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APPROXIMATION BY RATIOS OF EXPONENTIALS

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Summary. We prove a global existence theorem for general rational approximation problems in the spaces X_p, where $X_p=(L_p(I), \|\cdot\|_p)$ for $1\leq p<\infty$ and $X_\infty=(C(I), \|\cdot\|_\infty)$, $I\subset\mathbb{R}$ compact interval. The well-known results concerning existence and approximate compactness for generalized rational approximation are included. But in more general cases difficulties may arise, due to the fact that the denominators do not belong to locally compact subsets of X_∞. As a typical example, we consider the approximation by ratios of exponentials. Since proximinal extensions of this family, lying in $C(I)$, cannot be described as its closure in some appropriate topology, we prove proximality for suitable subsets.

1. Introduction. Let $I\subset\mathbb{R}$ be a compact interval and denote by X_p the space $X_p=(L_p(I), \|\cdot\|_p)$ for $1\leq p<\infty$ or $X_\infty=(C(I), \|\cdot\|_\infty)$, respectively. For a set $R\subset X_p$ we have the following:

1.1. Approximation problem. Given $f\in X_p$, find $r_0\in R$, called "best approximation to f", such that $\|f-r_0\|=\inf\{\|f-r\|:r\in R\}=d(f, R)$. R is called "proximinal" or "existence set" iff for all $f\in X_p$ the best approximation $r_0\in R$ exists. If R can be written in the form

$$R=U/V=\{u/v: u\in U, \quad v\in V-\{0\}\}\cap X_p$$

with appropriate sets U and V, we will refer to problem 1.1 as "rational approximation problem".

Our aim is to look for the existence of best approximations, where R is a set of ratios of exponentials. To be more precise, we define for fixed integers n, $n\geq 1$, m, $m\geq 0$:

1.2. Definition.

i) $$V_{n,m}^{(0)}=\{\sum_{j=1}^{n} a_j \exp(p_j): a_j\in\mathbb{R}, \quad p_j\in P_m\},$$

where P_m denotes the set of all polynomials of degree not exceeding m and (see [10])

$$V_{n,m}=cl_{\|\cdot\|_\infty} V_{n,m}^{(0)}=\{\sum_{j=1}^{s} q_j \exp(p_j): p_j\in P_m, q_j\in P_{nm}, \sum_{j=1}^{s}(\partial q_j+m)\leq nm, s\leq n\},$$

here ∂q denotes the degree of the polynomial q. For simplicity, we refer to the elements of $V_{n,m}$ or $V_{n,m}^{(0)}$ as "exponential sums" or "property exponential sums" respectively, thereby extending the terminology for the case $m=1$ (see [4, 11, 14, 15, 16]).

ii) $$R^{n,m}_{\underline{n},\underline{m}} = V_{n,m}/V_{\underline{n},\underline{m}} = \{u/v : u \in V_{n,m}, \quad v \in V_{\underline{n},\underline{m}} \setminus \{\theta\}\} \cap C(I).$$

The elements in $R^{n,m}_{\underline{n},\underline{m}}$ we will call "rational exponential sums". Note that for the definition of $R^{n,m}_{\underline{n},\underline{m}}$ it does not matter whether we take the *intersection* by $C(I)$ or $L_p(I)$; this is an immediate consequence of the analyticity of exponentials.

There are many papers concerning the approximation by elements of $R^{n,1}_{1,0} = V_{n,1}$ (approximation by (ordinary) exponentials). Existence proofs are given for the case $p=\infty$ ([12, 14, 15, 16]) as well as in more general terms for the case $1 \leq p \leq \infty$ ([1, 5, 9, 11]). There are also some results concerning Tschebyscheff-approximation in $V_{n,m} = R^{n,m}_{1,0}$ when $m \geq 2$ ([2, 10]). But it appears that the case of approximation by ratios of exponentials has not been treated yet in the literature. This kind of functions are used in connection with computations of the magnetic field in neutron transport problems.

For our investigations we make use of a strong compactness property of exponentials (p-boundedly Φ_∞-compactness, see definition 2.1). But as it is shown in [7] this is a property of a large class of functions and it seems reasonable to consider the rational approximation problem for a more general set $R = U/V$, where U and V have the compactness property mentioned above. Thus, the theorems given in section 3 will include the well-known results concerning existence and approximately compactness for generalized rational approximation (see e. g. [3]).

In our general notation V does not belong to a locally compact subset of $C(I)$. From this difficulties may arise, which are not known in the common rational approximation problem. As a consequence, a global existence theorem will not hold for arbitrary R, but we give a necessary and sufficient condition for $x \in X_p$ to possess the best approximation in R (section 3).

