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SOME PROBLEMS OF THE SUMMATION OF ORTHOGONAL SERIES WITH RESPECT TO UNIFORMLY BOUNDED ORTHOGONAL SYSTEM

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Summary. Let $\{m_N\}$ be a subsequence of natural numbers, then the following statements are true:

a) $\|a(m_N, \infty), K\| = \sup_{\varphi\in\Omega(K)} \{\int_0^1 \sup_{N\leq i<j<\infty} |a_{m_i+1}\varphi_{m_{i+1}}(x)+\cdots+a_{m_j}\varphi_{m_j}(x)|^2dx\}^{1/2}\to 0$

as $N\to\infty$ iff $S_{m_N}(x)=\sum_{n=1}^{m_N} a_n\varphi_n(x)$ converges a. e. on [0,1] where $\Omega(K)$ denotes the set of K-bounded ON-systems on (0,1);

b) $\|a(m_N, \infty), K\|\to 0$ as $N\to\infty$ iff
$\|a(m_N, \infty), 1\|\to 0$ as $N\to\infty (1<K<\infty)$;

c) $\|a(2^N, \infty), K\|\geq C(K)\{\sum_{n=2^N}^{\infty} a_n^2 (\log\log n)^2\}^{1/2}$, $(1<K\leq\infty)$.

From a), b) and c) it follows specially the following theorem, which is strengthening of a previous result of L. Csernyák and L. Leindler (1966).

Theorem: If $\{a_n\}$ is a sequence for which $|na_n|\geq|(n+1)a_{n+1}|$ for every $n\geq 1$ and $\Sigma a_n^2(\log\log n)^2<\infty$ then there exist a $\varphi\in\Omega(1)$ such that $\lim\sup_{N\to\infty}(S_2N(x)|=\infty$ a. e. on (0,1).

For any given $K(1\leq K\leq\infty)$ denote $\Omega(K)$ the set of orthonormal systems $\varphi=\{\varphi_n\}$ defined on (0,1) such that $|\varphi_n(x)|\leq K$ for almost every $x\in(0,1)$ and for $n=1, 2, \ldots$. K. Tandori defined for arbitrary sequence $\{a_n\}$ of real numbers the following norm

$$\|\{a_n\}_N^\infty;\ K\| = \sup_{\varphi\in\Omega(K)} \{\int_0^1 \sup_{N\leq i<j<\infty} |a_i\varphi_i(x)+\cdots+a_j\varphi_j(x)|^2dx\}^{1/2}$$

and proved in [3], [4]:

A) The series $\Sigma a_n\varphi_n(x)$ converges almost everywhere on (0,1) for every $\varphi\in\Omega(K)\,(1\leq K\leq\infty)$ if and only if $\lim_{N\to\infty}\|\{a_n\}_N^\infty;\ K\|=0$;

B) At $1<K<\infty \lim_{N\to\infty}\|\{a_n\}_N^\infty;\ 1\|=0$ if and only if $\lim_{N\to\infty}\|\{a_n\}_N^\infty;\ K\|=0$;

C) If $|a_n|\geq|a_{n+1}|\,(n=1, 2, \ldots;\ 1<K<\infty)$ then

$$\|\{a_n\}_N^\infty;\ K\|\geq C_1(K)\{\sum_{n=N}^{\infty} a_n^2\log^2 n\}^{1/2};\ N=1, 2, \ldots.$$

Here $C_1(K)$ (and later on $C_2(K)$) denotes a constant depending only on K.

From A), B) and C) follows immediately the following result, which is strengthening of the latest result of B. S. Kašin [2]:

D) If $|a_n|\geq|a_{n+1}|$ and $\Sigma a_n^2 \log n=\infty$ are fulfilled, then there exists a $\varphi\in\Omega(1)$ for which $\lim_{N\to\infty}\sup\{|\sum_{i=1}^k a_i\varphi_i(x)|:\ 1\leq k\leq N\}=\infty$ a. e. on [0,1].

