КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София. 1980, с. 281—291

CONSTRUCTIVE FUNCTION THEORY'77
Sofia. 1980, p. 281—291

EQUIVALENCE AND SHIFT PROPERTY OF SPLINE BASES IN L_p SPACES

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Summary. There is known relation between the orthonormal Haar (h_n, $n=1, 2, \ldots$) and the Walsh-Paley (w_n, $n=1, 2, \ldots$) systems. A new system of orthonormal splines ($f_n^{(r)}$, $n=2-r, \ldots$) of order $r>0$ is being defined as follows: 1. Integrate the Haar system $(r-1)$ times. 2. Complete set of functions obtained in this way with the monomials $1, t, \ldots, t^{r-2}$. 3. Apply to this set the Schmidt orthonormalization procedure. It is assumed that $(f_n^{(1)})=(h_n)$. Let now $w_n^{(r)}=f_n^{(r)}$ for $n=2-r, \ldots, 0$, and let for $n>0$ $w_n^{(r)}$ be obtained from $f_n^{(r)}$, $n>0$, by the same matrix transformations as (w_n) is being obtained from (h_n). Clearly $(w_n^{(1)})=(w_n)$. For each $r>0$ the orthonormal set $(w_n^{(r)})$ is uniformly bounded. The main object of this paper is to prove that the shift operator $S: w_n^{(r)} \to w_{n+1}^{(r)}$ is for each $r>0$ bounded in $L_p(0, 1)$, $1<p<\infty$. Consequently, it is proved that any two bases $(w_n^{(r)})$ and $(w_n^{(r')})$ are equivalent in $L_p(0,1)$ and in the Sobolev space $W_p^{(m)}(0,1)$ whenever $0 \leqq m$ $\min(r, r')$ and $1<p<\infty$.

1. Introduction. In the case of Walsh-Paley system $(w_n)=(w_n^{(1)})$ the boundedness of the shift was established in [5]. The uniformly bounded set of polygonals $(c_n)=(w_n^{(2)})$ was introduced in [1]. The very definition of (c_n) and Paley's result that (w_n) is a basis in $L_p(I)$, $I=\langle 0, 1\rangle$, $1<p<\infty$, imply immediately (cf. [10]) that (c_n) is a basis in $L_p(I)$, $1<p<\infty$, as well. With the help of the boundedness of the shift of (w_n) it is proved in [5] that the bases (w_n) and (c_n) are equivalent in $L_p(I)$, $1<p<\infty$. The aim of this paper is to extend these results to higher orders of uniformly bounded o. n. (orthonormal) spline systems $(w_n^{(r)})$, $r>0$. Moreover, we are going to construct equivalent interpolating spline bases in the Sobolev spaces $W_p^{(m)}(I)$.

2. Preliminaries. In the interval $I=[0, 1]$ we consider the sequence of dyadic partitions $\pi_n=(s_{n,i}, i=0, 1, \ldots, n)$ defined as follows: for $n=1$ we put $s_{n,0}=0$, $s_{n,1}=1$ and for $n>1$: $n=2^k+p$, $k\geqq 0$, $1\leqq p\leqq 2^k$;

$$s_{n,i}=i\overline{2}^{k-1} \text{ for } i=0, \ldots, 2p; \quad s_{n,i}=(i-p)\overline{2}^{k} \text{ for } i=2p+1, \ldots, n.$$

For each n π_n is being extended to a partition of $R=(-\infty, \infty)$ by the formula $s_{n,i+jn}=s_{n,i}+j$ for $i=0, \ldots, n-1$, and for any integer j. To each integer $r>0$ and to the knots π_n there corresponds the space $S_n^r=S_n^r(R)$ of splines of order r (of degree $r-1$) with multiplicities 1 at each knot. The set of all restrictions to I of f from S_n^r is denoted by $S_n^r(I)$, $n>0$. For $1-r<n\leq 0$ the space $S_n^r(I)$ is simply defined as the linear span of $1, t, \ldots, t^{n+r-2}$. Thus, $\dim S_n^r(I)=n+r-1$ and for $n>1-r$ $S_n^r(I)\subset S_{n+1}^r(I)$.

Natural basis in $S_n^r(I)$ do form the B-splines

$$N_{n;i}^{(r)}(t)=(s_{n,i+r}-s_{n,i})[s_{n,i},\ldots,s_{n,i+r};\ (s-t)_+^{r-1}],$$

$i=1-r,\ldots,n-1$. For the entries of the inverse to the Gram matrix of these B-splines we have the following estimate (cf. [6])

$$|a_{n;\ i,j}^{(r)}|\leq C_r n q_r^{|i-j|},\ i,j=1-r,\ldots,n-1;\ n>0, \tag{2.1}$$

where $C_r>0$ and q_r, $0<q_r<1$, depend on r only. For the dual basis with respect to the L_2 scalar product we have the formula

$$\underline{N}_{n;\ j}^{(r)}=\sum_{i=1-r}^{n-1} a_{n;\ j,i}^{(r)} N_{n;\ i}^{(r)},\ j=1-r,\ldots,n-1. \tag{2.2}$$

