign} M(\mathbf{x}; x_{k_i}-0) = \operatorname{sign} M(\mathbf{x}; x_{k_i}+0) > 0 & \text{for even } r, \\ \operatorname{sign} M(\mathbf{x}; x_{k_i}-0) = -\operatorname{sgn} M(\mathbf{x}; x_{k_i}+0) > 0 & \text{for odd } r. \end{cases} \tag{3.25}$$

According to the definition of a zero of a monospline, this means that $M(\mathbf{x}; t)$ has not a zero at x_{k_i} if r is even and $M(\mathbf{x}; t)$ has a zero at x_{k_i} if r is odd. Therefore

$$Z(M; [a, b]) \geq Z(M; [a, x_{k_1})) + Z(M; (x_{k_m}, b])$$
$$+ \sum_{i=1}^{m-1} Z(M; (x_{k_i}, (x_{k_{i+1}})) + \sigma_{k_1} + \cdots + \sigma_{k_m} \geq r + \sum_{k=1}^{n} (\nu_k + \sigma_k).$$

This inequality and (3.18) prove our assertion.

Now suppose that $\mathfrak{B}=\text{PBC}$. If $M(\mathbf{x}; t)$ has at least one knot ξ of multiplicity r we consider the $(b-a)$- periodic extension of $M(\mathbf{x}; t)$ over the interval $[\xi, \xi+b-a]$ and prove the assertion in the same way as in the case $\mathfrak{B}=\text{ZBC}$, using Lemma 4. Let $\nu_k < r$ for all $k=1, \ldots, n$. By virtue of Lemma 1, $F(\mathbf{x}; t)$ has $N=\nu_1+\cdots+\nu_n$ zeros at least in (a, b). But $F(\mathbf{x}; t)$ is a periodic function. Then, by Rolle's theorem, $F^{(r)}(\mathbf{x}; t)$ and consequently $M(\mathbf{x}; t)$ must have N simple zeros at least in (a, b). This and (3.21) show that $M(\mathbf{x}; t)$ has a maximal number of zeros. The lemma is proved.

Corollary 1. *Let the multiplicities* (ν_k) *satisfy the evenness condition* (3.23). *Then*

$$F(\mathbf{x}; t) \geq 0 \text{ for all } t \in [a, b], \tag{3.26}$$
$$F(\mathbf{x}; t) > 0 \text{ for } t \notin (x_1, \ldots, x_n),$$
$$F^{(\nu_k)}(\mathbf{x}; x_k) > 0 \text{ for } \nu_k < r.$$

(3.27) $F^{(\nu)}(\mathbf{x}; a)>0$
(3.28) $(-1)^{\mu} F^{(\mu)}(\mathbf{x}; b)>0$ } when $\mathfrak{B}=\mathrm{ZBC}$.

Proof. We first observe that $F(\mathbf{x}; t)$ does not change sign in the knots $x_{k_1}, \ldots, x_{k_m}$ of odd multiplicities. Indeed, if ν_{k_i} is odd, then $\nu_{k_i}=r$ and r is an odd number, according to (3.23). Then, as we proved in (3.25), $M(\mathbf{x}; t)$ and consequently $F^{(r)}(\mathbf{x}; t)$ changes sign in x_{k_i}. This, together with the property $F^{(\lambda)}(\mathbf{x}; x_{k_i})=0$, $\lambda=0, \ldots, r-1$, shows that $F(\mathbf{x}; t)$ does not change sign at x_{k_i}. Now let us assume that $F(\mathbf{x}; t_0)=0$ for some $t_0 \in (x_1, \ldots, x_n)$ or $F^{(\nu_k)}(\mathbf{x}; x_k)=0$ for some $\nu_k<r$. Then applying Rolle's theorem we find that $M(\mathbf{x}; t)$ has at least $r+1+\Sigma_{k=1}^{n}(\nu_k+\sigma_k)$ zeros in $[a, b]$ if $\mathfrak{B}=\mathrm{ZBC}$ and $1+\Sigma_{k=1}^{n}(\nu_k+\sigma_k)$ zeros in (a, b) if $\mathfrak{B}=\mathrm{PBC}$. This contradicts (3.19) and (3.22), respectively. Therefore $F(\mathbf{x}; t)$ does not change sign in (a, b). Because $I(F(\mathbf{x}; \cdot))=R_q(\mathbf{x})>0$ we get $F(\mathbf{x}; t)\geqq 0$ for $t \in [a, b]$. Analogously one proves that $F^{(\nu)}(\mathbf{x}; a)\neq 0$ and $F^{(\mu)}(\mathbf{x}; b)\neq 0$. Then (3.26) implies (3.27) and (3.28).

Theorem 1. *Let the multiplicities $(\nu_k)_1^n$ satisfy the evenness condition (3.23). Then the coefficients A_j, B_j, $a_{k\lambda}$ of the optimal quadrature formula of the type $(\nu_1, \ldots, \nu_n; \mathfrak{B})$ for the class $W_q^r[a, b]$ $(1<q\leqq\infty)$ fulfil the relations*

$$a_{k, \nu_k-1}=0, \ a_{k\lambda}>0, \ \lambda=0, 2, \ldots, \nu_k-2 \ \textit{if} \ \nu_k \ \textit{is even};$$

$$a_{k\lambda}>0, \ \lambda=0, 2, \ldots, \nu_k-1 \ \textit{if} \ \nu_k \ \textit{is odd};$$

$$\left.\begin{array}{l} A_j>0, \ j=0, \ldots, \nu-1 \\ (-1)^j B_j>0, \ j=0, \ldots, \mu-1 \end{array}\right\} \ \textit{for} \ \mathfrak{B}=\mathrm{ZBC}.$$

Proof. Taking into account (3.25) and (3.12) we find

$$M(\mathbf{x}; x_k-0)=M(\mathbf{x}; x_k+0)>0 \quad \text{if } \nu_k=r \text{ and } r \text{ is even},$$

$$M(\mathbf{x}; x_k-0)=-M(\mathbf{x}; x_k+0)>0 \text{ if } \nu_k=r \text{ and } r \text{ is odd}.$$

This and (2.2) give

$$a_{k, \nu_k-1}=0 \text{ if } \nu_k=r \text{ and } r \text{ is even},$$

$$a_{k, \nu_k-1}>0 \text{ if } \nu_k=r \text{ and } r \text{ is odd}.$$

Now suppose that $\nu_k<r$. According to (3.11) $a_{k, \nu_k-1} F^{(\nu_k)}(\mathbf{x}; x_k)=0$. By virture of Corollary 1 this implies $a_{k, \nu_k-1}=0$. The other assertions of the theorem follow immediately from Lemmas 5, 3, 4 and formulas (2.2), (2.4). The proof is complete.