Given a monotone increasing sequence of natural numbers $\{m_N\}$, define the following generalization of Tandori's norm mentioned above:

$$\|\{a_n\}_{m_N}^{\infty};\ K\|^*=\sup_{\varphi\in\Omega(K)}\{\int_0^1\sup_{N\leq i<j<\infty}|a_{m_i}\varphi_{m_i}(x)+\cdots+a_{m_j}\varphi_{m_j}(x)|^2dx\}^{1/2}.$$

One can prove the analogous statements to A), B), C) and D), namely we shall prove the following theorems:

Theorem 1. *The sequence* $S_{m_N}(x)=\Sigma_{n=1}^{m_N}a_n\varphi_n(x)$ *converges as* $N\to\infty$, *a. e. on* (0,1), *for every* $\varphi\in\Omega(K)$; $1\leq K\leq\infty$ *if and only if* $\lim_{N\to\infty}\|\{a_n\}_{m_N};\ K\|^*=0$.

Theorem 2. *At* $1<K<\infty$ $\lim_{N\to\infty}\|\{a_n\}_{m_N}^{\infty};\ 1\|^*=0$ *if and only if* $\lim_{N\to\infty}\|\{a_n\}_{m_N}^{\infty};\ K\|^*=0$.

Teorem 3. *If*

(1) $$n|a_n|\geq(n+1)|a_{n+1}|;\ n=1,2,\ldots$$

and $1<K<\infty$, *then* $\|\{a_n\}_{2^N}^{\infty};\ K\|^*\geq C_2(K)\{\Sigma_{n=2^N}^{\infty}a_n^2(\log\log n)^2\}^{1/2}$.

Corollary: *If* $n|a_n|\geq n\ a_{n+1}|$ *and* $\Sigma a_n^2(\log\log n)^2=\infty$ *then there exists a* $\varphi\in\Omega(1)$ *for which* $\lim\sup_{N\to\infty}|S_{2^N}(x)|=\infty$ *a. e. on* (0,1). This is refining of a previous result of L. Csernyák and L. Leindler [1].

The proof of Theorem 1 and Theorem 2 follows Tandori's method and it does not need new ideas, so we omit it. We must prove only Theorem 3. For this we need some lemmas.

Lemma 1. (K. Tandori [3] Lemma VI). *Let* $K>1$ *and* $p\geq2$ *be an arbitrary natural number, further* $1\leq c\leq p/4$. *Then there exist an orthonormal system* $g_l(c,p;\ x)$; $l=1,\ldots,p^2$ *of stepfunctions on* (0,1) *and an interval* $E\leq(0,1)$ *such that* $|g_l(c,p;\ x)|\leq K$; $0\leq x\leq1$; $l=1,\ldots,p^2$, *further for every* $x\in E$ *there exists an index* $m(x)<p^2$ *for which*

$$g_l(c,p;\ x)\geq0;\ l=1,2,\ldots,m(x)$$

and

(2) $$\sum_{i=1}^{m(x)}g_l(c,p;\ x)\geq C_3(K)\sqrt{c}\,p\log p,$$

(3) $$m(E)\geq C_4(K)c^{-1}.$$

Lemma 2. (D. Mensov [5]). *Let* d *and* q *be positive integers,* $0<d<q$. *Given for every pair* $1\leq i,j\leq q$ *such that* $|i-j|=d$ *a real number* $a_{i,j}$. *Denote* $B_d=\max\{|a_{i,j}|:\ i/j\}$. *Then for every interval* (u,v) *with* $v-u>2B_d$ *there exists a sequence* $\varphi_l(x)\ (l=1,\ldots,q)$ *of stepfunctions having the following properties:*

(4) $$|\varphi_l(x)|=1;\ u<x<v;\ l=1,\ldots,q,$$

$$\int_u^v \varphi_i(x)\varphi_j(x)dx = -a_{i,j};\ |i-j|=d;\ 1\leq i,j\leq q,$$

$$\int_u^v \varphi_i(x)\varphi_j(x)dx = 0;\ i\neq j;\ |i-j|\neq d;\ 1\leq i,j\leq q.$$

L e m m a 3. (L. Csernyák, L. Leindler [1]). *Let* $\{c_n\}$ *be an arbitrary sequence of real numbers. Denote* $E_{n,m}$ *the set of those* $x\in(0,1)$ *for which*

$$\left|\sum_{i=n+1}^{n+m} c_i r_i(x)\right| \geq C_5 \left\{\sum_{i=n+1}^{n+m} c_i^2\right\}^{1/2}$$

s fulfilled, where $\{r_n(x)\}$ *denotes the Rademacher-system. Then the set* $E_{m,n}$ *is simple (i. e. the union of finite many intervals) and* $m(E_{m,n})\geq C_5^2/4>0$.