Clearly, $(N_{n;\ i}^{(r)}, \underline{N}_{n;\ j}^{(r)})=\delta_{i,j}$. It now follows from (2.1) and (2.2) that the following inequalities

$$\|\sum_{i=1-r}^{n-1} a_i N_{n;\ i}^{(r)}\|_p \sim M_{n,r}^{(p)}(a) \tag{2.3}$$

and

$$\|\sum_{j=1-r}^{n-1} a_j \underline{N}_{n;\ j}^{(r)}\|_p \sim n M_{n,r}^{(p)}(a), \tag{2,4}$$

take place uniformly in $n>1-r$ and $1\leq p\leq\infty$ with (see [4])

$$M_{n,r}^{(p)}(a)=\left(\frac{1}{n+r-1}\sum_{j=1-r}^{n-1}|a_j|^p\right)^{1/p}.$$

Let us now define the o. n. spline basis in $S_n^r(I)$. We denote by $f_{2-r}^{(r)}, \ldots, f_0^{(r)}, f_1^{(r)}$ the first r o. n. Legendre polynomials on I, with degree of $f_j^{(r)}$ equal to $j+r-2$. For $n>1$ the function $f_n^{(r)}$ is defined as the unique element in $S_n^r(I)$ orthogonal in the L_2 sense to $S_{n-1}^r(I)$ such that $\|f_n^{(r)}\|_2=1$ and $\|f_n^{(r)}\|_\infty=f_n^{(r)}(s_n)$ at some point $s_n\in I$. Notice, for $n>1$, $n=2^m+k$ with $1\leq k\leq 2^m$, $m\geq 0$,

$$f_n^{(r)}=\sum_{j=r-1}^{n-1}(N_{n;\ j}^{(r)}, f_n^{(r)})\underline{N}_{n;\ j}^{(r)},$$

and since $(N_{n;\ j}^{(r)}, f_n^{(r)})=0$ for the indices j for which $N_{n;\ j}^{(r)}$ is in $S_{n-1}^r(I)$ it follows that

$$f_n^{(r)}=\sum_{j=2k-1-r}^{2k-1}(N_{n;\ j}^{(r)}, f_n^{(r)})\underline{N}_{n;\ j}^{(r)}. \tag{2.5}$$

In connection with the investigation of the spline Fourier series $\sum_{j=2-r}^{\infty}(f, f_j^{(r)})f_j^{(r)}$ in the Sobolev spaces it is necessary to look at the related systems $(f_j^{(r,k)}, j>|k|+1-r)$ with integer k: $|k|<r$. These systems are defined as follows

$$f_j^{(r,k)}=\begin{cases} D^k f_j^{(r)} & \text{for } 0\leq k<r,\\ H^{-k} f_j^{(r)} & \text{for } -r<k<0; \end{cases}$$

where $j>|k|+1-r$, D is the differentiation operator and $Hf(t)=\int_t^1 f(s)ds$. One checks easily that $(f_i^{(r,k)}, f_j^{(r,-k)})=\delta_{i,j}$ for $i,j>|k|+1-r$. The properties of the systems $(f_j^{(r,k)})$, $|k|<r$, are discussed in [2, 3 and 4]*).

Theorem 2.1, (cf. [3]). *Let* $|k|<r, 1<p<\infty$. *Then* $(f_j^{(r,k)}, j>|k|+1-r)$ *is an unconditional basis in* $L_p(I)$.

To compare the coefficient spaces of the various sytems we normalize them as follows

$$h_j^{(r,k)}=f_j^{(r,k)}/\|f_j^{(r,|k|)}\|_2^{\operatorname{sgn} k}.$$

Clearly, $(h_j^{(r,k)}, h_i^{(r,-k)})=\delta_{i,j}$ for $i,j>|k|+1-r$, and it follows from [2] that for large j $\|h_j^{(r,k)}\|_2\sim 1$.

Theorem 2.2, (cf. [3]). *Let two pairs* $(r,k)(r',k')$ *be given with* $|k|<r, |k'|<r'$. *Then for* $1<p<\infty$ *the bases* $(h_j^{(r,k)}, j>|k|+1-r)$ *and* $(h_r^{(r',k')}, j>|k'|+1-r')$ *are equivalent in* $L_p(I)$.

Theorem 2.3, (cf. [3]). *Let* $(h_j^{(r,k)})$ *and* $(h_j^{(r',k')})$ *be given as in Theorem 2.2. Then the following series* $\Sigma_j a_j h_j^{(r,k)}$ and $\Sigma_j a_j h_j^{(r',k')}$ are equiconvergent in $L_p(I)$, $1<p<\infty$.

