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ON THE HISTORY OF APPROXIMATION

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Summary. In this contribution documents shall be shown which allow to retrace, in retrospect, the approximation of unknown functions since the 3rd millennium B. C. Introducing the (sexagesimal) positional system, the Sumerians, after having devised methods to compute the reciprocal values, putting at disposal all rational functions, developed powerful methods for the computation of roots of equations of algebraic — and in one instance of transcendental — character. Linear interpolation provided them with an iterational scheme, equivalent to Newton-Ralphson procedures. The deduction of these procedures — being the same as the modern ones — do not need such deep "theoretical" considerations as might be expected! Approximation of functions has been a question mainly raised by astronomy. This explains why E. Halley was the first of modern scientists who improved methods, affected by second-order errors, to such with third-order errors, and better. How the attempts to represent the "sine-functions" of the irregularity of planetary motion by zig-zag-linear functions, though leading to sufficiently exact representations, had to stop on the "trial and error"-level is explained: in order to obtain the "best approximation" one has to know the sine-function. Fundamental became the determination of the value of 2π and the establishing of a table for the sine-function. The comparatively rough method of Archimedes and the elementary computation of Ptolemy (though leading to 7-place decimal values) were improved. An inequality stated by Heron, later rediscovered by Huygens (1654) allows to easily obtain high accuracies. With the sine-function is related the computation of the sides of a regular polygon, inscribed into a circle. This problem was solved for any n-gon in an iterational scheme, needing at each step the solution of one quadratic equation only. Once the sine-function is available, the application in the "epicycle-theory" — equivalent to the Fourier series in its developments — solved the representation of observed, experimentally given functions.

When the great invention of the Sumerians in the third millennium B.C., the positional system, was more or less forgotten, numerical analysis became more difficult.

By means of a few tables of powers—preserved in Istanbul, registered as Kiš 9 and Kiš 19—the Babylonians were in a position[1] to compute reciprocals of regular numbers quickly and straightforward. For irregular numbers the computation was reduced by $A=R(1+a)$, where R is regular and a is small. The reciprocals of $1+a$ were computed[2] according to

$$1/(1+a)=1-a+a^2-a^3\ldots$$

just as in "modern times"!

The first irrational values to be determined were the square roots. In fact, also in modern times, the only functions which can be computed straightforward are rational algebraic functions, the quotients of polynomia. The first transcendental equation to be solved, for compound interest, was the exponential function which was computed by linear interpolation (AO 6770, TMB 146).[3]

The old methods for the computation of a square root are based on much simpler relations than modern treatises frequently do suggest. From $(d-a)\,(b-d)=\varepsilon$ follows $d=(d^2+ab+\varepsilon)/(a+b)$ and taking in a positional system an upper and a lower bound differing in the n-th digit, computing the number $d_1=(d^2+ab)/(a+b)$ has more than double a number of exact digits. Archimedes uses this relation in his "Measurement of the circle" using the bounds a and $a+1$ in

$$d_1=[d^2+a(a+1)]/(2a+1).$$

It must be remarked that the more sophisticated linear interpolation[4] leads to the very same result:

$$a+(d^2-a^2)\,(b-a)\,/\,(b^2-a^2)=(d^2+ab)/(a+b).$$

For $b=a$ the "Heron approximation" results, which method, however, is attested for the Old Babylonian Period (IM 52301) with the indication of the error resulting in the square, deduced geometrically.

In non-positional systems the rational values were obtained along the same line of thought.

Example: $\sqrt{3}$.

$$1<\sqrt{3}<2 \quad d_1=\frac{3+2}{3}=\frac{5}{3}<\sqrt{3}$$

$$\left(\left(\frac{5}{3}\right)^2+3\right)/\frac{10}{3}=\frac{26}{15}>\sqrt{3}$$

$$\frac{3+\frac{5}{3}\times\frac{26}{15}}{5/3+26/15}=\frac{265}{153}<\sqrt{3}$$

$$\frac{\left(\frac{26}{15}\right)^2+3}{52/15}=\frac{1351}{780}>\sqrt{3},$$

in which the last bounds are those used by Archimedes in his "Measurement of the circle".

Triplication of the number of exact digits was first obtained by the astronomer Edmund Halley in 1694. Using

$$d^2=a^2+2ax+x^2=a^2+b$$

in a first approximation, neglecting x^2, the "Heron-approximation" is obtained for $x=b/2a$. From the identity

$$x=b/(2a+x)$$

Halley deduces a better approximation by inserting in the denominator the value $x=b/2a$ and reducing the rational results

$$d_1=a(3d^2+a^2)/(a^2+3d^2)$$

is obtained.

With a few divisions an enormous number of decimals can be obtained which aroused Halley's enthusiasm.

Example:

$a=1.4$ Heron: $(2+1.96)/2.8=1.41428$, four digits exact;

Halley: $1.4\ (6+1.96)/(2+5.88)=1.4142131$

in which six digits are exact.