4. Continuity of $F(\mathbf{x}; t)$ with Respect to the Nodes x. Evidently every system of nodes

$$\mathbf{x}=\begin{pmatrix} x_1, \ldots, x_n \\ \nu_1, \ldots, \nu_n \end{pmatrix}$$

can be considered as a point $\mathbf{x}=\{(x_1, \nu_1), \ldots, (x_n, \nu_n)\}$ from $\mathbb{R}^N$, where $N=\nu_1+\cdots+\nu_n$ and (t, ϱ) stands for $t, \ldots, t$ (ϱ—times).

We have defined the monospline $M(\mathbf{x}; t)$ for every system of nodes $\mathbf{x}$ from the class $\Omega(\nu_1, \ldots, \nu_n)$ (with $1 \leq \nu_k \leq r$, $k=1, \ldots, n$) with respect to fixed boundary conditions $\mathfrak{B}$. Now we extend this definition for all points $\mathbf{x} \in \Omega_N = \{\mathbf{y} = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N : a \leq \tau_1 \leq \cdots \leq \tau_N \leq b\}$. Let $\mathbf{x} \in \Omega_N$. Suppose that

$$\mathbf{x} = \{(a, \varrho_0), (x_1, \varrho_1), \ldots, (x_m, \varrho_m), (b, \varrho_{m+1})\}, \tag{4.1}$$

where $\varrho_0 \geq 0$, $\varrho_{m+1} \geq 0$, $\varrho_k > 0$, $k=1, \ldots, m$ and $a < x_1 < \cdots < x_m < b$. Set $\mu_k = \min(r, \varrho_k)$, $k=1, \ldots, m$. With $\mathbf{x}$ we associate the nodes

$$[\mathbf{x}]_r = \begin{pmatrix} x_1, \ldots, x_m \\ \mu_1, \ldots, \mu_m \end{pmatrix}$$

and the boundary conditions $\mathfrak{B}(\mathbf{x})$ determined as follows: $\mathfrak{B}(\mathbf{x}) = \mathfrak{B}$ if $\mathfrak{B} = \mathrm{PBC}$;

$$\mathfrak{B}(\mathbf{x}) : \begin{cases} M^{(j)}(a) = 0, \; j = 0, \ldots, r - \nu - 1 - \varrho_0 \\ M^{(j)}(b) = 0, \; j = 0, \ldots, r - \mu - 1 - \varrho_{m+1} \end{cases}$$

for $\mathfrak{B} = \mathrm{ZBC}$, by convention $r - \nu - 1 - \varrho_0 < 0$ $(r - \mu - 1 - \varrho_{m+1} < 0)$ signifies $M(t)$ has free left (right) end. Henceforth, for every $\mathbf{x} \in \Omega_N$, $M(\mathbf{x}; t)$ will denote the monospline of degree r with knots $[\mathbf{x}]_r$ which has a minimum L_p-norm in $[a, b]$ among all monosplines from the class $\mathfrak{W}([\mathbf{x}]_r; \mathfrak{B}(\mathbf{x}))$. The determination of $R_q(\cdot)$ and $F(\cdot; t)$ for $\mathbf{x} \in \Omega_N$ is also given in an obvious way.

For every $\mathbf{y} = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N$ we set $\|\mathbf{y}\| = \max_{1 \leq k \leq N} |\tau_k|$.

Denote by $\bar{\Omega}(\nu_1, \ldots, \nu_n)$ the closure of $\Omega(\nu_1, \ldots, \nu_n)$.

Lemma 6. *Suppose that* $1 \leq p < \infty$. *Let* $\mathbf{x}$ *(say, given by* (4.1)) *belong to* $\bar{\Omega}(\nu_1, \ldots, \nu_n)$ *and let*

$$\mathbf{x}_i = \begin{pmatrix} x_{i1}, \ldots, x_{in} \\ \nu_1, \ldots, \nu_n \end{pmatrix}$$

be a sequence of nodes from $\Omega(\nu_1, \ldots, \nu_n)$ *such that* $\lim_{i \to \infty} \|\mathbf{x}_i - \mathbf{x}\| = 0$. *Then there exists a subsequence of* $\{M(\mathbf{x}_i; t)\}$ *which converges uniformly to* $M(\mathbf{x}; t)$ *on compact subsets of* $(a, b) \setminus \{\xi_1, \ldots, \xi_s\}$, *where* $\xi_1, \ldots, \xi_s$ *are the points of discontinuity of* $M(\mathbf{x}; t)$.

Proof. Let $\varepsilon = (1/3) \min(x_1 - a, b - x_m)$. Consider the monosplines $\widetilde{M}(\mathbf{x}_i; t)$ defined on $[a - \varepsilon, b + \varepsilon]$ as follows:

$$\widetilde{M}(\mathbf{x}_i; t) = \begin{cases} (t-a)^r / r! & \text{for} \quad t < a, \\ (t-b)^r / r! & \text{for} \quad t > b, \\ M(\mathbf{x}_i; t) & \text{for} \quad a \leq t \leq b \end{cases}$$

in the case $\mathfrak{B} = \mathrm{ZBC}$ and $\widetilde{M}(\mathbf{x}_i; t)$ coincides with the $(b-a)$-periodic extension of $M(\mathbf{x}_i; t)$ on $[a - \varepsilon, b + \varepsilon]$ if $\mathfrak{B} = \mathrm{PBC}$. As we saw before $\|M(\mathbf{x}_i; \cdot)\|_p = R_q(\mathbf{x}_i) < \text{const}$ for $i = 1, 2, \ldots$. Therefore the sequence $\{\|\widetilde{M}(\mathbf{x}_i; \cdot)\|_p\}$ is bounded too. Then it follows from [13, p. 69, Proposition 1] (see also [14-16]) that there exists a subsequence of $\{\mathbf{x}_i\}$ (denote it again by $\{\mathbf{x}_i\}$) such that

