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A PROBLEM OF STEČKIN ON TRIGONOMETRIC APPROXIMATION

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Summary. In reply to a question of S. B. Stečkin an example is given of a periodic function g, which is not infinitely differentiable, yet $\lim_{n\to\infty} \omega_r(g, 1/n)/E_n(g)=\infty$ for every natural number r. Here $E_n(g)$ denotes the order of approximation of the function g by trigonometric polynomials of degree n. Furthermore, a theorem is proved about comparison between generalized moduli of continuity $\omega_\sigma(f, t)$ (σ a measure in R^n), and this theorem is shown to imply the existence of a function g with the first mentioned properties.

1. The following inverse theorem for trigonometric approximation was proved by S. B. Stečkin in 1951 [4]. For any positive integer q there is a constant C_q, depending only on q, such that

$$\omega_q(f;\ 1/n)\leq C_q n^{-q}\sum_{k=1}^{n} k^{q-1}E_k(f),\ n=1, 2, \ldots, \tag{1}$$

for 2π-periodic and contionuous functions f. Here $\omega_q(f, t)$ is the q-th order modulus of continuity, and $E_n(f)=\inf\{\|f-p\|: p\in T_n\}$, where T_n is the set of trigonometric polynomials of degree n, $\|\cdot\|$ is the supremum norm. In his lecture to the conference on Constructive Function Theory in Blagoevgrad in June 1977 Professor Stečkin posed the following problem. Given a continuous and 2π-periodic function f, which is *not* infinitely differentiable, does there always exist a positive integer r and a constant C such that

$$\omega_r(f;\ 1/n)\leq CE_n(f),\ n=1, 2, \ldots. \tag{2}$$

If the answer to this question were "yes", one would have a strengthening of (1) for functions that are not infinitely differentiable. For the well-known inequality

$$\omega_q(f;\ t)\leq C_{q,r}t^r\int_t^1 \omega_r(f, u)u^{-r-1}du,\ 0<t<1/2,\ q<r,$$

implies

$$\omega_q(f;\ 1/n)\leq C_{q,r}n^{-q}\sum_{k=1}^{n} k^{q-1}\omega_r(f, 1/k),\ q<r,$$

which together with (2) gives (1).

The purpose of this is *first* to give a negative answer to this question through the construction of a specific function, and *second* to prove a general theorem on generalized moduli of continuity, from which the solution to Stečkin's problem can be deduced as a corollary.

2 Our negative answer to Stečkin's question can be formulated as follows.

Theorem 1. *There exists a continuous and periodic function g on R such that g is not continuously differentiable, but*

$$\overline{\lim_{n\to\infty}}\ \omega_q(g;\ 1/n)\big/E_n(f)=\infty,\ q=1,2,\dots \tag{3}$$

Proof. Set

$$g(x)=\sum_{j=1}^{\infty} m_j^{-1}\cos m_j x, \tag{4}$$

where $m_j=2^{j^2}$. Then g is not continuously differentiable, because the derivative of g in the sense of the theory of distribution must have the Fourier expansion $\Sigma_{j=1}^{\infty}(-\sin m_j x)$, and this series cannot be the Fourier series of a continuous function. Moreover, it is obvious that

$$E_{m_k}(g)\leq\sum_{j=k+1}^{\infty}1/m_j<2/m_{k+1}. \tag{5}$$

Denote the q-th order difference operator with step length t by Δ_t^q. Since

$$\Delta_t^q\cos ax=\operatorname{Re}\Delta_t^q e^{iax}=\operatorname{Re}\{e^{iax}(e^{iat}-1)^q\},$$

it is clear that $\|\Delta_t^q\cos ax\|=|e^{iat}-1|^q$, and hence

$$\|\Delta_t^q\cos ax\|\leq(at)^q,\ a>0,\ t>0, \tag{6}$$

and

$$\|\Delta_t^q\cos ax\|>(at/2)^q,\ 0<t\leq 1/a. \tag{7}$$

Taking $a=m_k$ and $t=t_k=1/m_k$ in (7), we get

$$\|\Delta_{t_k}^q\cos m_k x\|>2^{-q}. \tag{8}$$

From (6) we get similarly

$$\|\Delta_{t_k}^q\cos m_j x\|\leq(m_j/m_k)^q. \tag{9}$$

We also need the trivial estimate

$$\|\Delta_{t_k}^q\cos m_j x\|\leq 2^q. \tag{10}$$

Combining (8), (9), and (10), using (9) for $j<k$ and (10) for $j>k$, we get after a little computation $\|\Delta_{t_k}^q g\|\geq(1-1/4-1/4)/m_k2^q=1/(m_k2^{q+1})$ for $q\geq 2$ and $k\geq k_0$ (depending on q), and hence

$$\omega_q(g;\ t_k)=\omega_q(g;\ 1/m_k)\geq 2^{-q-1}/m_k. \tag{11}$$

Since $\omega_2(g;\ t)\leq 2\omega_1(g;\ t)$ an analogous estimate must hold also for $q=1$. Combining (5) and (11) we obtain (3).