For *cubic* roots Heron interpolated between an upper and a lower bound according to

$$a+x=b-y,\ x+y=b-a$$

$$K-a^3=D_1,\quad b^3-K=D_2$$

$$x=a+bD_1/(bD_1+aD_2),$$

which in a positional system triplicates the number of exact digits. For $b=a$ the iteration

$$a(2K+a^3)/(K+2a^3)$$

Heron's method becomes identical with that deduced by E. Halley and De Lagny in the 17th century A. D.

For higher exponents the reductions by Halley lead to approximation of n-th roots by square roots. For the fifth root Halley's method leads to quintuplation of the number of exact digits!

The great discovery by Halley, however, is that in this way a solution of any polynomial equation can be approximated by square roots, triplicating the number of exact digits at each step in an iteration scheme. In modern symbols it corresponds to

$$0=F(x+h)=F+F'h+\frac{1}{2}F''h^2+\cdots,\ h_0=-F/F',$$

$$h=-\frac{F'}{F''}+\sqrt{\frac{F'^2-2FF''}{F''^2}}\sim h_0-\frac{F''}{2F'}h_0^2.$$

Tschebyscheff[5] inverted the function $y=f(x)$ into $x=F(y)$ by differentiation in 1838—only to be published in 1951!—but this general method is hardly ever used for higher than the second order approximation and *then* it is identical with Halley's method.

In astronomy the motion of the planets, with their "harmonic oscillation" about the positions in exact uniform circular motion, led the Babylonians to try to represent such a "sine-function" by *linear* functions. For many phenomena, due to the small amplitude, such a simple zig-zag function rendered good services. Defining the best approximation of $y=mx$ to $\sin x$, minimizing the sum of the squares of the errors, leads to

$$m=24/\pi^3=0.7741.$$

The following table shows that as soon as the amplitude becomes greater than 2°30′ the deviations near the maximum can grow to about $\frac{1}{2}$°!

a	$m\,a\pi$	$\sin a\pi$	D
0.1	0.24317	0.30902	−0.066
0.2	0.48634	0.58779	−0.101
0.3	0.72951	0.80902	−0.080
0.4	0.97268	0.95106	+0.022
0.5	1.21585	1.00000	+0.216

At the maximum an error of some 20 per cent does occur. This asks for improvement!

The Chaldean technique went over to a polygonal tract... and this *had* to be done by *trial and error*... as is evident from modern analysis: in order to solve the problem with only one change in direction of the line, one has to know the sine function already!

Minimizing

$$\int_0^{a\pi} (mx-\sin x)^2\,dx+\int_{a\pi}^{\pi/2}[(m-m_1)p+m_1x-\sin x]^2dx=\Phi(p, m, m_1)$$

one obtains three linear equations m, m_1, the determinant of which — $a=0$ and $a=\frac{1}{2}$ being of no trivial interest, — leads to an equation for a, viz.

$$\pi\cos a\pi\,[8a^4+4a^3-10a^2+3a]-\sin a\pi\,[24\,a^3+12a^2-18a+3]+48a^3-24a=0.$$

Determining the root $a=0.2523$, corresponding to $45°24'$... the values $m=0.9401$ and $m_1=0.40320$ lead to

$$\left.\begin{aligned}y&=2.95350a\ ,\ 0<\ a\ <0.2500\\ y&=0.73838+1.2666(a-0.2500)\end{aligned}\right\}F(a).$$

The next table shows that the accuracy is increased to such an extent that for the Babylonian experiments it should be sufficient.

a	$F(a)$	$\sin a\pi$	D
0.1	0.29535	0.30902	0.0136
0.2	0.59070	0.58779	−0.0029
0.3	0.80171	0.80902	+0.0073
0.4	0.92838	0.95106	+0.0227
0.5	1.05504	1.00000	+0.0550

The consideration of the uniform motion in circles leads to ask for the exact relations between the arc of a circle and its subtending chord.

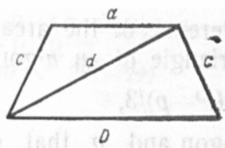
The cuneiform texts of Susa[6] show that the regular pentagon, hexagon and heptagon were drawn and numerically treated about 2000 B. C. A constant for the circle $\pi=3$, and a better one $\pi=3^1/_8$ is contained in a table of constants which gives also data about the area, the arc, the chord and the sagitta of a segment of a circle.

It is well known that Ptolemaios, having studied the "halving of the angle", the "doubling of the number of sides of an inscribed polygon", computed a table of "sines" and demonstrating that

$$\text{if } x>y \text{ then } x:y>\sin x:\sin y$$

could interpolate linearly between the chords of $3/4^c$ and $1\frac{1}{2}^c$ in order to obtain the chord of 1°.