$\{\widetilde{M}(\mathbf{x}_i; t)\}$ converges uniformly on compact subsets of $(a-\varepsilon, b+\varepsilon)\setminus\{\xi_1, \ldots, \xi_s\}$ to a monospline $\widetilde{M}(t)$ with knots

$$\{(a, \min(r, \nu+\varrho_0)), (x_1, \mu_1), \ldots, (x_m, \mu_m), (b, \min(r, \mu+\varrho_{m+1})\}.$$

Here $\xi_1, \ldots, \xi_s$ denote the points of discontinuity of the monospline $\widetilde{M}(t)$. Clearly $\{\xi_1, \ldots, \xi_s\} \subset \{a, x_1, \ldots, x_m, b\}$. Let $M(t)$ be the monospline with knots $x_1, \ldots, x_m$ that equals $\widetilde{M}(t)$ on (a, b). We claim that $M(t)$ satisfies the boundary conditions $\mathfrak{B}(\mathbf{x})$. The assertion is evident for $\mathfrak{B}=\mathrm{PBC}$ or $\mathfrak{B}=\mathrm{ZBC}$ with $\varrho_0=\varrho_{m+1}=0$ or $r-\nu-1<\varrho_0$, $r-\mu-1<\varrho_{m+1}$. Suppose that $\mathfrak{B}=\mathrm{ZBC}$ and $0<\varrho_0\leq r-\nu-1$. Since $\widetilde{M}(t)\in C^{(r-\nu-1-\varrho_0)}[a-\varepsilon, a+\varepsilon]$, we get $M^{(j)}(a)=\widetilde{M}^{(j)}(a-0)=0$ for $j=0, \ldots, r-\nu-1-\varrho_0$. Our claim is proved.

It remains to show that $M(t)$ has a minimal L_p-norm in $\mathfrak{M}([\mathbf{x}]_r; \mathfrak{B}(\mathbf{x}))$, i. e., that $M(t)=M(\mathbf{x}; t)$. Note that we do not need this fact in the sequel. Nevertheless we prefer to present the lemma in this more interesting form.

Let $\mathfrak{B}=\mathrm{ZBC}$. Suppose that $\|f^{(r)}\|_q\leq 1$ and $f(t_1)=\cdots=f(t_r)=0$ for some points $a\leq t_1\leq\cdots\leq t_r\leq b$. Then the known (see [17] or [18]) integral representation of the divided difference of f based on the points $t, t_1, \ldots, t_r$ implies

$$f(t)=(t-t_1)\ldots(t-t_r)\int_a^b u(\tau) f^{(r)}(\tau)\,d\tau,$$

where $u(\tau)\geq 0$ in (a, b) and $\int_a^b u(\tau)\,d\tau=1/r!$. Applying Hölder's inequality we get $|f(t)|<C$ for all $t\in[a, b]$, where C depends only on $[a, b]$. Then it is clear from Lemma 1 and stipulation (2.9) that $|F(\mathbf{x}_i; t)|<\mathrm{const}$ for $i=1, 2, \ldots$. This entails the uniform boundedness in $[a, b]$ of the polynomial sequence $\{P(\mathbf{x}_i; t)\}_1^\infty$ (see (3.7) for the definition of $P(\mathbf{x}; t)$). Hence, after going to subsequence if necessary, we may assume that $\{F^{(\lambda)}(\mathbf{x}_i; t)\}$ ($\lambda=0, \ldots, r$) converges uniformly on compact subsets of $(a, b)\setminus\{x_1, \ldots, x_m\}$ to the λ-th derivative of a function $F(t)$ which has a form

$$F(t)=P(t)+\sigma\frac{(-1)^r}{(r-1)!}\int_a^b (t-\tau)_+^{r-1}\,|M(\tau)|^{p-1}\operatorname{sgn} M(\tau)\,d\tau,$$

where $P\in\pi_{r-1}$ and $\sigma=(-1)^r(\int_a^b |M(t)|^p dt)^{-1/q}$. We show that

$$\begin{aligned}
&F^{(\lambda)}(x_k)=0,\quad k=1, \ldots, m,\ \lambda=0, \ldots, \mu_{k-1}\\
(4.2)\qquad &F^{(j)}(a)=0,\quad j=0, \ldots, \min(r-1, \nu-1+\varrho_0)\\
&F^{(j)}(b)=0,\quad j=0, \ldots, \min(r-1, \mu-1+\varrho_{m+1}).
\end{aligned}$$

Indeed, for each $h>0$ there is an integer $i(h)>0$ such that $F(\mathbf{x}_i; t)$ has μ_k zeros at least in $J_k(h) := [x_k-h, x_k+h]$ if $i\geq i(h)$. Then $F^{(\lambda)}(\mathbf{x}_i; t)$ ($\lambda=0, \ldots, \mu_k-1$) has at least one zero t_λ in $J_k(h)$. Evidently

$$|F^{(\lambda)}(\mathbf{x}_i; t)|=\left|\int_{t_\lambda}^{t} F^{(\lambda+1)}(\mathbf{x}_i; \tau)\,d\tau\right|\leq 2h\max\{|F^{(\lambda+1)}(\mathbf{x}_i; \tau)| : \tau\in J_k(h)\}$$

for every $t\in J_k(h)$ and $\lambda=0, \ldots, \min(r-2, \mu_k-1)$. Further, making use of Hölder's inequality we get $|F^{(r-1)}(\mathbf{x}_i; t)|=|\int_{t_{r-1}}^t F^{(r)}(\mathbf{x}_i; \tau)\,d\tau|\leq(2h)^{1/p}$ for $t\in J_k(h)$ if $\mu_k=r$. These inequalities imply $\max\{|F^{(\lambda)}(\mathbf{x}_i; t)| : t\in J_k(h)\}$ $(2h)^{\mu_k-1-\lambda+1/p}\cdot c$, for $\lambda=0, \ldots, \mu_k-1$, $c=\mathrm{const}$. Since $F_{(\lambda)}(\mathbf{x}_i; \mathbf{x}_k+h)$ tends to

$F^{(\lambda)}(x_k+h)$, we get

$$|F^{(\lambda)}(x_k+h)|\leq 2(2h)^{\mu_k-1-\lambda+1/p}.c$$

for every number $h>0$. Hence $F^{(\lambda)}(x_k)=0$ for $\lambda=0,\ldots,\mu_k-1$. In the same fashion one proves the rest of (4.2). But, as we showed in the proof of Lemma 1, the conditions (4.2) are equivalent to the classical necessary and sufficient (see [19], p. 213) conditions for $M(t)$ to have a minimal L_p-norm in the set $\mathfrak{M}([\mathbf{x}]_r;\mathfrak{B}(\mathbf{x}))$. Therefore $M(t)=M(\mathbf{x};t)$. The proof is entirely similar in the case $\mathfrak{B}=$PBC. The lemma is proved.