3. We now want to fit Theorem 1 into the framework of the theory of generalized moduli of continuity. We first recall some definitions (see [3], chapter 9). Denote by $M(R^d)$, $d\geqq 1$, the set of complex-valued, bounded, regular measures on R^d. For $\sigma \in M(R^d)$ and f a complex-valued, uniformly continuous and bounded function on R^d the convolution $\sigma * f$ can be defined and is a uniformly continuous and bounded function. For $t>0$ the dilation $\sigma_{(t)}$ of σ is defined by $\widehat{\sigma_{(t)}}(\xi)=\widehat{\sigma}(t\xi)$, $\xi \in R^d$, where $\widehat{}$ denotes Fourier transform. The σ-modulus of continuity of f, $\omega_\sigma(f, t)$ is defined by

$$\omega_\sigma(f;\, t)=\sup\{\,\|\sigma_{(u)}*f\|: 0<u<t\},\ t>0.$$

The purpose of introducing those concepts has been to establish so-called comparison theorems, i. e. for a measures σ, τ to obtain an estimate of $\omega_\tau(f, t)$ in terms of $\omega_\sigma(f, t)$ as $t\to 0$ under appropriate assumptions on σ and τ. These assumptions are expressed in terms of the Fourier transforms $\widehat{\sigma}$ and $\widehat{\tau}$. Concerning applications of such results see [2] and [3]. As an example let us just mention here that Stečkin's inequality (1) is a corollary of a general comparison theorem of this kind (see [1], p. 109-110, or [2]).

Here we will give a *necessary* condition on the pair of Fourier transforms $\widehat{\sigma}$ and $\widehat{\tau}$ for the validity of a certain type of inequality between the coresponding moduli of continuity.

Theorem 2. *Let $\sigma, \tau \in M(R^d)$, $\widehat{\sigma}(0)=\widehat{\tau}(0)=0$, let $B>0$, and assume that for each uniformly continuous and bounded function f there exists a constant C and $t_0>0$ such that*

$$\omega_\tau(f;\, t)\leqq C\omega_\sigma(f;\, Bt),\ 0<t<t_0. \tag{12}$$

Then there exists a constant C_1 and a neighbourhood V of the origin in R^d, such that

$$|\widehat{\tau}(\xi)|\leqq C_1 \sup\{|\widehat{\sigma}(u\xi)|: 0<u\leqq B\},\ \xi\in V. \tag{13}$$

Note that if $u\to|\widehat{\sigma}(u\xi)|$ is increasing in $[0, B]$ for all $\xi\in V$, as is often the case in applications, then (13) is equivalent to

$$|\widehat{\tau}(\xi)|\leqq C_1|\widehat{\sigma}(B\xi)|,\ \xi\in V. \tag{14}$$

Proof of Theorem 2. Assume that (13) does not hold for any C_1. Then there exist $\xi_j\in R^d\setminus\{0\}$ tending to $0\in R^d$, such that

$$|\widehat{\tau}(\xi_j)|>j\sup\{|\widehat{\sigma}(\xi_j u)|: 0<u\leqq B\},\ j=1, 2, \ldots. \tag{15}$$

Choose a decreasing sequence of positive numbers a_j such that $\sum_{j=1}^\infty a_j\leqq 1$, and

$$\sum_{j=k+1}^{\infty} a_j< a_k|\widehat{\tau}(\xi_k)|/(3k\max(\sup|\widehat{\sigma}|,\ \sup|\widehat{\tau}|)),\ k=1, 2, \ldots. \tag{16}$$

We will set $f(x)=\sum_{j=1}^\infty a_j e^{i(x,\,\xi_j)/t_j}$, where $t_1>t_2>t_3>\ldots$ the decreasing sequence of positive numbers t_j will now be chosen inductively. Let $t_1=1$ and assume that $t_1, \ldots, t_{k-1}$ are chosen. Then we choose $t_k<t_{k-1}/2$ such that

$$|\widehat{\tau}(t_k\xi_j/t_j)|<a_k|\widehat{\tau}(\xi_k)|/3,\ j=1, \ldots, k-1, \tag{17}$$

and

$$(18)\qquad \sup_{0<u\leq B}|\widehat{\sigma}(ut_k\xi_j/t_j)|<a_k|\widehat{\tau}(\xi_k)|/k,\ j=1,\ldots,k-1.$$

