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CONVOLUTIONS AND APPROXIMATION

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Summary. The generalized convolutions for an arbitrary linear right inverse operator L of the differentiation operator d/dt, found by the author, and, independently, by L. Berg, are used for explicit characterization of cyclic elements of L and for representing in explicit form all the commuting operators to L. An application to the remainder term of the corresponding Taylor formula for L is indicated.

Both the "one-sided" convolution and the "periodic" convolution have many applications in approximation theory. Now we intend to show that a generalization of these operations, found independently by the author [1] and [2] and by L. Berg [3], could also be applied for studying approximation problems and related topics.

It is well known that the one-sided convolution

(1) $$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau,$$

considered in the space $C[0, T]$ of continuous functions, is a bilinear, commutative and associative operation. It is closely connected with the integration operator

(2) $$lf(t)=\int_0^t f(\tau)d\tau$$

by the relation

(3) $$lf=\{1\}*f,$$

where by $\{1\}$ is denoted the constant function 1.

But the integration operator (2) is only one of the right inverse operators of the differentiation operator d/dt in $C[0, T]$. The most general linear right inverse operator of d/dt has the form

(4) $$Lf(t)=\int_0^t f(\tau)d\tau+\Phi(f),$$

where $\Phi: C[0, T]\to \mathbb{C}$ is a linear functional.

It is natural to ask for a bilinear, commutative and associative operation $f * g$ in $C[0, T]$ or in a subspace of it, for which instead of (3), the relation

(5) $$Lf = \{1\} * f$$

to hold true.

An answer in the affirmative had been given by the author in [1] and [2] for the subspace $C^1[0, T]$ of the smooth functions of $C[0, T]$. There had been proven the following

Theorem 1. *The operation*

(6) $$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau + \Phi_x\{\frac{\partial}{\partial x}\int_t^x f(x+t-\tau)g(\tau)d\tau\},$$

where the subscript x of Φ denotes that the linear functional Φ should be applied to the corresponding expression, considered as a function of x only, leaving t as a parameter, is a bilinear, commutative and associative operation in $C^1[0, T]$, for which (5) *holds true.*

Then it is natural the operation (6) to be called a convolution for L. It can be represented in the following equivalent form.

Theorem 2. *The operation* (6) *has the representation*

(7) $$(f * g)(t) = F_x\{\int_x^t f(x+t-\tau)g(\tau)d\tau\},$$

where

(8) $$F(f) = f(0) - \Phi(f').$$

Here F, in general, is a functional in $C^1[0, T]$ with $F(\{1\}) = 1$. If F $C[0, T] \to \mathbb{C}$ is an arbitrary linear functional with $F(\{1\}) = 1$, then L. Berg [3] had shown that the operation (8) is a convolution for the right inverse operator of d/dt, given by $Lf(t) = \int_0^t f(\tau)d\tau - F(lf)$ in the whole space $C[0, T]$.

Now we will show that in the general case of (4) there also exists a convolution for L in the whole space $C[0, T]$, understood as a bilinear, commutative and associative operation $f \tilde{*} g$ in $C[0, T]$, for which the convolution property

(9) $$L(f \tilde{*} g) = (Lf) \tilde{*} g$$

holds (for such general definition of convolution, see [4]).

Theorem 3. *The operation*

(10) $$(f * g)(t) = \int_0^t d\tau \int_0^\tau f(\tau-\sigma)g(\sigma)d\sigma + \Phi(f)lg$$

$$+ \Phi(g)lf + \Phi(f)\Phi(g) + \Phi_x\{\int_t^x f(x+t-\tau)g(\tau)d\tau\}$$

is a convolution for (4) *in* $C[0, T]$ *with*

(11) $$L^2 f = \{1\} * f.$$

Proof. If $f, g \in C^1[0, T]$, then it is easy to see that $f \tilde{*} g = L(f * g)$ and hence $f \tilde{*} g$ is a convolution of L in $C^1[0, T]$ with (11). The commutativity and bilinearity of (10) in $C[0, T]$ are evident. In order to prove the associativity relation $(f \tilde{*} g) \tilde{*} h = f \tilde{*} (g \tilde{*} h)$ for arbitrary $f, g, h \in C[0, T]$, we approximate them with uniformly convergent sequences $f_n, g_n, h_n, n=1, 2, \ldots$ from $C^1[0, T]$. Letting $n \to \infty$ in $(f_n \tilde{*} g_n) \tilde{*} h_n = f_n \tilde{*} (g_n \tilde{*} h_n)$ and using the continuity of (10) with respect to the uniform convergence, we obtain the associativity relation in $C[0, T]$. The convolution property (9) can be proven in the same manner.