Corollary 2. *Under the same assumptions as in Lemma 6 the following statement is valid. There exists a subsequence of* $\{\mathbf{x}_i\}$, *which we denote again by* $\{\mathbf{x}_i\}$, *such that* $\lim_{i\to\infty}\|F^{(j)}(\mathbf{x}_i;\cdot)-F^{(j)}(\mathbf{x};\cdot)\|_{C[a,p]}=0$ *for* $j=0,\ldots,r-1$.

Indeed, since $|F^{(r)}(\mathbf{x}_i;\cdot)|$ is bounded and $\{F^{(r)}(\mathbf{x}_i;t)\}$ converges uniformly on compact subsets of $(a,b)\setminus\{x_1,\ldots,x_m\}$ to $F^{(r)}(\mathbf{x};t)$, we conclude that $\lim_{i\to\infty}\|F^{(r)}(\mathbf{x}_i;\cdot)-F^{(r)}(\mathbf{x};\cdot)\|_1=0$. This together with the convergence of $\{F^{(\lambda)}(\mathbf{x}_i;t)\}$ $(\lambda=0,\ldots,r-1)$ on compact subsets of $(a,b)\setminus\{x_1,\ldots,x_m\}$ proves our assertion.

5. Existence. The following auxiliary lemma is an improvement of a known result due to M. Powell [1].

Lemma 7. *Let* $M(\mathbf{x};t)$ *be a monospline of least* L_p *deviation in* $[a,b]$ $(1\leq p<\infty)$ *of the type* $(\nu_1,\ldots,\nu_n;\mathfrak{B})$. *Suppose that* $M(\mathbf{x};t)$ *is discontinuous at* x_k. *Then, for every pair of positive integer numbers* m_1, m_2 *satisfying* $m_1+m_2=r$ *there exists* $\varepsilon>0$ *such that* $\|M(\mathbf{x}_\varepsilon;\cdot)\|_p<\|M(\mathbf{x};\cdot)\|_p$ *where*

$$\mathbf{x}_\varepsilon=\begin{pmatrix}x_1,\ldots,x_{k-1}, & x_k-\varepsilon, & x_k, & x_{k+1},\ldots,x_n\\ \nu_1,\ldots,\nu_{k-1}, & m_1, & m_2, & \nu_{k+1},\ldots,\nu_n\end{pmatrix}.$$

Proof. Clearly $\nu_k=r$ and r is odd, since $M(\mathbf{x};t)$ is discontinuous at x_k. Then, according to Lemma 2, $M(\mathbf{x};x_k-0)=-M(\mathbf{x};x_k+0)\neq 0$. Without loss of generality we may assume that $M(\mathbf{x};x_k+0)=h>0$. We construct a monospline $\widetilde{M}(t)$ defined by the condition that it is to have knots of multiplicities m_1, m_2 respectively at $x_k-\varepsilon$, x_k and such that $\widetilde{M}(t)=M(\mathbf{x};t)$ for $t\notin(x_k-\varepsilon,x_k)$. Therefore, for $x_k-\varepsilon<t<x_k$, $\widetilde{M}(t)=M(\mathbf{x};t)+Q(t)$ where $Q(t)$ is the unique polynomial from π_{r-1} which satisfies the interpolation conditions

$$Q^{(j)}(x_k-\varepsilon)=0,\ j=0,\ldots,r-m_1-1$$

$$Q^{(j)}(x_k)=M^{(j)}(\mathbf{x};x_k+0)-M^{(j)}(\mathbf{x};x_k-0),\ j=0,\ldots,r-m_2-1.$$

Let us put for convenience $l=r-m_1-1$, $m=r-m_2-1$. Using Hermite interpolation formula we find

$$Q(t)=\{(t-(x_k-\varepsilon))/\varepsilon\}^{l+1}\sum_{j=0}^{m}(-1)^j\frac{(x_k-t)^j}{j!}Q^{(j)}(x_n)\sum_{i=0}^{m-j}\binom{l+i}{i}((x_k-t)/\varepsilon)^i.$$

In order to show that $\widetilde{M}(t)$ deviates in L_p less than $M(\mathbf{x};t)$ we just have to compare the deviation over $[x_k-\varepsilon,x_k]$. We have $\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_{x_k-\varepsilon}^{x_k}|M(\mathbf{x};t)|^p dt=h^p$. On the other hand,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_{x_k-\varepsilon}^{x_k}|\widetilde{M}(t)|^p\,dt=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{x_k-\varepsilon}^{x_k}|M(\mathbf{x};t)+Q(t)|^p\,dt=h^p\int_0^1|-1+H(t)|^p\,dt,$$

where $H(t)=2t^{l+1}\sum_{i=0}^{m}\binom{l+i}{i}(1-t)^i$. It is easy to verify that $H^{(j)}(0)=0$, $j=0,\dots,l$, $H(1)=2$, $H^{(j)}(1)=0$, $j=1,\dots,m$. Then Rolle's theorem yields that $H'(t)>0$ in $(0,1)$. Therefore $H(t)$ is strictly monotone in $(0,1)$. Thus $|-1+H(t)|<1$ for $t\in(0,1)$. This inequality implies $\int_0^1|-1+H(t)|^p dt<1$. Now it is clear that $\|\widetilde{M}(\cdot)\|_p<\|M(\mathbf{x};\cdot)\|_p$ for sufficiently small ε and consequently $\|M(\mathbf{x}_\varepsilon;\cdot)\|_p<\|M(\mathbf{x};\cdot)\|_p$. The proof is complete.