P r o o f of Theorem 3. We may suppose that $a_n>0$ for every n. Let $m_k=2^{2k}$; $k=0,1,\ldots$, $a_1^*=a_1$; $a_n^*=a_{2^i}$; $n=2^{i-1}+1,\ldots,2^i$; $i=1,2,\ldots$; $A_i^*=\{\Sigma_{j=2^{i-1}+1}^{2i} a_i^{*2}\}^{1/2}$.

According to (1) the sequence $\{A_i^*\}$ is monotone non-increasing. To prove Theorem 3 it is enough to show that for every $N\geq1$ there exists an orthonormal system $\{\Phi_n(x)\}$ for which the following conditions are fulfilled:

$$\left\{\int_0^1 \left(\max_{1\leq i\leq j\leq m_N} |a_{2^i+1}\Phi_{2^i+1}(x)+\cdots+a_{2^j}\Phi_{2^j}(x)|\right)^2 dx\right\}^{1/2}$$

$$\geq C_2(K)\left\{\sum_{i=1}^{2^{m_N}} a_i^2[\log\log(i+2)]^2\right\}^{1/2},\quad |\Phi_n(x)|\leq K;\ n=1,2,\ldots,2^{m_N}.$$

Let k^* be the smallest integer for which: $2^{2^{k^*}}>2^4C_5^{-2}$ (C_5 taken from lemma 3)

$$J_k=(2^{-k},2^{-k-1});\ k=k^*+1,\ldots,m.$$

We shall define on $(-1,2)$ an orthonormal system $\psi_n(x)$ $(n=1,\ldots,2^{m_N})$ of stepfunctions having the following properties:

(5) a) $|\psi_n(x)|\leq K;\ 0\leq x\leq1;\ n=1,2,\ldots,2^{m_N}$,

(6) b) $\psi_n(x)=0;\ x\in J_k\ (n\leq 2^{m_{k-1}};\ n>2^{m_k};\ k^*<k\leq N)$

c) for every $k^*<k\leq N$ there exists a simple set $E_k(CJ_k)$ such that, for every $x\in E_k$

(7) $a_{2^{m_{k-1}}+1}^*\psi_{2^{m_{k-1}}+1}(x)+\cdots+a_{2^{m(x)}}\psi_{2^{m(x)}}(x)\geq 2^{-4}C_3(K)C_5\sqrt{3k2^{3k}A_{2m_{k-1}}^{*2}}$

is fulfilled at some index $m_{k-1}<m(x)\leq 2m_{k-1}$, further

(8) $$m(E_k)\geq C_4(K)C_5^2 2^{-k-5}.$$

(9) d) $$\int_{-1}^0 |a_1\psi_1(x)+\cdots+a_{2^{m_{k^*}}}\psi_{2^{m_{k^*}}}(x)|\,dx\geq \sum_{i=1}^{m_{k^*}} A_i^{*2}.$$

Let

$$\psi_n(x)=\begin{cases} r_n(x); & x\in(-1,0),\\ 0 & ;\ \text{otherwise};\ n=1,\ldots,2^{m_{k^*}}.\end{cases}$$