First, we will use the convolution (10) for a characterization of the linear operators $M: C[0, T] \to C[0, T]$ which commute with the general operator (4), i. e.

$$ML = LM. \tag{12}$$

Theorem 4. *If a linear operator $M: C[0, T] \to C[0, T]$ commutes with L, then M is a multiplier of the convolution* (10), *i. e. it satisfies the identity*

$$M(f \tilde{*} g) = (Mf) \tilde{*} g. \tag{13}$$

Proof. We shall use the evident fact that the constant function $h = \{1\}$ is a cyclic element of operator L in $C[0, T]$ with respect to the uniform convergence. This means that the linear combinations of the functions $L^n h, n=0, 1, 2, \ldots$ are everywhere dense in $C[0, T]$. In fact $L^n h = B_n(t)$ is a polynomial of n-th degree, and this follows simply from the Weierstrass' approximation theorem. Sometimes $B_n(t), n=0, 1, 2, \ldots$ are called generalized Bernoulli polynomials.

Let m and n be arbitrary nonnegative integers. Then, using (19) and (12), we can write

$$ML^p h \tilde{*} L^q h = L^p h \tilde{*} ML^q h, \; p, q = 0, 1, 2, \ldots \tag{14}$$

If $f, g \in C[0, T]$ are arbitrarily chosen, we take sequences

$$f_n = \sum_{i=0}^{n} a_{ni} L^i h, \quad g_m = \sum_{j=0}^{m} b_{mj} L^j h,$$

converging uniformly to f and g, correspondingly. From the bilinearity of the operation $f \tilde{*} g$ and from (14), we have $Mf_n \tilde{*} g_m = f_n \tilde{*} Mg_m$. Now it remains to use to continuity of the operation $f \tilde{*} g$ with respect to the uniform convergence in order to obtain the relation $(Mf) \tilde{*} g = f \tilde{*} (Mg)$, which, in fact, is equivalent to (13) (see [4]).

The theorem just proved states that the set of the multipliers of the convolution (10) coincides with the set of all the commuting linear operators with L.

A similar result is true for the convolution (6) [or (7)] too, but for $C^1[0, T]$, considered as an algebra with this multiplication.

Now we will use the convolution (10) to characterize in explicit form the cyclic elements of the operator (4).

Theorem 5. *If $k \in C[0, T]$ is a cyclic element of the operator L, then k is not a divisor of zero for the convolution* (10).

Proof. Let $k \tilde{*} f=0$ for some $f \in C[0, T]$. Then $L^n k * f=0$, $n=0, 1, 2, \ldots$. Let $h_n=\Sigma_{j=0}^{n} a_{nj} L^j k$ be an uniformly converging sequence to the constant function $h=\{1\}$. From $h_n \tilde{*} f=0$ and the continuity of (10) it follows $h \tilde{*} f=0$, or $L^2 f=0$. Hence $f=0$, and thus k is not a divisor of zero for the convolution (10).

Theorem 6. *If $k \in C^2[0, T]$ is not a divisor of zero for* (10) *and if $F(k)=k(0)-\Phi(k')\neq 0$, then k is a cyclic element for L in $C[0, T]$.*

Proof. Let f be an arbitrary function from $C[0, T]$. We determine the function $g \in C[0, T]$ as a solution of the Fredholm second kind integral equation $k' * g+F(k)g=f$, where $*$ denotes the operation (6). According to Fredholm's alternative, the existence [and uniqueness] of g is ensured provided the homogeneous equation

$$k' * x+F(k)x=0 \tag{15}$$

has only the trivial solution $x=0$. In order to show this, let us apply the operator L to both sides of (15). Using the identity

$$k=Lk'+F(k) \tag{16}$$

and the convolution property of $*$,we get $k * x=0$. But k is not a divisor of zero for $*$, and hence $x=0$. Thus the existence of such g is ensured.