In the same way we prove

Lemma 8. *Suppose that the multiplicities $(\nu_k)_1^n$ satisfy the evenness condition* (3.24). *Let $M(\mathbf{x};t)$ be a monospline of least L_p deviation in $[a,b]$ $(1\le p<\infty)$ of the type $(\nu_1,\dots,\nu_n;\mathfrak{B})$, where $\mathfrak{B}=$ ZBC with $\nu=r$, i. e., with free left end. Then, for every pair of positive integer numbers m_1, m_2 satisfying $m_1+m_2=r$, there exists $\varepsilon>0$ such that $\|M_\varepsilon(\cdot)\|_p<\|M(\mathbf{x};\cdot)\|_p$, where $M_\varepsilon(t)$ is the monospline of least L_p deviation in $\mathfrak{M}(\mathbf{x}_\varepsilon;\widetilde{\mathfrak{B}})$. Here*

$$\mathbf{x}_\varepsilon=\begin{pmatrix}a+\varepsilon, & x_1,\dots,x_n\\ r-m_2, & \nu_1,\dots,\nu_n\end{pmatrix},$$

and

$$\widetilde{\mathfrak{B}}:\begin{cases}M^{(j)}(a)=0, & j=0,\dots,m_1-1\\ M^{(j)}(b)=0, & j=0,\dots,r-\mu-1.\end{cases}$$

Proof. By virtue of Lemma 5, $M(\mathbf{x};t)$ has a maximal number of zeros. Then, according to (3.21), $M(a)\ne 0$. Without loss of generality we assume that $M(a)=h>0$. Next we construct a monospline $\widetilde{M}(t)$ with knot at $a+\varepsilon$ of multiplicity $r-m_2$, which satisfies the boundary conditions $\widetilde{\mathfrak{B}}$ and equals $M(t)$ in $(a+\varepsilon,b)$. Therefore $\widetilde{M}(t)=M(\mathbf{x};t)-P(t)$ for $t\in(a,a+\varepsilon)$, where $P\in\pi_{r-1}$ and

$$P^{(j)}(a)=M^{(j)}(\mathbf{x};a),\ j=0,\dots,m_1-1,\quad P^{(j)}(a+\varepsilon)=0,\ j=0,\dots,m_2-1.$$

Then, as in the proof of the previous lemma, we show that

$$\int_a^{a+\varepsilon}|\widetilde{M}(t)|^p dt<\int_a^{a+\varepsilon}|M(\mathbf{x};t)|^p\,dt$$

for a sufficiently small $\varepsilon>0$. The lemma is proved.

We need the following simple fact proved in [20].

Lemma 9. *Let h be an arbitrary positive number and let $f\in C^r[\tau-h,\tau+h]$. Suppose that f has exactly r zeros in $[\tau-h,\tau+h]$ and $f(\tau-h)=f(\tau+h)=0$. If $0<m<f^{(r)}(t)<M$ for all $t\in[\tau-h,\tau+h]$, then $\beta-\alpha>\sqrt{m}\,(Mr!\,2^{r-2})^{-1/2}.h$, where α, β are the zeros of $f^{(r-2)}(t)$ in $[\tau-h,\tau+h]$.*

Corollary 3. *Under the same assumptions as in Lemma 9 the following statement is valid. There exists $C>0$ independent of h such that* $\min(\xi-a,b-\xi)>Ch$, *where ξ is the zero of $f^{(r-1)}(t)$ in $(\tau-h,\tau+h)$.*

Proof. Indeed, $a\le\alpha<\xi<\beta\le b$ and $m|t-\xi|\le|f^{(r-1)}(t)|\le M|t-\xi|$ for $t\in[\tau-h,\tau+h]$. Since $f^{(r-2)}(\alpha)=f^{(r-2)}(\beta)=0$, we have

$$\frac{M}{2}(\xi-\alpha)^2 \geqq \left|\int_\alpha^\xi f^{(r-1)}(t)\,dt\right| = \left|\int_\xi^\beta f^{(r-1)}(t)dt\right| \geqq \frac{m}{2}(\beta-\xi)^2.$$

Analogously $M(\beta-\xi)^2 \geqq m(\xi-\alpha)^2$. These inequalities together with the obvious relation $\max(\xi-\alpha, \beta-\xi) \geqq (\beta-\alpha)/2$ imply

$$\min(\beta-\xi, \xi-\alpha) \geqq (m/M)^{1/2}(\beta-\alpha)/2.$$

Then the assertion follows from Lemma 9.

Theorem 2. *Suppose that the boundary conditions $\mathfrak{B}$ are of the kind ZBC or PBC. Let the multiplicities $(\nu_k)_1^n$ be fixed satisfying the inequalities $1 \leqq \nu_k \leqq r$, $k=1, \ldots, n$ and postulate* (2.9). *Let $1 \leqq p < \infty$. Then there exists a monospline $M(t)$ of degree r of least L_p deviation of the type $(\nu_1, \ldots, \nu_n; \mathfrak{B})$. Moreover, the knots $(x_k)_1^n$ and the coefficients $(a_{k\lambda})$ of $M(t)$ satisfy the relations*

$$a < x_1 < \cdots < x_n < b,$$

(5.1) $$a_{k,\nu_k-1}=0,\ a_{k\lambda}>0,\ \lambda=0,2,\ldots,\nu_k-2, \text{ if } \nu_k \text{ is even}$$

$$a_{k\lambda}>0,\ \lambda=0,2,\ldots,\nu_k-1, \text{ if } \nu_k \text{ is odd}.$$

Proof. We first assume that the multiplicities $(\nu_k)_1^n$ satisfy the evenness condition (3.24).