Theorem 2.4. *Let be given* r, k *and* p *such that* $|k|<r, 1<p<\infty$. *Moreover, let us define the linear shift by the formula*: $Tf_j^{(r,k)}=f_{j+1}^{(r,k)}$, $j>|k|+1-r$. *Then* $T: L_p(I)\to L_p(I)$ *is bounded.*

This result is implicitly contained in [3] and it is being obtained from Theorom 2.1—2.3 as follows. We know that $\|f_n^{(r,k)}\|_2\sim n^k$ for large n. Thus Theorems 2.1 and 2.2 imply equiconvergence in $L_p(I)$ of the series

$$\Sigma_{j=|k|+1-r}^{\infty} a_j f_{j+1}^{(r,k)} \tag{2.6}$$

and

$$\Sigma_{j=|k|+1-r}^{\infty} a_j f_j^{(r+1,k)}. \tag{2.7}$$

Now, Theorems 2.1 and 2.3 give equiconvergence of (2.7) with

$$\Sigma_{j=|k|+1-r}^{\infty} a_j f_j^{(r,k)}. \tag{2.8}$$

Consequently, (2.8) is equiconvergent with (2.6) and this proves Theorem 2.4.

Theorem 2.5. *Let* $1<p<\infty$ *and let* $|k|<\min(r,r')$. *Then the bases* $(f_j^{(r,k)}, j>|k|+1-r)$ *and* $(f_j^{(r',k)}, j>|k|+1-r')$ *are equivalent in* $L_p(I)$.

The situation with this theorem is similar as with the previous one, i. e. it follows from [3]. It is simply a consequence of Theorems 2.1 and 2.3 and the fact that the asymptotic $\|f_n^{(r,k)}\|_2\sim n^k$ is independent of r (cf. [2]).

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* In the earlier author's works the notation for the present $(f_j^{(r,k)})$ is a little different. Namely, $(f_j^{(r,k)})$ for $r>k\geq 0$ is identified with $(f_j^{(m,k)})$ and for $-r<k<0$ with $(g_j^{(m,k)})$, where $m=r-2$.

3. **The Bounded Orthonormal Spline Systems.** The Haar $(h_n, n=1, 2, \dots)$ and the Walsh-Paley $(w_n, n=1, 2, \dots)$ systems we define e. g. as in [1]. Thus, the relation between (h_n) and (w_n) is the following: $h_1=w_1=1$ and for $m\geq 0$

$$\operatorname{span}(h_{2^m+j}, j=1, \dots, 2^m)=\operatorname{span}(w_{2^m+j}, j=1, \dots, 2^m).$$

Consequently, to each $m\geq 0$ there exists orthogonal matrix $A^{(m)}_{i,j}$, $i, j=1, \dots, 2^m$ such that

$$w_{2^m+j}=\Sigma_{i=1}^{2^m} A^{(m)}_{j,i} h_{2^m+i}, \tag{3.1}$$

and it can be seen that $(A^{(m)}_{i,j})$ is symmetric (cf. [1]). Moreover

$$A^{(m)}_{i,j}=(h_{2^m+i}, w_{2^m+j})=\pm 2^{-m/2}. \tag{3.2}$$

For the given o. n. spline system $(f^{(r)}_j, j>1-r)$ we define a new o. n. system $(w^{(r)}_j, j>1-r)$ in exactly the same way as the Walsh-Paley system in terms of Haar system using (3.1). More precisely, let us define: $w^{(r)}_n=f^{(r)}_n$ for $1-r<n<2$ and for $m\geq 0$, $i=1, \dots, 2^m$,

$$w^{(r)}_{2^m+i}=\sum_{j=1}^{2^m} A^{(m)}_{i,j} f^{(r)}_j, \tag{3.3}$$

where $(A^{(m)}_{i,j})$ is given by (3.2). Since (cf. [2])

$$\|\sum_{j=1}^{2^m} |f^{(r)}_{2^m+j}|\|_\infty \leq C_r 2^{m/2},$$

it follows from (3.3) and (3.2) that $(w^{(r)}_n, n>1-r)$ is uniformly bounded o. n. set of splines of order r on I. Moreover, to each $(f^{(r,k)}_n)$, $|k|<r$, there corresponds in a natural way the spline system $(w^{(r,k)}_n\ n>|k|+1-r)$ defined as follows: $w^{(r,k)}_n=f^{(r,k)}_n$ for $|k|+1-r<n<2$, and for $m\geq 0, i=1, \dots, 2^m$,

$$w^{(r,k)}_{2^m+i}=\Sigma_{j=1}^{2m} A^{(m)}_{i,j} f^{(r,k)}_{2^m+j}. \tag{3.4}$$

It should be clear that $(w^{(r,k)}_n)$ can be obtained by $|k|$-fold differentiation or integration of $(w^{(r)}_n)$ for k positive or negative, respectively.

Theorem 3.1, (cf. [10]). *Let the integers k and r, $|k|<r$, be given. Then $(w^{(r,k)}_n, n>|k|+1-r)$ is a basis in $L_p(I)$, $1<p<\infty$.*

The next result follows from Theorem 2.1.

Theorem 3.2. *Let k, r and p be given such that $|k|<r$ and $1<p<\infty$. Then for some constant $C_{p,r}>0$ we have for $f\in L_p(I)$*

$$\|f\|_p C^{-1}_{p,r}\leq \left(\int_I\left(\sum_{m=-1}^{\infty}(D_m(f))^2\right)^{p/2}dt\right)^{1/p}\leq C_{p,r}\|f\|_p,$$

where $D_{-1}(f)=\Sigma^1_{j=|k|+2-r}(f, w^{(r,-k)}_j)w^{(r,k)}_j$ and for $m=0,1,\dots$ $D_m(f)=\Sigma^{2^{m+1}}_{j=2^m+1}(f, w^{(r,-k)}_j)w^{(r,k)}_j$.