The relations between chords of different arcs are also obtainable by means of the theorem of the isosceles trapezium



$$d^2=c^2+aD$$

next to the halving of the number of sides in a circle with radius $=1$ by (BM 85194, TMB 64, 65)

$$c=a,$$

which leads to the triplication of the number of sides by

$$3a-a^3=D.$$

For $D=1=a_6$ the substitution of $a={}^1/_3+x$ leads to $x\geqq 1/72$ and

$$\pi>9a/8>3{}^1/_8.$$

Starting from a_{10}, determined by $a^2=1-a$, one comes in two steps to $2\sin 2°$. (The relation was used in mediaeval times to determine $\sin 1°$.)

The general isosceles trapezium formula is equivalent to a relation for sine functions:

$$d^2_{p+q+1}=d^2_p+d_q\,d_{2p+q+3}$$

corresponds

$$\sin^2(p+q+2)\alpha=\sin^2(p+1)\alpha+\sin(q+1)\alpha\sin(2p+q+3)\alpha,$$

where $d_1, d_2, d_3, \ldots$ denote the different, increasing in length, diagonals of a regular n-gon.

The system of equations — as John Wallis rediscovered in the 17th century—leads to

$$d_1^2=a^2+ad_2,\quad d_2^2=d_1^2+ad_4$$

$$d_2^2=a^2+d_1d_3,$$

$$d_3^2=a^2+d_2d_4.$$

This systems allows the determining of "chords" for an arbitrary value of n-, by linear interpolation, at each step solving *one* quadratic equation only: As for $k=\frac{1}{2}(n-3)$ subsequent diagonals are equal and putting that length equal to unity, chosing a value of a one can deduce the length of the diagonals by rational values until one arrives at the first equation[7], which gives an equation for a. Interpolating according to the differences between the initial and the final value of a a rapidly converging computational scheme arises. . . and from pre-Pythagorean times a drawing using an eleven-gon with a fourteen-gon has been preserved.[8]

To modern mathematicians the simple inequalities used in ancient times tend to escape the attention.

Heron (Metrica 1.30) states that the area of a segment of a circle is greater than four-thirds of the inscribed isosceles triangle. It is proved applying the exhaustion method of Archimedes, who applied this to the parabola. It is, however, evident remarking that the parabola lies totally inside the circle if the axis, the top and the basis of the segments are the same.

As the triangle quoted is the difference of the area of two central triangles of a $2n$-gon and one central triangle of an n-gon, which means

$$2\pi > P + (P - p)/3,$$

if P denotes the perimeter of the $2n$-gon and p that of the n-gon. This means an important improvement is given with respect to Archimedes'

$$2\pi > P.$$

Huygens rediscovered in 1654 Heron's inequality seeing that it *doubled* the number of exact digits as compared to Archimedes method and he derived—applying the theorem of Guldin for surfaces of revolution — a relation *triplicating* the number of exact digits and found

$$2\pi < P + {}^1/_3(P-p)\ (4P+p)/(2P+3p).$$

It is possible to improve Huygens' result, *quadruplating* the number of exact digits by

$$2\pi > P + {}^1/_3(P-p)\ (34P+p)/(20P+15p),$$

which means that the series for 2π is gradually improved in

$$2\pi/P = 1 + q/3 + 2q^2/15 + 2q^3/35 + 8q^4/315 + \cdots,$$

where $q = (P-p)/P$.

If the arc of the segment is $2x$, in radians, the inequality of Heron corresponds to

$${}^4/_3x > \sin x + \sin x(1-\cos x)/3$$

which corresponds to

$$\sin x > x - x^3/6..$$

Once the sine-function had been mastered, the description of the motion of celestial bodies could be effectuated with any degree of accuracy by the "epicycles", which are simply Fourier-series-developments with varying, eventually, multiples.

NOTES

1. E. M. Bruins. La construction de la grande table de valeurs réciproques. Acta R. A. I., Bruxelles, 1969.
2. E. M. Bruins. Tables of Reciprocals with Regular Entires, Centaurus, XVII, 177—188.
3. TMB indicates: Thureau—Dangin. Textes mathématiques babyloniens. Leiden, 1938.
4. If $d^2 = a^2 + b^2$, $a < d < a + b$ is evident and

$$d_1 = [a^2 + b^2 + a(a+b)]/(2a+b) = a + b^2/(2a+b),$$

which lower bound is attested on VAT 6598, TMB 233, following the computation of an upper bound by the "Heron-formula" in TMB 232.

5. P. L. Tschebyscheff, V, pag. 7 seq. Moscow, 1951.
6. Textes mathématiques de Suise, edited by E. M. Bruins and M. Rutten. Paris, 1961.
7. Eliminating all intermediate diagonals the procedure is equialent to the modern relation

$$d_{2n-2} = (2n-1)a \sum_{k=0}^{n-1}{}' (-1)^k \binom{n+k-1}{2k+1} \frac{a^{2k}}{n-k-1}$$

with $d = 2\sin(2n-1)\alpha$, $a = 2\sin\alpha$.

8. Musée des Beaux Arts, Budapest, Inv. No. 52.820; comp. JANUS, 1976, page 73.