Let $\{\mathbf{x}_i\}_{i=1}^\infty$ be a minimizing sequence for the extremal problem

(5.2) $$\inf\{\|M(\mathbf{z};\cdot)\|_p : \mathbf{z} \in \Omega(\nu_1, \ldots, \nu_n)\}.$$

After going to a subsequence if necessary, we may assume that

(5.3) $$\lim_{i\to\infty} \|\mathbf{x}_i - \mathbf{x}\| = 0$$

for some $\mathbf{x}=((a,\varrho_0), (x_1,\varrho_1), \ldots, (x_m,\varrho_m), (b,\varrho_{m+1}))$, where $0 \leqq m \leqq n$. Then, by virtue of Lemma 6, $\{\|M(\mathbf{x}_i;\cdot)\|_p\}$ converges to $\|M(\mathbf{x};\cdot)\|_p$. We have to show that $\mathbf{x} \in \Omega(\nu_1, \ldots, \nu_n)$. Evidently, it suffices to prove that $a < x_1$, $x_m < b$ and $m=n$. We begin by showing that the knots of the minimizing sequence do not stream to the endpoints a, b. Note that in the periodic case, i. e., when $\mathfrak{B}=$PBC, every monospline $M(t)$ of the form $M(\mathbf{x}; t+\text{const})$ is extremal for (5.2). Thus, translating $\mathbf{x}$ if necessary we can find a system of interior extremal knots. Now suppose that $\mathfrak{B}=$ZBC. Let the sequences $\{x_{i1}\}, \ldots, \{x_{is}\}$ $(s \geqq 1)$ tend to a when $i \to \infty$. Then $\varrho_0 = \nu_1 + \cdots + \nu_s > 0$. It follows from Lemma 8 that $\varrho_0 < r-\nu$. Since $M(\mathbf{x}; t)$ satisfies $\mathfrak{B}(\mathbf{x})$, we have

(5.4) $$F^{(\lambda)}(\mathbf{x}; a)=0,\ \lambda=0, \ldots, \nu+\varrho_0-1.$$

Put $h_0=(x_1-a)/3$. Let h be an arbitrary number from the interval $[0, h_0]$. Define $\mathbf{x}_h=((a,\varrho_0-\nu_s), (a+h,\nu_s), (x_1,\varrho_1), \ldots, (x_m,\varrho_m), (b,\varrho_{m+1}))$. Evidently $\mathbf{x}_h \in \bar{\Omega}(\nu_1, \ldots, \nu_n)$. Therefore

(5.5) $$\|M(\mathbf{x};\cdot)\|_p \leqq \|M(x_h;\cdot)\|_p.$$

Set $\mu_k=\min(\varrho_k, r)$, $k=1, \ldots, n$, $\mu_0=\nu+\varrho_0$, $\mu_{m+1}=\min(\varrho_{m+1}+\mu, r)$. Let

(5.6) $$I(f) \approx \sum_{j=0}^{\mu_0-1} A_j f^{(j)}(a) + \sum_{j=0}^{\mu_{m+1}-1} B_j f^{(j)}(b) + \sum_{k=1}^{m} \sum_{\lambda=0}^{\mu_k-1} a_{k\lambda} f^{(\lambda)}(x_k) = : Q(f)$$

be the best quadrature formula in $W_q^r[a, b]$ among all formulae of the same type. Denote by $R_q(\mathbf{x})$ the error of (5.6). Obviously

$$(5.7)\qquad I(F(\mathbf{x}_h;\cdot))-Q(F(\mathbf{x}_h;\cdot))\leq R_q(\mathbf{x})=\|M(\mathbf{x};\cdot)\|_p.$$

Making use of Lemma 1 and the relation between quadrature formulae and monosplines, we get $Q(F(\mathbf{x}_h;\cdot))=\sum_{j=0}^{\mu_0-1}A_jF^{(j)}(\mathbf{x}_h;a)=:\varepsilon(h)$, $I(F(\mathbf{x}_h;\cdot))=\|M(\mathbf{x}_h;\cdot)\|_p$. Then (5.7) gives

$$(5.8)\qquad \|M(\mathbf{x}_h;\cdot)\|_p-\varepsilon(h)\leq\|M(\mathbf{x};\cdot)\|_p.$$

We next obtain a contradiction with (5.5) by showing that $\varepsilon(h)<0$ for sufficiently small h. Indeed, the knots $\mathbf{x}$ are optimal of the type $(\mu_1,\ldots,\mu_m$ $\mathfrak{B}(\mathbf{x}))$. Then, by virtue of Theorem 1,

$$(5.9)\qquad A_{\mu_0-1}>0.$$

Further, by Corollary 1, $F^{(\mu_0)}(\mathbf{x};a)>0$. Then it follows from the continuity of $F^{(\mu_0)}(\mathbf{x};t)$ in $[a,a+h]$ and from Corollary 2 that there exist constants $C_1>0$, $C_2>0$ and a positive number $h_1\leq h_0$ such that

$$(5.10)\qquad C_2\geq F^{(\mu_0)}(\mathbf{x}_h;t)\geq C_1>0,\ t\in[a,a+h_1]$$

for every $h\in[0,h_1]$. We showed in Lemma 1 that $F(\mathbf{x}_h;t)$ has at least μ_0 zeros in $[a,a+h]$. Now making use of Rolle's theorem and (5.10) we conclude that $F(\mathbf{x}_h;t)$ has precisely μ_0 zeros in $[a,a+h]$. Let ξ be the unique zero of $F^{(\mu_0-1)}(\mathbf{x}_h;t)$ in $[a,a+h]$. The bounds (5.10) give $-F^{(\mu_0-1)}(\mathbf{x}_h;a)>C_1(\xi-a)$. On the other hand, in view of Corollary 3, there exists a constant $C_3>0$ such that $\xi-a\geq C_3$ h. Therefore

$$(5.11)\qquad -F^{(\mu_0-1)}(\mathbf{x}_h;a)\geq C_1.C_3.h.$$

Finally, using (5.10) and the property of $F^{(j)}(\mathbf{x}_h;t)\,(j=0,\ldots,\mu_0)$ to vanish $\mu_0-\lambda$ times in $[a,a+h]$ we conclude that there exists a constant $C>0$ such that

$$(5.12)\qquad |F^{(j)}(\mathbf{x}_h;a)|\leq Ch^{\mu_0-j},\ j=0,\ldots,\mu_0-1,$$

Thus, on the basis of (5.9), (5.11) and (5.12),

$$\varepsilon(h)=\sum_{j=0}^{\mu_0-2}A_jF^{(j)}(\mathbf{x}_h;a)-A_{\mu_0-1}(-F^{(\mu_0-1)}(\mathbf{x}_h;a))$$

$$<-A_{\mu_0-1}C_1C_3h+C\sum_{j=0}^{\mu_0-2}|A_j|h^{\mu_0-j}.$$

Now it is seen that $\varepsilon(h)<0$ for sufficiently small h and (5.8) leads to the inequality $\|M(\mathbf{x}_h;\cdot)\|_p<\|M(\mathbf{x};\cdot)\|_p$, which contradicts (5.5). Therefore $\varrho_0=0$. Similarly one can show that $\varrho_{m+1}=0$.