For later use the following decomposition lemma is needed.

L e m m a 3.1. *Let k, r and p be given as in theorem 3.2. Then the projection*

$$P^{(r,k)}f=\sum_{m=0}^{\infty}(f, w_{2m+1}^{(r,-k)})w_{2m+1}^{(r,k)}$$

in the $L_p(I)$ space is bounded.

P r o o f. According to Theorem 3.2

$$\| P^{(r,k)}f\|_p\sim\|(\sum_{m=0}^{\infty} b_{2m+1}^2(w_{2m+1}^{(r,k)})^2)^{1/2}\|_p\leqq C_{p,r}(\sum_{m=0}^{\infty} b_{2m+1}^2\, 2^{2k(m+1)})^{1/2},$$

where $b_j=(f, w_j^{(r,-k)})$. If now

(3.5) $$f=\Sigma_{j=|k|+2-r}^{\infty}a_j f_j^{(r,k)}$$

with $a_j=(f, f_j^{(r,-k)})$, then by (3.4) $b_{2m+1}=\Sigma_{j=1}^{2m}A_{2m,j}^{(m)}a_{2m+j}$, whence (3.2) implies $(2^{k(m+1)}\, b_{2m+1})^2\leqq 2^{2k(m+1)}\Sigma_{j=1}^{2m}(a_{2m+j})^2$. Thus for some constant $C_{p,r}$

$$\| P^{(r,k)}f\|_p\leqq C_{p,r}(\sum_{n=|k|+2-r}^{\infty}((n+r)^k a_n)^2)^{1/2}.$$

On the other hand $(f_n^{(r,k)})$ is a Riesz basis in $L_2(I)$ (cf. [9]) and therefore by (3.5)

$$\|f\|_2\sim(\sum_{n=|k|+2-r}^{\infty}((n+r)^k a_n)^2)^{1/2}.$$

Consequently, $\| P^{(r,k)}f\|_p\leqq C_{p,r}\|f\|_2$ which for $p>2$ implies the required inequality

$$\| P^{(r,k)}f\|_p\leqq C_{p,r}\|f\|_p, |k|<r.$$

Since the conjugate operator is $P^{(r,-k)}$: $L_q(I)\to L_q(I)$, $q(p-1)=p$, Banach's theorem gives $\| P^{(r,-k)}f\|_q\leqq C_{p,r}\|f\|_q, |k|<r$, and clearly this completes the proof.

T h e o r e m 3.3. *Let k,r and p be given as in Theorem 3.2. Moreover, let S be the shift operator, i. e. $Sw_n^{(r,k)}=w_{n+1}^{(r,k)}$ for $n>|k|+1-r$. Then S: $L_p(I)\to L_p(I)$ is bounded.*

Proof. For given f from $L_p(I)$ let

(3.6) $$f_0=\sum_{n=|k|+2-r}^{1}(f, w_n^{(r,-k)})w_n^{(r,k)}$$

(3.7) $$f_1=P^{(r,k)}f=\sum_{m=0}^{\infty}(f, w_{2m+1}^{(r,-k)})w_{2m+1}^{(r,k)},$$

(3.8) $$f_2=\sum_{m=1}^{\infty}\sum_{i=1}^{2^m-1}(f, w_{2m+i}^{(r,-k)})w_{2m+i}^{(r,k)}.$$

It is a consequence of Theorem 3.1 and Lemma 3.1 that the series (3.7) and (3.8) converge in $L_p(I)$ and therefore in that space we have the following decomposition $f=f_0+f_1+f_2$. Formula (3.6) gives immediately for some constant $C_{p,r}$

(3.9) $$\| Sf_0\|_p\leqq C_{p,r}\|f\|_p.$$

The next step is to estimate the norm of Sf_1. For simplicity let $b_m=(f, w_{2^m}^{(r,-k)})$ and then

$$Sf_1=\sum_{m=1}^{\infty} b_m w_{2^m+1}^{(r,k)}.$$

Using now (3.4) and Theorem 2.1 we obtain

$$\| Sf_1 \|_p \sim \| (\sum_{m=1}^{\infty} b_m^2 (w_{2^m+1}^{(r,k)})^2)^{1/2} \|_p.$$

For the Hardy maximal function Mg of g we have the following trivial inequalities $\| g \|_1 \leq Mg(t) \leq \| g \|_\infty$ satisfied for all t in I. On the other hand (cf. [10])

$$\| w_n^{(r,k)} \|_1 \sim \| w_n^{(r,k)} \|_\infty \sim n^k,$$

and therefore for some positive constants C and C'

$$\| (\sum_{m=1}^{\infty} b_m^2 (w_{2^m+1}^{(r,k)})^2)^{1/2} \|_p \leq C (\sum_{m=1}^{\infty} (b_m 2^{km})^2)^{1/2} \leq C' \| (\sum_{m=1}^{\infty} (b_m M w_{2^m}^{(r,k)})^2)^{1/2} \|_p.$$