In section 4 we apply the general results of section 3 to the approximation problem in $R^{n,m}_{\underline{n},\underline{m}}$. Since for no $p (1 \leq p \leq \infty) R^{n,m}_{\underline{n},\underline{m}}$ is proximinal ($\underline{n} \geq 2$, $\underline{m} \geq 1$), we have to decide whether to restrict or to extend $R^{n,m}_{\underline{n},\underline{m}}$ in order to guarantee existence. But proximinal extensions, lying in $C(I)$ cannot be described as usual as closure of $R^{n,m}_{\underline{n},\underline{m}}$ in an appropriate topology. Therefore we prove proximinality for suitable subsets.

2. **Definition, Notations.** Let $I \subset \mathbb{R}$ be a compact interval. For a finite subset $E \subset I$ and $\delta > 0$ we define $E_\delta = \bigcup_{t \in E} (t-\delta, t+\delta) \cap I$ ($=\emptyset$ if $E=\emptyset$) and

$$\chi_\delta = \chi_{I \setminus E_\delta} \quad \text{with} \quad \chi_{I \setminus E_\delta} = \begin{cases} 1 & \text{for} \quad t \in T \setminus E_\delta, \\ 0 & \text{for} \quad t \in E_\delta. \end{cases}$$

We consider the following family of seminorms on X_p $(1 \leq p \leq \infty)$:

$$\Phi_p(E) = \{\varphi_\delta : X_p \to \mathbb{R} : \varphi_\delta(x) = \| x \varkappa_\delta \|_p, \ \delta > 0\}$$

and define:

2.1. Definition ([5]). Let X be a linear space and Φ a family of seminorms on X. A sequence $\{x_k\}\subset X$ is called "Φ-convergent" to $x\in X$, iff $\lim_{k\to\infty}(x-x_k)=0$ for all $\varphi\in\Phi$. For simplicity we will write $x=\Phi\text{-}\lim x_k$.

In the case $X=X_p(1\leqq p\leqq\infty)$ we are interested in the following compactness property:

2.2. Definition. Let $V\subset X_p$ be a (nonvoid) subset $(1\leqq p\leqq\infty)$. V is called "p-boundedly Φ_∞-compact" iff

$$\underset{N\in\mathbb{N}}{\exists}\quad \underset{\substack{\{v_k\}\subset V\\ \|\cdot\|_p-bd.}}{\forall}\quad \underset{\substack{\{w_k\}\subset\{v_k\}\\ \text{subseq.}}}{\exists}\quad \underset{\substack{E\subset I\\ |E|\leqq N}}{\exists}\quad \underset{v\in V}{\exists}\quad v_0=\Phi_\infty(E)\text{-}\lim w_k,$$

here $|E|$ denotes the number of elements in E. We say the set E is related to the sequence $\{w_k\}$. In the notion given above we will speak of Φ_∞-topology.

The p-boundedly Φ_∞-compactness holds, for instance, for sets $V\subset C(I)$, which are the union of solutions of differential equations with certain regularity properties (see [7]).

We now introduce some technical notations.

For $x,\ u,\ v\in C(I)$ we define: $Z_x=\{t\in I\mid x(t)=0\}$, $\|u/v\|_\infty=\sup_{t\in I\setminus Z_v}|u(t)/v(t)|$. We say $r=(u/v)\in C(I)$ if $r(t)=u(t)/v(t)$, $t\notin Z_v$ can be extended by continuity to the points of Z_v.

3. Existence Theorem for General Rational Approximation Problems. As motivated in the previous sections, we will now consider the problem 1.1 making the following assumptions for R:

(A1) $R=U/V:=\{u/v: u\in U,\quad v\in V\setminus\{\theta\}\}\cap X_p$, where $U,\ V\subset C(I)$ homogeneous

$$\text{(i. e. } \alpha\in\mathbb{R},\ u\in U,\ v\in V\Longrightarrow \alpha u\in U,\ \alpha v\in V),$$

(A2) U p-boundedly Φ_∞-compact, V ∞-boundedly Φ_∞-compact.

(A3) There is an $N\in\mathbb{N}$ such that every $v\in V\setminus\{\theta\}$ has at most N zeros in I

(A4) $\|u/v\|_\infty<\infty\Longrightarrow(u/v)\in C(I)$.

It has been shown in [5], that p-boundedly Φ_∞-compact subsets of X_p are proximinal. Thus, it is natural to ask whether assumption (A2) would imply this compactness property for R. But this is not true in general, which may be seen by the following

3.1 Example.

$I=[-1,1]$; for $k\in\mathbb{N}$ define $r_k(t)=(1+e^{kt})^{-1}\in R^{1,0}_{2,1}$. This sequence converges for $1\leqq p<\infty$ in $\|\cdot\|_p$ and for $p=\infty$ $\Phi_\infty(\{0\})$ to r_0, defined as follows:

$$r_0(t)=\begin{cases}1 & \text{for}\quad t\in[-1,0),\\ 1/2 & \text{for}\quad t=0,\\ 0 & \text{for}\quad t\in(0,1].\end{cases}$$

r_0 does not belong to $R^{1,0}_{2,1}$. But assumptions (A1)—(A4) hold for $R^{1,0}_{2,1}$: (A2) follows from [14, 15, 16] and the others are obvious.