We next prove that $m=n$. Let us assume that $m<n$. Then there exists a knot x_k such that $\varrho_k=\nu_k+\cdots+\nu_{k+s}$ where $1\leq s$. It is easily seen that $\max(\nu_k,\ldots,\nu_{k+s})<r$. Indeed, suppose that $\max(\nu_k,\ldots,\nu_{k+s})=\nu_k=r$. Construct the knots

$$\mathbf{z}=((x_1,\varrho_1),\ldots (x_{k-1},\varrho_{k-1}),(x_k,\nu_k),(x_{k+1},\varrho_{k+1}),\ldots,$$
$$(x_m,\varrho_m),(t_0,\nu_{k+1}+\cdots+\nu_{k+s})),$$

where $t_0\in(x_m,b)$. Evidently $\mathbf{z}$ can be represented as a point of condensation of a sequence $\{\mathbf{z}_i\}_{i=1}^{\infty}$ of knots $\mathbf{z}_i\in\Omega(\nu_1,\ldots,\nu_n)$. Then, in view of the definition of $\mathbf{x}$, we have

$$R_q(\mathbf{x})=\|M(\mathbf{x};\cdot)\|_p\leq\|M(\mathbf{z};\cdot)\|_p=R_q(\mathbf{z}). \tag{5.13}$$

On the other hand, by virtue of Lemma 1,

$$R_q(\mathbf{z})=\int_a^b F(\mathbf{z};t)\,dt\leq\max_{f\in W_0([\mathbf{x}]_r;\,\mathfrak{B}^*)} I(f)=\int_a^b F(\mathbf{x};t)\,dt=R_q(\mathbf{x}). \tag{5.14}$$

Moreover, $F(\mathbf{x};t)$ is the unique function from $W_0([\mathbf{x}]_r;\mathfrak{B}^*)$ for which the maximal error $R_q(\mathbf{x})$ is achieved. Since $F(\mathbf{z};t_0)=0$ and $F(\mathbf{x};t_0)\neq0$, we have $F(\mathbf{z};\cdot)\neq F(\mathbf{x};\cdot)$. Hence the inequality in (5.14) is strict. This contradicts (5.13). Our claim is proved.

Recall that the knots $[\mathbf{x}]_r$ are optimal of the type $(\mu_1,\ldots;\mathfrak{B})$. This and Lemma 7 imply $\varrho_k<r$ for odd r. Indeed, if $\varrho_k\geq r$ and r is an odd number, then $M(\mathbf{x};t)$ must be discontinuous at x_k. Then Lemma 7 leads to contradiction with the optimality of the knots $[\mathbf{x}]_r$. Therefore $\mu_k=\min(r,\varrho_k)\leq r$ and μ_k is even. Corollary 1 (for $\mu_k<r$) and (3.13), (3.26) (for $\mu_k=r$) yield .

$$F^{(\mu_k)}(\mathbf{x};x_k)>0. \tag{5.15}$$

Next we employ the same idea as in the proof of the equality $\varrho_0=0$. We use the same notations for the analogous quantities and notions. With any h, $0\leq h\leq h_0:=\min\{x_k-x_{k-1}:1\leq k\leq m+1\}$ $(x_0=a,\ x_{m+1}=b)$ we associate the nodes

$$\mathbf{x}_h=\begin{pmatrix}x_1,\ldots,x_{k-1},\ \tau-h,\ \tau+h,\ x_{k+1},\ldots,x_m\\ \mu_1,\ldots,\mu_{k-1},\ \nu_k,\ \mu_k-\nu_k,\ \mu_{k+1},\ldots,\mu_m\end{pmatrix},$$

where τ is a point from $[x_k-h,\ x_k+h]$ which we shall collocate later. It follows from the optimality of the knots $\mathbf{x}$ that

$$R_q(\mathbf{x})=\|M(\mathbf{x};\cdot)\|_p\leq\|M(\mathbf{x}_h;\cdot)\|_p=R_q(\mathbf{x}_h). \tag{5.16}$$

Let (A_j), (B_j), $(a_{k\lambda})$ be the coefficients of the best quadrature formula

$$I(f)\approx Q(f) \tag{5.17}$$

based on the nodes $\mathbf{x}$. Applying (5.17) for the function $F(\mathbf{x}_h;t)$, we get

$$I(F(\mathbf{x}_h;\cdot))-Q(F(\mathbf{x}_h;\cdot))\leq R_q(\mathbf{x}). \tag{5.18}$$

Denote as above the term $Q(F(\mathbf{x}_h;\cdot))$ by $\varepsilon(h)$. From Lemma 1 $\varepsilon(h)=\sum_{\lambda=0}^{\mu_k-1}a_{k\lambda}F^{(\lambda)}(\mathbf{x}_h;x_k)$. Our aim now is to prove that $\varepsilon(h)<0$ for sufficiently small h. First we note that, by virtue of Corollary 2,

$$\lim_{h\to0}\|F^{(\mu_k)}(\mathbf{x}_h;\cdot)-F^{(\mu_k)}(\mathbf{x};\cdot)\|_{C[x_k-h_0,\,x_k+h_0]}=0 \tag{5.19}$$

for $\mu_k<r$. But $F^{(\mu_k)}(\mathbf{x};t)$ is continuous at x_k, since μ_k is an even number. Then Lemma 6 implies that (5.19) remains true for $\mu_k=r$. Taking into account (5.15), we conclude that there exist constants $C_1>0$, $C_2>0$ and a number $h_1\in(0,h_0)$ such that

$$C_1\leq F^{(\mu_k)}(\mathbf{x}_h;t)\leq C_2,\quad t\in[x_k-2h_1,\ x_k+2h_1] \tag{5.20}$$

for every $h\in[0,h_1]$. Since $F^{(\lambda)}(\mathbf{x}_h;t)\ (\lambda=0,\ldots,\mu_k-1)$ has precisely $\mu_k-\lambda$ zeros in $[x_k-2h_1,\ x_k+2h_1]$, it is not difficult to verify that

$$|F^{(\lambda)}(\mathbf{x}_h;\ x_k)|\leq Ch^{\mu_k-\lambda},\quad \lambda=0,\ldots,\mu_k \tag{5.21}$$

for all $h\in[0,h_1]$, where C does not depend on h.