Application of the Fefferman-Stein maximal inequality (cf. [7]) and of Theorem 2.1 and Lemma 3.1 gives for some constants C, C' and C''

$$\| (\sum_{m=1}^{\infty} (b_m M w_{2^m}^{(r,k)})^2)^{1/2} \|_p \leq C \| (\sum_{m=1}^{\infty} (b_m w_{2^m}^{(r,k)})^2)^{1/2} \|_p \leq C' \| f_1 \|_p \leq C'' \| f \|_p.$$

Combining this with the previous inequality we get

$$\| Sf_1 \|_p \leq C_{p,r} \| f \|_p. \tag{3.10}$$

The estimate for $\| Sf_2 \|_p$. We know from (3.8) that $f_2=\Sigma_{n=3}^{\infty} d_n w_n^{(r,k)}$, with $d_{2^m}=0$ for $m=2, 3, \ldots$ For later convenience let $f_2=\Sigma_{n=3}^{\infty} a_n f_n^{(r,k)}$, and $Sf_2=\Sigma_{n=4}^{\infty} d'_n w_n^{(r,k)}=\Sigma_{n=4}^{\infty} a'_n f_n^{(r,k)}$, where $d'_{2^m+i}=d_{2^m+i-1}$ for $2 \leq i \leq 2^m$, and $d'_{2^m+1}=0$ for $m=2, 3, \ldots$ It now follows from (3.4) that $a_{2^m+j}=\Sigma_{i=1}^{2^m} A_{j,i}^{(m)} d_{2^m+i}$, and $a'_{2^m+j}=\Sigma_{i=1}^{2^m} A_{j,i}^{(m)} d'_{2^m+i}$.

Now, $\| f_n^{(r,|k|)} \|_2 \sim n^{|k|}$, $f_n^{(r,k)}=h_n^{(r,k)} \| f_n^{(r,|k|)} \|_2^{\operatorname{sgn} k}$, and therefore by Theorem 2.1 Sf_2 is in $L_p(I)$ if and only if the sum $\Sigma_{m=1}^{\infty} \Sigma_{i=1}^{2^m} a'_{2^m+i} 2^{mk} h_{2^m+i}^{(r,k)}$, $a'_3=0$, is in L_p. According to Theorem 2.3, since $h_n^{(1,0)}=f_n^{(1)}$, the last series is equiconvergent with

$$\sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} a'_{2^m+i} f_{2^m+i}^{(1)} = \sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} d'_{2^m+i} w_{2^m+i}^{(1)} = \sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m-1} d_{2^m+i} w_{2^m+i+1}^{(1)}.$$

We know, however, that the Walsh-Paley system $(w_n^{(1)})$ has bounded shift in L_p (cf. [5]), and therefore the last series is equiconvergent in L_p with

$$\sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} d_{2^m+i} w_{2^m+i}^{(1)} = \sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} a_{2^m+i} f_{2^m+i}^{(1)} = \sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} a_{2^m+i} h_{2^m+i}^{(1,0)},$$

and this according to Theorem 2.3 equiconverges with

$$\sum_{m=1}^{\infty} 2^{mk} \sum_{i=1}^{2^m} a_{2^m+i} h_{2^m+i}^{(r,k)}.$$

Application of Theorem 2.1 shows that the last series is equiconvergent with $\Sigma_{m=1}^{\infty}\Sigma_{i=1}^{2^m} a_{2^m+i} f_{2^m+i}^{(r,k)} = f_2$, and this implies

$$\| Sf_2 \|_p \leq C'_{p,r} \| f_2 \|_p \leq C_{p,r} \| f \|_p. \tag{3.11}$$

To complete the proof it remains to combine the inequalities (3.9), (3.10) and (3.11).

We are now in a position to prove the analogue of Theorem 2.5 for the bounded o. n. spline systems $(w_n^{(r)})$. This result for $k=0$, $r=1$ and $r'=2$ was proved earlier in [5].

Theorem 3.4. *Let $1<p<\infty$ and let $|k|<\min(r, r')$. Then the bases $(w_j^{(r,k)}, j>|k|+1-r)$ and $(w_j^{(r',k)}, j>|k|+1-r')$ are equivalent in $L_p(I)$. In particular, the bounded o. n. spline systems $(w_j^{(r)}, j>1-r)$ and $(w_j^{(r')}, j>1-r')$ are equivalent bases in $L_p(I)$.*

Proof. It follows by Theorem 3.3 that the series $\Sigma_{j=|k|+2-r}^{\infty} b_j w_j^{(r,k)}$ equiconverges with

$$\sum_{j=2}^{\infty} b'_j w_j^{(r,k)} = \sum_{j=2}^{\infty} a'_j f_j^{(r,k)}, \tag{3.12}$$

where $b'_j = b_{j+r'-r}$ for $j \geq 2$, and for $1 \leq i \leq 2^m$, $m \geq 0$, $a'_{2^m+i} = \Sigma_{j=1}^{2^m} A_{i,j}^{(m)} b'_{2^m+j}$. However Theorems 2.4 and 2.5 imply the equiconvergence of (3.12) with $\Sigma_{j=2}^{\infty} a'_j f_j^{(r',k)} = \Sigma_{j=2}^{\infty} b'_j w_j^{(r',k)} = \Sigma_{j=2}^{\infty} b_{j+r'-r} w_j^{(r',k)}$, and this completes the proof.