From (16) we have $L^n k=k' * B_n+F(k)B_n$, $n=0, 1, 2, \ldots$, where $B_n=L^n\{1\}$ are the generalized Bernoulli polynomials, corresponding to L. Let us take a sequence $g_n(t)=\Sigma_{j=0}^{n} a_{nj} B_j(t)$, $n=0, 1, 2, \ldots$ uniformly converging to g. Then the sequence $f_n=\Sigma_{j=0}^{n} a_{nj} L^j k=k' * g_n+F(k)g_n$ converges uniformly to f.

Hence, k is a cyclic element for L in $C[0, T]$.

Now we will use the convolution (6) for an explicit representation of the operators M, which commute with L.

Theorem 7. *A linear operator $M: C[0, T] \to C[0, T]$ with $m=M\{1\} \in C^2[0, T]$ commutes with the operator L if and only if*

$$Mf=F(m)f+m' * f, \tag{17}$$

where by $*$ is denoted the operation (6) *or, equivalently* (7).

Proof. Let M commute with L, i. e. $ML=LM$. Then, by theorem 4, M is a multiplier of the convolution (10) in $C[0, T]$. Then, applying M to both sides of the identity $L^2 f=\{1\} \tilde{*} f$, and using the commutating relation $ML^2=L^2 M$, we get $L^2 Mf=m \tilde{*} f$ with $m=M\{1\}$. Hence $Mf=d^2/dt^2(m \tilde{*} f)$. But for $m \in C^2[0, T]$ we have $m \tilde{*} f=L(m * f)$, and therefore

$$Mf(t)=\frac{d}{dt}(m * f)=F(m)f(t)+F_x\{\int_x^t m'(x+t-\tau) f(\tau)d\tau\},$$

which is exactly the representation (17).

Conversely, it is easy to verify that an integral operator $M: C[0, T] \to C[0, T]$ of the form (17) commutes with L.

Corollary. *If $\Phi(t)\equiv 0$, then formula* (17) *gives*

$$Mf(t)=m(0)f(t)+\int_0^t m'(t-\tau)f(\tau)d\tau$$

with $m(t)\in C^1[0, T]$ as an explicit representation of all linear operators $M: C[0, T]\to C[0, T]$, which commute with the integration operator (2). Similar result for a space of analytic functions was obtained by I. Raichinov [5].

The above results are easily extendable in the complex domain. Of special interest is the possibility to express all the multiplier operators for the Dirichlet expansions in a convex domain, containing the origin in the form (17). But this is worked out in detail in another paper by the author [6].

Now we shall indicate another use of the convolutions (6), namely, for explicit representation of the remainder term of the general Taylor formula for L. According to D. Przeworska-Rolewicz [7] under "the Taylor formula for L" should be understood the formal identity

$$f=\sum_{k=0}^{N} L^k F D^k f+L^{N+1}D^{N+1}f \tag{18}$$

in the domain $C_{D^{N+1}}$ of the operator $D^{N+1}=(d/dt)^{N+1}$.

The remainder term $R_N f(t)=L^{N+1}f^{(N+1)}(t)$ expressed by means of the convolution (7), takes the form $R_N f=B_n * f^{(N+1)}$.

Then the generalized Taylor formula (18) takes the explicit form

$$f(t)=\sum_{k=0}^{N} F(f^{(k)})B_k(t)+F_x\{\int_x^t B_n(x+t-\tau)f^{(N+1)}(\tau)d\tau\} \tag{19}$$

with $F(g)=g(0)-\Phi(g')$. Along with the usual Taylor formula, (19) embraces Euler-Mac Laurin and Boole formulas. The formula (19) is due to J. Delsarte [8] and was found in another way.

All the above considerations support strongly enough the convinction of usefulness of the convolutions just introduced for studying other problems of the approximation theory as well.

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