This is possible since $\widehat{\tau}$ and $\widehat{\sigma}$ are continuous and vanish at the origin. In order to prove that (12) does not hold we first estimate $\|\tau_{(t)}*f\|$ for $t=t_k$. Using the identity

$$(19)\qquad \tau_{(t)}*f(x)=\sum_{j=1}^{\infty}a_j\widehat{\tau}(t\xi_j/t_j)e^{i\langle x,\xi_j\rangle/t_j}$$

we get

$$(20)\qquad \|\tau_{(t_k)}*f\|\geq a\,|\widehat{\tau}(\xi_k)|-|\sum_{j<k}|-|\sum_{j>k}|.$$

By (17) we have

$$(21)\qquad |\sum_{j<k}|\leq(\sum_{j<k}a_j)a_k|\widehat{\tau}(\xi_k)|/3<a_k|\widehat{\tau}(\xi_k)|/3,$$

and from (16) we get

$$(22)\qquad |\sum_{j>k}|<a_k|\widehat{\tau}(\xi_k)|/3.$$

Combination of (20), (21) and (22) gives

$$(23)\qquad \|\tau_{(t_k)}*f\|>a_k|\widehat{\tau}(\xi_k)|/3.$$

Next we must estimate $\omega_\sigma(f,Bt_k)$. Using the identity

$$\sigma_{(t)}*f(x)=\sum_{j=1}^{\infty}a_j\widehat{\sigma}(t\xi_j/t_j)e^{i\langle x,\xi_j\rangle/t_j}$$

we get

$$\omega_\sigma(f,Bt_k)\leq|\sum_{j<k}|+a_k\sup_{0<u\leq B}|\widehat{\sigma}(u\xi_k)|+|\sum_{j>k}|.$$

The middle term on the right-hand side is majorized by $a_k|\widehat{\tau}(\xi_k)|/k$ because of (15). Estimating the first and last term in a similar way as we estimated the corresponding terms in the series for $\tau_{(t)}*f$ we obtain using (16) and (18)

$$(24)\qquad \omega_\sigma(f,Bt_k)<3a_k|\widehat{\tau}(\xi_k)|/k,\ k=1,2,\ldots.$$

Now (23) and (24) together show that (12) does not hold. This completes the proof of the theorem.

R e m a r k s. It is easily seen from the proof that the function f can be chosen not to be continuously differentiable. In fact the sequence t_j, which is chosen after the sequence a_j, may be made to tend to zero arbitrarily fast. We also note that if $d=1$ we may take the function f periodic; for in this case we can choose t_j so that ξ_j/t_j are integers.

4. We now show how Theorem 1 can be deduced from Theorem 2. Take $\tau\in M(R)$ so that $\widehat{\tau}(\xi)=e^{-1/|\xi|}$ for $0<|\xi|<1$. Then for each integer $q\geq1$ the function $h(\xi)=\xi^{-q}\widehat{\tau}(\xi)$, $h(0)=0$, is infinitely differentiable for $|\xi|<1$. Denoting the q-th order difference measure by λ_q (i. e. $\widehat{\lambda}_q(\xi)=(e^{-i\xi}-1)^q$), this implies that $\widehat{\tau}$ is divisible by $\widehat{\lambda}_q$ in the ring of germs at $\xi=0$ of Fourier transforms of elements of $M(R)$. Hence it follows from Theorem 3 in [1] (see also [2]) that there are constants C_q depending only on q, such that

(25) $$\omega_\tau(f, t) \leqq C_q \omega_q(f, t), \quad t>0,$$

for uniformly continuous and bounded functions f. Take $\sigma \in M(R)$ such that $\widehat{\sigma}(\xi)=0$ for $0<|\xi|<1/2$, $\widehat{\sigma}(\xi)=1$ for $|\xi|>1$. Then using the fact $1-\widehat{\sigma}(\xi)=0$ for $|\xi|>1$ it is easy to see that for continuous and 2π-periodic f

(26) $$E_n(f) \leqq C\omega_\sigma(f;\ 1/n), \quad t>0,$$

(see e. g. [1], p. 108, formula (5)). Now, the pair of measures σ and τ does not satisfy (13) for any B. Taking $B=1$, we conclude that there exists a bounded and uniformly continuous function g such that

(27) $$\overline{\lim_{t\to 0}}\ \omega_\tau(g;\ t)/\omega_\sigma(g;\ t)=\infty.$$

According to the remarks at the end of the preceding section we can in fact choose g periodic and non-continuously differentiable. Combining (25) and (27), we get

(28) $$\overline{\lim_{t\to 0}}\ \omega_q(g;\ t)/\omega_\sigma(g;\ t)=\infty$$

for every q. Using the fact that $\omega_\sigma(g;\ t)$ is increasing together with the well-known inequality $\omega_q(f;\ At) \leqq (1+A)^q \omega_q(f;\ t)$, we easily pass from (28) to

(29) $$\overline{\lim_{n\to \infty}}\ \omega_q(g;\ 1/n)/\omega_\sigma(g;\ 1/n)=\infty.$$

Combining (26) and (29), we finally obtain (3), which completes the proof

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