Corollary 3.1. *Let $1<p<\infty$ and let $|k|<\min(r, r')$: Then the following two series are equiconvergent in $L_p(I)$:*

$$\sum_{j=2}^{\infty} a_j w_j^{(r,k)} \text{ and } \sum_{j=2}^{\infty} a_j w_j^{(r',k)}.$$

This follows from Theorems 3.3 and 3.4.

Corollary 3.2. *Let $1<p<\infty$ and let r be given positive integer. Then the bounded o. n. spline system $(w_n^{(r)})$ is not an equivalent basis in $L_p(I)$ to the trigonometric basis.*

It has been proved in [11] that the Walsh-Paley basis is not equivalent to the trigonometric basis in $L_p(I)$. On the other hand, Theorem 3.4 gives the equivalence of the Walsh-Paley basis with $(w_n^{(r)})$, and this completes the proof.

For the completeness of our discussion of the bounded o. n. spline systems $(w_n^{(r)})$ we introduce on analogy to $h_j^{(r,k)}$ the following functions

$$v_j^{(r,k)} = w_j^{(r,k)} \text{ for } 1-r<j<2,$$

$$v_j^{(r,k)} = w_j^{(r,k)} / \| w_n^{(r,|k|)} \|_2^{\operatorname{sgn} k} \text{ for } j \geq 2 \text{ and } n = 2^{[\log_2(j-1)]},$$

where $[x]$ is the integer part of x. It should be clear that $(v_i^{(r,k)}, v_j^{(r,-k)}) = \delta_{i,j}$ for $i, j > |k|+1-r$ and $|k|<r$. Moreover, $\| v_j^{(r,k)} \|_2 \sim 1$ for large j

Theorem 3.5. *Let* $1<p<\infty$ *and let* $|k|<r$, $|k'|<r'$. Then,

(a) $(v_j^{(r,k)}, j>|k|+1-r)$ *is a basis in* $L_p(I)$.

(b) *The linear shift* $T: v_j^{(r,k)} > v_{j+1}^{(r,k)}$ *is bounded in* L_p.

(c) *The bases* $(v_j^{(r,k)}, j>|k|+1-r)$ *and* $(v_j^{(r',k')}, j>|k'|+1-r')$ *are equivalent in* $L_p(I)$.

(d) *The following two series are equiconvergent in* $L_p(I)$:

$$\sum_{j=2}^{\infty} a_j v_j^{(r,k)} \text{ and } \sum_{j=2}^{\infty} a_j v_j^{(r',k')}.$$

Proof. Since $(w_j^{(r,k)})$ is a basis in $L_p(I)$ property (a) follows by the very definition of $(v_j^{(r,k)})$. Checking (b) we repeat the argument used in the proof of Theorem 3.3. For the proof of (c) let

$$f=\Sigma_{j=|k|+2-r}^{\infty} b_j v_j^{(r,k)} \tag{3.13}$$

be in $L_p(I)$ and let $g=\Sigma_{j=|k'|+2-r'}^{\infty} b_{j+|k|-|k'|+r'-r} v_j^{(r',k')}$. We need to show that g is in $L_p(I)$. The shift property reduces this problem to the equiconvergence of (3.13) and $\Sigma_j b_j v_j^{(r',k')}$. To see this let us write $\Sigma_{j=2}^{\infty} b_j v_j^{(r,k)} = \Sigma_{j=2}^{\infty} b_j' w_j^{(r,k)} = \Sigma_{j=2}^{\infty} a_j' f_j^{(r,k)} = \Sigma_{j=2}^{\infty} a_j'' h_j^{(r,k)}$, which equiconverges with

$$\sum_{j=2}^{\infty} a_j'' h_j^{(r',k')} = \sum_{j=2}^{\infty} a_j' \varepsilon_j f_j^{(r',k')}),$$

where for $n=2^{[\log_2(j-1)]}$

$$\varepsilon_j = \| f_j^{(r,|k|)} \|_2^{\operatorname{sgn} k} / \| f_j^{(r',|k'|)} \|_2^{\operatorname{sgn} k'} \sim \| w_n^{(r,|k|)} \|_2^{\operatorname{sgn} k} / \| w_n^{(r',|k'|)} \|_2^{\operatorname{sgn} k'} = \delta_j.$$

Thus the last series is equiconvergent with

$$\sum_{j=2}^{\infty} \delta_j a_j' f_j^{(r',k')} = \sum_{j=2}^{\infty} \delta_j b_j' w_j^{(r',k')} = \sum_{j=2}^{\infty} b_j v_j^{(r',k')},$$

and this completes the proof of (c).

Finally, (a), (b) and (c) imply (d).

4. **Bases in Sobolev Spaces.** On the interval $I=[0,1]$ we consider the Sobolev space $W_p^m(I)$, $m\geq 0$, $1\leq p<\infty$, with the norm $\|f\|_{p,m}=\Sigma_{k=0}^{m} \|D^k f\|_p$.