New suppose that h is fixed in the interval $[0,h_1]$. We choose the parameter $\tau=\tau(h)$ to satisfy the requirement

$$F^{(\mu_k-1)}(\mathbf{x}_h;\ x_k)=0. \tag{5.22}$$

This may be done. Indeed, let $\xi(\tau)$ denote the unique zero of $F^{(\mu_k-1)}(\mathbf{x}_h;\ t)$ in $[x_k-2h_1,\ x_k+2h_1]$. Obviously $\xi(\tau)$ is a continuous function of τ for fixed h. In addition $\xi(x_k-h)<x_k$ and $\xi(x_k+h)>x_k$. Therefore, there exists $\tau\in[x_k-h,\ x_k+h]$ for which $\xi(\tau)=x_k$. In what follows we assume that the point τ is chosen in this way. It is easy to derive from (5.20) and (5.22) that $|F^{(\mu_k-2)}(\mathbf{x}_h;\ x_k)|\geq C_1(\beta-\alpha)^2/4$ for every $h\in[0,h_1]$, where α, β are the zeros of $F^{(\mu_k-2)}(\mathbf{x}_h;\ t)$ in $[x_k-2h_1,\ x_k+2h_1]$. Then, according to Lemma 9, there exists a constant $C_3>0$ such that

$$|F^{(\mu_k-2)}(\mathbf{x}_h;\ x_k)|\leq C_3h^2 \tag{5.23}$$

for all $h\in[0,h_1]$. Finally note that

$$F^{(\mu_k-2)}(\mathbf{x}_h;\ x_k)<0, \tag{5.24}$$

since $\alpha<x_k<\beta$ and $F^{(\mu_k)}(\mathbf{x}_h;t)>0$ in (α,β). Now we are ready to show that $\varepsilon(h)<0$. Indeed, by virtue of Theorem 1, $a_{k,\mu_k-1}=0$, $a_{k,\mu_k-2}>0$. This and (5.21)-(5.24) give

$$\varepsilon(h)\leq\sum_{\lambda=0}^{\mu_k-3}|a_{k\lambda}|\,|F^{(\lambda)}(\mathbf{x}_h;\ x_k)|-a_{k,\mu_k-2}(-F^{(\mu_k-2)}(\mathbf{x}_h;\ x_k))$$

$$\leq C\sum_{\lambda=0}^{\mu_k-3}|a_{k\lambda}|\,h^{\mu_k-\lambda}-a_{k,\mu_k-2}\,C_3h^2=-a_{k,\mu_k-2}\,C_3h^2+O(h^3).$$

Therefore $\varepsilon(h)<0$ for sufficiently small h. Then (5.18) contradicts (5.16). So $m=n$ and consequently $\mu_k=\nu_k$, $k=1,\ldots,n$. The existence of optimal knots is proved in the case of multiplicities satisfying the evenness condition.

Consider the case of arbitrary multiplicities $(\nu_k)_1^n$. Denote $\mu_k=\min\{2[(\nu_k+1)/2],r\}$, $k=1,\ldots,n$, where $[\cdot]$ is the greatest integer function. Evidently $(\mu_k)_1^n$ satisfy (3.24). Therefore there exist knots $\mathbf{x}$ which are optimal of the type $(\mu_1,\ldots,\mu_n;\mathfrak{P})$. But, according to Theorem 1, $a_{k,\mu_k-1}=0$ if $\mu_k>\nu_k$. Thus the knots $\mathbf{x}$ are optimal of the type $(\nu_1,\ldots,\nu_n;\mathfrak{P})$ too.

The last assertion (5.1) of the theorem follows immediately form Theorem 1. The proof is complete.

Using the relation between quadrature formulae and monosplines we formulate the main result in another form.

Theorem 3. *Let $1<q\leq\infty$ and let $(\nu_k)_1^n$ be arbitrarily fixed integer numbers satisfying the inequalities $1\leq\nu_k\leq r$, $k=1,\ldots,n$, $\nu_1+\cdots+\nu_k\geq r$. Then there exists an optimal quadrature formula of the form*

$$I(f)\approx\sum_{j=0}^{\nu-1}A_jf^{(j)}(a)+\sum_{j=0}^{\mu-1}B_jf^{(j)}(b)+\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}f^{(\lambda)}(x_k)$$

in the class $W_q^r[a,b]$. Moreover, the nodes and the coefficients of this quadrature formula satisfy the relations

$$a<x_1<\cdots<x_n<b$$

$$(5.25)\qquad \begin{cases} a_{k,\nu_k-1}=0,\ a_{k\lambda}>0,\ \lambda=0,2,\ldots,\nu_k-2, & \text{if } \nu_k \text{ is even,}\\ a_{k\lambda}>0,\ \lambda=0,2,\ldots,\nu_k-1 & \text{if } \nu_k \text{ is odd}\end{cases}$$

$$A_j>0,\ j=0,\ldots,\nu-1,$$

$$(-1)^{r-j}B_j>0,\ j=0,\ldots,\mu-1.$$

Theorem 4. *Let $1<q\leq\infty$ and let $(\nu_k)_1^n$ be arbitrary fixed integer numbers satisfying the inequalities $1\leq\nu_k\leq r$, $k=1,\ldots,n$, $\nu_1+\cdots+\nu_n\geq 1$. Then there exists an optimal quadrature formula of the form $I(f)\approx\Sigma_{k=1}^n\Sigma_{\lambda=0}^{\nu_k-1}a_{k\lambda}f^{(\lambda)}(x_k)$ in the class* $\widetilde{W}_q^r[a,b]:=\{f\in W_q^r[a,b]:f$ is $(b-a)$-periodic$\}$. *Moreover, the coefficients of the optimal quadrature formula satisfy* (5.25).

This statement was given in [21] without proof. A special case of Theorem 3 ($\nu=0$, $\mu=0$) was announced in [22].

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