Then (9) is obviously fulfilled. Suppose, for some $k^* \leq k_0 < N$ the orthonormal system $\psi_n(x)$; $n=1, 2, \ldots, 2^{m_{k_0}}$ of stepfunctions and (if $k_0 > k^*$) the simple sets $E_{k^*+1}, \ldots, E_{k_0}$ are defined so that (5), (6), (7) and (8) are fulfilled. Apply Lemma 1, at $p=2^{k_0}$, $c=1$, and denote the resulting functions and set $g_n(x)$; $n=m_{k_0}, \ldots, 2m_{k_0}-1$ and E, respectively. Then according to (2) and (3) for every $x \in E$ there exists an index $m_{k_0} \leq m(x) \leq 2m_{k_0}-1$ such that $g_l(x) \geqq 0$; $l=m_{k_0}, \ldots, m(x)$,

$$(10) \qquad \sum_{l=m_{k_0}}^{m(x)} g_l(x) \geq C_3(K) 2^{k_0} k_0,$$

$$(11) \qquad m(E) \geq C_4(K)$$

are fulfilled. Using the notation $Q_i = 2^{2^{i-k_0-2}}$; $i=m_{k_0}, \ldots, 2m_{k_0}-1$, denote $I_{i,q}$; $q=1, \ldots, Q_i$ subintervals of $[0, 1]$ of lengths $1/Q_i$ such that on each of them the functions $r_1(x), r_2(x), \ldots, r_{2^{i-k_0-2}}(x)$ are constant. As it is well known: $r_n((1/2)-x) = -r_n((1/2)+x)$; $0 \leq x \leq 1/2$; $n=1, 2, \ldots$, so by Lemma 3 the sum of lengths ot those intervals $I_{i,q}$, on which

$$(12) \qquad \sum_{j=1}^{2^{i-k_0-2}} \Big(\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l \Big) r_j(x) > C_5 \Big\{ \sum_{j=1}^{2^{i-k_0-2}} \Big[\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l \Big]^2 \Big\}^{1/2}$$

is greater than $C_5^2 \cdot 2^{-3}$. So the number of those intervals $I_{i,q}$, on which (12) is fulfilled, is greater than $[C_5^2 2^{-3} Q_i] = d_i \geqq 2$. Change the intervals $I_{i,q}$ for which (12) is not fulfilled and $q \leq d_i$ with those ones for which (12) is fulfilled and $q > d_i$. So we obtain a set of functions $r_{i,j}(x)$ ($j=1, \ldots, 2^{i-k_0-2}$, such that

$$\sum_{j=1}^{2^{i-k_0-2}} \Big(\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l \Big) r_{i,j}(x) > C_5 \Big\{ \sum_{j=1}^{2^{i-k_0-2}} \Big[\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l \Big]^2 \Big\}^{1/2},$$

if $x \in (0, C_5^2 2^{-4})$. Let

$$\varphi_{2^i+(j-1)2^{k_0+2}+1}(x) = \sum_{q=1}^{Q_{2m_{k_0}-1}} r_{i,j}(x) g_i(I_{2m_k-1,q};\ x)$$

$$x \in (0, 1);\ j=1, 2, \ldots, 2^{i-k_0-2};\ i=m_{k_0}, \ldots, 2m_{k_0}-1,$$

where for any interval $I=[a, b] \subset [0, 1]$ we have $g(I;\ x) = g((x-a)/b-a))$ if $x \in I$ and $g(I;\ x)=0$ otherwise.

By an easy calculation, the system $\{\varphi_n(x)\}$ is orthonormal on $[0, 1]$. Next we follow Tandori's method. Let

$$(13) \qquad \bar{G}_q = E(I_{2m_{k_0}-1,q});\ q=1, \ldots, d_{2m_{k_0}-1},\ G_{k_0+1} = \bigcup_{q=1}^{d_{2m_{k_0}-1}} \bar{G}_q.$$

Then using (11) we have

$$(14) \qquad m(G_{k_0+1}) \geq C_4(k) C_5^2 2^{-4}.$$

Let $\bar{\psi}_n(x)=\varphi_{2^i+(j-1)2^{k_0+2}+1}(J_{k_0+1};\ x);\ n=2^i+(j-1)2^{k_0+2}+l;\ l=1,\ldots,2^{k_0+2};$ $j=1,\ldots,2^{i-k_0-2};\ i=m_{k_0},\ldots,2m_{k_0}-1$. Obviously

$$(15)\qquad a_{m,n}=\int_0^1\bar{\psi}_n(x)\bar{\psi}_m(x)dx\leq m(J_{k_0+1})=2^{-k_0-2};\ 2^{m_{k_0}}<m,\ n\leq 2^{2m_{k_0}}$$