It is known that $(f_j^{(r)}, j>1-r)$ is a basis in $W_p^m(I)$ for $m<r$ and $1\leq p<\infty$ (cf. [4]) and that it is unconditional basis in the same space for $1<p<\infty$ (cf. [3]). The bounded system $(w_j^{(r)}, j>1-r)$ is a basis in $W_p^m(I)$ for $m<r$ and $1<p<\infty$ as well (cf. [10]).

Theorem 4.1. *Let* $1<p<\infty$ *and let* $r>0$. *Then for each* m, $0\leq m<r$, *the linear shift operators* $T: f_j^{(r)} \to f_{j+1}^{(r)}$ *and* $S: w_j^{(r)} \to w_{j+1}^{(r)}$ *are bounded in* $W_p^m(I)$.

Proof. For the polynomial $f=\Sigma_{j=2-r}^{n} a_j f_j^{(r)}$ we have for $k=0,1,\ldots,m$, $D^k f=\Sigma_{j=k+2-r}^{n} a_j f_j^{(r,k)}$ and $D^k Tf=\Sigma_{j=j_0}^{n} a_j f_{j+1}^{(r,k)}$; $j_0=\max(1,k)+2-r$. It now follows from Theorem 2.4 that for some constant $C_{p,r}$ $\|D^k Tf\|_p \leq C_{p,r} \|D^k f\|_p$ for $k=0,1,\ldots,m$, and this implies boundedness of T in $W_p^m(I)$. Similar argument gives the continuity of S.

Theorem 4.2. *Let the integers m, r and r' be such that $0 \leq m < r < r'$. Then, for $1 < p < \infty$,*

(a) *the systems $(f_n^{(r)}, n > 1-r)$ and $(f_n^{(r')}, n > 1-r')$ are equivalent bases in $W_p^m(I)$,*

(b) *the bounded systems $(w_n^{(r)}, n > 1-r)$ and $(w_n^{(r')}, n > 1-r')$ are equivalent bases in $W_p^m(I)$.*

Proof. In case (a) Theorem 4.1 reduces the problem to the question of equiconvergence in $W_p^m(I)$ of the following two series $\Sigma_j a_j f_j^{(r)}$ and $\Sigma_j a_j f_j^{(r')}$. However, this property follows immediately from Theorems 2.4 and 2.5.

Statement (b) is being proved by similar argument.

It may be of some interest to prove a theorem on equivalence of interpolating spline bases in $W_p^m(I)$. For given $f \in W_p^r(I)$, $1 \leq p < \infty$, $r > 0$, let us denote by $H^{(2r)}f$ the algebraic polynomial of degree $2r-1$ solving the following two-point Hermite interpolating problem $D^j H^{(2r)} f(s) = D^j f(s)$ for $s = 0, 1$; $j = 0, \ldots, r-1$. The basic interpolating spline functions are now defined by the formula

$$g_n^{(2r)}(t) = \int_0^t f_n^{(r, -r+1)}(s)ds, \, t \in I, \, n > 1.$$

Clearly $g_n^{(2r)}$ is in $S_n^{2r}(I)$. It has been essentially shown in [2] that $(H^{(2r)} g_n^{(2r)}, n > 1)$ is a basis in $W_p^m(I)$ for $r \leq m < 2r$ and $1 \leq p < \infty$, i. e. for each $f \in W_p^m(I)$ we have in this space

$$f = H^{(2r)} f + \Sigma_{n=2}^{\infty} a_n(f) g_n^{(2r)} \quad \text{with} \quad a_n(f) = \int_I f_n^{(r, r-1)}(s) df(s).$$

The basis $(H^{(2r)}, g_n^{(2r)}, n > 1)$ is interpolating. Namely, for $n > 1$, $n = 2^q + k$ $1 \leq k \leq 2^q$, $g_n^{(2r)}(s_{n,j}) = g_n^{(2r)}(s_{n, 2k-1}) \delta_{j, 2k-1}$, $j = 0, \ldots, n$.

Lemma 4.1. *Let $n = 2^q + k$, $1 \leq k \leq 2^q$, $0 \leq q$. Now, for given $r > 0$ there are constants $C_r > 0$ and q_r, $0 < q_r < 1$, such that*

$$|g_n^{(2r)}(t)| \leq C_r n^{1/2r} q_r^{n|t-t_n|}, \tag{4.1}$$

where $t_n = s_{n, 2k-1}$. Moreover,

$$\|g_n^{(2r)}\|_\infty \sim |g_n^{(2r)}(t_n)| \sim n^{1/2-r}. \tag{4.2}$$

Proof. We know that for some constants C_r and q_r, $0 < q_r < 1$, (see [2]) the following inequality holds

$$|f_n^{(r, -r+1)}(t)| \leq C_r n^{3/2-r} q_r^{n|t-t_n|}. \tag{4.3}$$

Now, for each $t \in I$ there is an i, $1 \leq i \leq n$, such that $s_{n, i-1} \leq t \leq s_{n,i}$, and either $g_n^{(2r)}(s_{n, i-1}) = 0$ or $g_n^{(2r)}(s_{n,i}) = 0$. Suppose that the first case takes place. Then by (4.3)

$$|g_n^{(2r)}(t)| = |g_n^{(2r)}(t) - g_n^{(2r)}(s_{n, i-1})|$$

$$\leq \int_{s_{n,i-1}}^{s_{n,i}} |f_n^{(r, -r+1)}(s)| ds \leq C_r \frac{2}{n} n^{3/2-r} \max_{s \in I_{n,i}} q_r^{n|s-t_n|} \leq C_r' n^{1/2-r} q_r^{n|t-t_n|},$$

where $I_{n,i}=[s_{n,i-1}, s_{n,i}]$. In the second case the argument i is almost identical.