What is the reason for this behaviour? Let us write r_k in the normalized form $r_k(t)=(1+e^k)^{-1}/(1+e^k)^{-1}(1+e^{kt})$, $k\in\mathbb{N}$. The $\|\cdot\|_\infty$-norm of the denominators equals $1 (I=[-1, 1])$, but $\lim_{k\to\infty}(1+e^k)^{-1}(1+e^{kt})=0$ for all $t\in[-1, 1)$. This fact leads us to the following

3.2. Definition. For R assumptions (A1)—(A4) may hold $(1\leq p\leq\infty)$.

i) A sequence $\{r_k\}\subset R$ is an "admissible sequence", if there are $u_k\in U$, $v_k\in V\setminus\{\theta\}$ such that $r_k=u_k/v_k$ and

(1) $\|r_k\|_p\leq M$ for all $k\in\mathbb{N}$ with $M\in\mathbb{R}^+$

(2) $\|v_k\|_\infty\leq C$ for all $k\in\mathbb{N}$ with $C\in\mathbb{R}^+$

(3) There is a $v_0\in V\setminus\{\theta\}$ and a subsequence of $\{v_k\}$ converging Φ_∞ to v_0.

ii) $\{r_k\}\subset R$ is an "admissible minimizing sequence" for $x\in X_p$, iff $\{r_k\}$ is an admissible sequence and $\{r_k\}$ is a minimizing sequence for x (that is: $\lim_{k\to\infty}\|x-r_k\|=d(x, R)$).

iii) R is "admissible" iff every $\|\cdot\|$-bounded sequence in R contains an admissible subsequence.

Remark. If $\{r_k\}\subset R$ is a $\|\cdot\|$-bounded sequence, then clearly (2) of definition 3.2i) can be fulfilled because of the homogenity of U and V ((A1)). From (A2) it is clear that the set of Φ_∞-limits of subsequences of $\{v_k\}$ is not empty. Thus, the main requirement for admissible sequences is that there exists a non-zero Φ_∞-limit.

The following theorem states the essential property of admissible sequences.

3.3 Theorem $(1\leq p\leq\infty)$. *For $R\subset X_p$ assumptions* (A1)-(A4) *may hold. Then every admissible sequence in R contains a subsequence converging Φ_∞ to a function of R.*

Proof. Let $\{r_k\}\subset R$ be admissible. We can choose $u_k\in U$, $v_k\in V\setminus\{\theta\}$ such that (1)-(3) of definition 3.2 i) are fulfilled. Because of (A3) we have for all $k\in\mathbb{N}$

$$|r_k(t)|=|u_k(t)|/|v_k(t)|\geq C^{-1}|u_k(t)|,\ t\in I\setminus Z_{v_k}$$

and therefore

(a) $$M\geq\|r_k\|\geq C^{-1}\|u_k\| \quad \text{for all} \quad k\in\mathbb{N},$$

where M and C are the constants defined in 3.2i) (1), (2). Because of definition 3.2i) (3) we may assume without loss of generality that there exists $v_0\in V\setminus\{\theta\}$ such that $\{v_k\}$ converges Φ_∞ to v_0.

Because of (a), $\{u_k\}$ is $\|\cdot\|$-bounded and due to (A2) we may assume (without loss of generality) that $\{u_k\}$ converges to some $u_0\in U$.

Define $r_0=u_0/v_0$; we show that

(1) $\{u_k/v_k\}$ converges Φ_∞ to r_0 and

(2) $r_0\in R$.

Proof of (1). Let E_{u_0} and E_{v_0} be the sets related to $\{u_k\}$ and $\{v_k\}$ respectively (see definition 2.2 i); define $E_0=Z_{v_0}$ and $E=E_0\cup E_{u_0}\cup E_{v_0}$. Let $\delta>0$ be given. Put $K=I\setminus E_\delta$; K is compact and we have

(b) $c=\inf\{|v_0(t)|: t\in K\}>0$ (v_0 continuous and $z_{v_0}\cap K=\emptyset$).