and $\int_0^1\bar{\psi}_n(x)\bar{\psi}_m(x)dx=0;\ 2^{m_{k_0}}<n,\ m\leq 2^{2m_{k_0}};\ |n-m|>2^{2m_{k_0}}$. We can apply Lemma 2 and so we obtain a sequence $\psi_n^*(x);\ n=2^{m_{k_0}}+1,\ldots,2^{2m_{k_0}}$ of step-functions, having the following properties:

$$(16)\qquad |\psi_n^*(x)|=\begin{cases}1 & \text{for } x\in(1,2)\\ 0 & \text{otherwise};\ n=2^{m_{k_0}}+1,\ldots,2^{2m_{k_0}},\end{cases}$$

$$(17)\qquad \int_1^2\psi_n^*(x)\psi_n^*(x)dx=-a_{n,m};\ 2^{m_{k_0}}<n,\ m\leq 2^{2m_{k_0}}.$$

If $I_r;\ j=1,\ldots,\varrho$ is a partitition of $(1,2)$ into disjoint intervals such that on every interval, each of the functions $\psi_1^*(x),\ldots,\psi_{2^{m_{k_0}}}(x)$ are constant, then consider the following fuctions:

$$(18)\quad \psi_n(x)=\frac{1}{D}\Big[\bar{\psi}_n(x)+\sum_{r=1}^{\varrho}\psi_n^*(I_r';\ x)-\sum_{r=1}^{\varrho}\psi_n^*(I_r'';\ x)\Big];\ n=2^{m_{k_0}}+1,\ldots,2^{2m_{k_0}},$$

where

$$(19)\qquad D^2=\int_{J_{k_0+1}}\bar{\psi}_n^2(x)dx+\int_1^2\psi_n^{*2}(x)dx;$$

$$\psi_n(x)=\begin{cases}r_n(x+1) & \text{for } x\in(-1,0)\\ 0 & \text{otherwise};\ n=2^{2m_{k_0}}+1,\ldots,2^{m_{k_0+1}},\end{cases}$$

$\psi_{2^{m_{k_0}}+1}(x),\ldots,\psi_{2^{m_{k_0+1}}}(x)$ are stepfunctions, and by (17), (18) and (19) they form an orthonormal system on $(-1,2)$. According to (15) and (16) $1\leq D^2\leq 2$ and hence

$$(20)\qquad |\psi_n(x)|\leq K\ \ (-1\leq x\leq 2;\ 2^{m_{k_0}}<n\leq 2^{m_{k_0+1}}).$$

It follows from the definition of the system $\{\psi_n(x)\}$ that

$$(21)\qquad \psi_n(x)=0;\ 2^{m_{k_0}}<n\leq 2^{m_{k_0+1}};\ x\in(0,1)\setminus J_{k_0+1}.$$

Let

$$(22)\qquad E_{k_0+1}=G_{k_0+1}(J_{k_0+1}).$$

From (14) follows

$$(23)\qquad m(E_{k_0+1})=2^{-4}C_4(K)C_5^2 2^{-k_0-2}$$

and if $x\in E_{k_0+1}$, then according to (13) there exists an index $1\leq q_0\leq d_{2m_{k_0}-1}$ such that $x\in\bar{G}_{q_0}(J_{k_0+1})$ and hence by (10) and (12) we have the following estimate:

$$\sum_{n=2^{m_{k_0}}+1}^{2^{m(x)}} a_n\psi_n(x) \geq \frac{1}{\sqrt{2}} \sum_{i=m_{k_0}}^{m(x)} \sum_{j=1}^{2^{i-k_0-2}} \Big(\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l\Big)\varphi_{2^i+(j-1)2^{k_0+2}+1}(x)$$

$$= \frac{1}{\sqrt{2}} \sum_{i=m_{k_0}}^{m(x)} g_i(I_{2m_{k_0}-1}, q_0;\ x) \sum_{j=1}^{2^{i-k_0-2}} \Big(\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l\Big) r_{i,j}(x)$$