It now follows by (4.1) that

$$|g_n^{(2r)}(t_n)|\leq \|g_n^{(2r)}\|_\infty \leq C_r n^{1/2-r}.$$

The estimates from below are a little more difficult to obtain. Since for $n>1$ the function $f_n^{(r)}$ is orthogonal to polynomials of degree $r-1$ we have

$$(4.4)\qquad g_n^{(2r)}(t)=-(g_n^{(2r)}(1)-g_n^{(2r)}(t))=-\int_t^1 f_n^{(r,\,-r+1)}(s)ds$$

$$=\frac{-1}{(r-1)!}\int_0^1 (s-t)_+^{r-1} f_n^{(r)}(s)ds=\frac{(-1)^{r-1}}{(r-1)!}\int_0^1 (t-s)_+^{r-1} f_n^{(r)}(s)ds.$$

With the help of formula (4.4) we extend $g_n^{(2r)}$ from I to R, clearly this is the same as extension of $g_n^{(2r)}$ outside of I by zero. Formula (4.4) gives now

$$(4.5)\qquad [s_{n,j},\ldots,s_{n,j+r};\, g_n^{(2r)}]=(-1)^{r-1}(N_{n,j}^{(r)}, f_n^{(r)})/(r-1)!\,(s_{n,j+r}-s_{n,j}).$$

For later convenience let $w_{j,r}(s)=(s-s_{n,j})\cdots(s-s_{n,j+r})$ and $a_{n,j}=(N_{n,j}^{(r)}, f_n^{(r)})$. Now, the properties of $g_n^{(2r)}$ for $2k-1-r\leq j\leq 2k-1$ imply

$$(4.6)\qquad [s_{n,j},\ldots,s_{n,j+r};\, g_n^{(2r)}]=g_n^{(2r)}(t_n)/(Dw_{j,r})(t_n).$$

Comparing (4.5) and (4.6) we get for $2k-1-r\leq j\leq 2k-1$

$$a_{n,j}=(-1)^{r-1}(r-1)!\,(s_{n,j+r}-s_{n,j})g_n^{(2r)}(t_n)/(Dw_{j,r})(t_n).$$

whence we infer

$$(4.7)\qquad \sum_{j=2k-1-r}^{2k-1} |a_{n,j}|\leq C_r' n^{r-1}|g_n^{(2r)}(t_n)|$$

with some constant C_r'. On the other hand, (2.5) and (2.4) corresponding to $p=1$ give

$$n^{-1/2}\leq C_r''\|f_n^{(r)}\|_1\leq C_r'''\Sigma_{j=2k-1-r}^{2k-1}|a_{n,j}|.$$

Combining these inequalities with (4.7) we complete the proof.

For comparing the interpolating bases $(H^{(2r)}, g_n^{(2r)}, n>1)$ and $(H^{(2r')}, g_n^{(2r')}, n>1)$ it is convenient to normalize them as follows $G_n^{(2r)}(t)=g_n^{(2r)}\times(t)/g_n^{(2r)}(t_n)$, $n>1$.

Theorem 4.3. *Let p, m, r and r' be given such that $1<p<\infty$, $1\leq r\leq m<2r$, $1\leq r'\leq m<2r'$. Then $(H^{(2r)}, G_n^{(2r)}, n>1)$ and $(H^{(2r')}, G_n^{(2r')}, n>1)$ are equivalent bases in $W_p^m(I)$.*

Proof. It is enough to check equiconvergence of the following two series in $W_p^m(I)$

$$\sum_j a_j G_j^{(2r)} \text{ and } \sum_j a_{j+2r'-2r} G_j^{(2r')},$$

and this is equivalent to equiconvergence in $L_p(I)$ of the series

$$\sum_j a_j D^m G_j^{(2r)} \quad \text{and} \quad \sum_j a_{j+2r'-2r} D^m G_j^{(2r')}.$$

Now, the first series equiconverges with

$$\sum_j a_j j^{m-1/2} h_j^{(r,\, m-r)}, \tag{4.8}$$

what follows by Lemma 4.1 and Theorem 2.1. The second series for the same reasons equiconverges with

$$\sum_j a_{j+2r'-2r} (j+2r'-2r)^{m-1/2} h_j^{(r',\, m-r')} . \tag{4.9}$$

To get the equiconvergence of (4.8) and (4.9) it is sufficient to apply Theorems 2.3, 2.1 and 2.4.

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Received November 8, 1977