$$(24) \qquad > \frac{C_5}{2\sqrt{2}} \sum_{i=m_{k_0}}^{m(x)} g_i(I_{2m_{k_0}-1}, q;\ x) \Big\{ \sum_{j=1}^{2^{i-k_0-2}} \Big[\sum_{l=2^i+(j-1)2^{k_0+2}+1}^{2^i+j2^{k_0+2}} a_l\Big]^2\Big\}^{1/2}$$

$$> \frac{C_5}{2\sqrt{2}} \sum_{i=m_{k_0}}^{m(x)} g_i(I_{2m_{k_0}-1}, q_0;\ x)\sqrt{2^{i-k_0-2}}\,2^{k_0+2} a_{2^i+1}$$

$$= \frac{C_5}{2\sqrt{2}} \sum_{i=m_{k_0}}^{m(x)} \sqrt{2^{k_0+2}} A^*_{i+1} g_i(I_{2m_{k_0}-1}, q;\ x) \geq \frac{C_3(K)C_5}{2\sqrt{2}} \sqrt{2^{k_0+2}}\,2^{k_0} k_0 A^*_{2m_{k_0}}$$

$$\geq 16^{-1} C_3(K) C_5 \sqrt{2^{k_0+2}}\sqrt{3\,.\,2^{2k_0} A^{*2}_{2m_{k_0}}}\,2(k_0+2)$$

(20), (21), (23), (24) gives that the conditions a) b) and c) are fulfilled at $k=k_0+1$, hence

$$\int_{-1}^{2} \Big(\max_{1 \leq i \leq j \leq m_N} | a_{2^i+1}\psi_{2^i+1}(x) + \cdots + a_{2^j}\psi_{2^j}(x) |^2 dx$$

$$\geq \int_{-1}^{0} | a_1\psi_1(x) + \cdots + a_{2^{m_{k^*}}}\psi_{2^{m_{k^*}}}(x) | \, dx$$

$$+ \sum_{k=k^*+1}^{N} \int_{J_k} (a_{2^{m_{k-1}}+1}\psi_{2^{m_{k-1}}+1}(x) + \cdots + a_{2m(x)}\psi_{2m(x)}(x))^2 dx$$

$$\geq \sum_{i=1}^{m_{k^*}} A_i^{*2} + 16^{-3} C_4(K) C_5^4 C_3(K) \sum_{k=k^*+1}^{N} 3\,.\,2^{2k-1} A_{2m_{k-1}}(2k)^2$$

$$\geq C_6(K) \sum_{k=1}^{N} (m_k - m_{k-1}) A^{*2}_{m_k} \log^2 m_k$$

$$\geq 8^{-1} C_5(K) \sum_{n=1}^{2^{m_N}} a_n^2 [\log\log(n+2)]^2.$$

In the last step we used (1).

Because $K>1$, there exists a real number $0<\alpha(K)<1$ such that $\alpha(K)/3+(1-\alpha(K))K^2=1$ is fulfilled.

It is obvious from the above that the functions

$$\Phi_n(x) = \begin{cases} \psi^n((3x/\alpha(K))-1) & \text{for } x \in (0, \alpha(K)), \\ K r_n((\alpha(K), 1);\ x) & \text{for } x \in (\alpha(K), 1), \\ 0 & \text{otherwise}; \ n=1, 2, \ldots, 2^{m_N}. \end{cases}$$

satisfy the condition 4, so Theorem 3 is proved.

REFERENCES

1. L. Csernyák, L., L. Leindler. Über die Divergenz der Partialsummen von Orthogonalreihen. *Acta Sci. Math.*, **27**, 1966, 55—62.
2. B. S. Kašin. On Weyl's multipliers for almost everywhere convergence of orthogonal series. *Analysis Math.*, **2**, 1976, 4, 249—266.
3. K. Tandori. Über die Konvergenz der Orthogonalreihen III. *Publications Math.*, **12**, 1965, 127—157.
4. K. Tandori. Über beschränkte orthonormierte Systeme. *Acta Math. Acad. Sci. Hung.*, **31**, 1978, 279—285.
5. D. Mensov. Sur la sommation des séries orthogonales *C. R. Acad. Sci. Paris*, **180**, 1925, 2011—2013.

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