Clearly $\lim_{k\to\infty}\|(v_k-v_0)\chi_\delta\|_\infty=0$ (Φ_∞-convergence of $\{v_k\}$) and there exists $k_0\in\mathbb{N}$ such that for all $k\geq k_0$ and all $t\in K$ we have $|v_k(t)-v_0(t)|\leq c/2$ and consequently $|v_0(t)|-c/2\leq|v_k(t)|$. Thus we have $|v_k(t)|\geq|v_0(t)|-c/2\geq\inf_{t\in K}|v_0(t)|-c/2$ and conclude

(c) $\forall_{t\in K}\ \forall_{k\geq k_0}\quad |v_k(t)|\geq c/2.$

For all $t\in K$ and all $k\geq k_0$ we have

$$|u_k(t)/v_k(t)-u_0(t)/v_0(t)|=|v_k(t)|^{-1}|v_0(t)|^{-1}|v_0(t)u_k(t)-u_0(t)v_k(t)|$$

and from (a) and (b) we get the following estimates

$$\begin{aligned}|r_k(t)-r_0(t)|&=|u_k(t)/v_k(t)-u_0(t)/v_0(t)|\\&\leq 2c^{-2}(|v_0(t)(u_k(t)-u_0(t))|+|u_0(t)(v_0(t)-v_k(t))|)\\&\leq 2c^{-2}(\|v_0\|_\infty\|(u_k-u_0)\chi_\delta\|_\infty+\|u_0\|_\infty\|(v_k-v_0)\chi_\delta\|_\infty).\end{aligned}$$

From the Φ_∞-convergence of $\{u_k\}$ to u_0 and $\{v_k\}$ to v_0, we get $\lim_{k\to\infty}\|(r_k-r_0)\chi_\delta\|_\infty=0$ which shows the Φ_∞-convergence of $\{r_k\}$ to r_0 and part (1) of the proof is complete.

Proof of (2) ($r_0\in R$). *Case 1*; $p=\infty$. Because of the $\|\cdot\|_\infty$-boundedness of $\{r_k\}$ we conclude that $\|u_0/v_0\|<\infty$. By (A4) we have $r_0\in C(I)$ and consequently $r_0\in R$.

Case 2; $1\leq p<\infty$. Because of the continuity of u_0 and v_0 we have $r_0\in C(I\setminus E)$, especially $r_0\in L_p(K)$ for all compact sets $K\subset I\setminus E$. From the boundedness of $\{\|r_k\|_p\}$ (see 3.2 i)(1)) we have $\|r_0\chi_K\|_p\leq M$ for all compact $K\subset I\setminus E$ (M independently from K). Consider a sequence $\{K_j\}$ of compact subsets of $I\setminus E$ with

$$K_1\subset K_2\subset\ldots\subset K_j\subset K_{j+1}\subset\ldots\quad\text{and}\quad\bigcup_{j\in\mathbb{N}}K_j=I\setminus E.$$

Define:

$$r_{K_j}(t)=\begin{cases}|r_0(t)| & \text{for}\quad t\in K_j,\\ 0 & \text{for}\quad t\in I\setminus K_j.\end{cases}$$

Clearly for all $j\in\mathbb{N}$ we have $r_{K_j}\in L_p(I)$, $r_{K_j}(t)\leq r_{K_{j+1}}(t)$ $(t\in I)$ and $\|r_{K_j}\|_p\leq M$. We can apply the theorem of Lebesgue [13, Th. (2.28), p. 180] to conclude $r_0\in L_p(I)$. Consequently, we have $r_0\in R$ q. e. d.

Remark. Assumption (A4) was only used in the case $p=\infty$. This result is fundamental for the theorems concerning existence of best approximations given below. First we quote two lemmas of [5] concerning Φ_p-convergence:

3.4 Lemma. *For $1\leq p\leq\infty$ let $\{x_k\}\subset X_p$ be a norm-bounded sequence, converging Φ_∞ to $x\in X_p$. Then for all $f\in X_p$ we have $\|f-x\|_p\leq\underline{\lim}_{k\to\infty}\|f-x_k\|_p$.*

Proof. From the boundedness of $\{x_k\}$ it is clear that the Φ_∞ convergence to x implies the Φ_p-convergence to x with the same related set E. Because of lemma (6), p. 168 in [5] we need to show that for all $g\in X_p$ there is $\|g\|_p=\sup\{\varphi(g):\varphi\in\Phi_p(E)\}$. But this is an immediate consequence of the definition of $\Phi_p(E)$. q. e. d.

3.5 Lemma. *For $1\leq p<\infty$ let $\{x_k\}\subset X_p$ be a sequence converging Φ_p to $x\in X_p$. Then we have $\lim_{k\to\infty}\|x_k\|_p=\|x\|_p\Longrightarrow\lim_{k\to\infty}\|x_k-x\|_p=0$.*

Proof [5, Lemma (8), p. 174]. We are now in the position to formulate the existence theorems for general rational approximation. But first we will give the following

3.6. Definition ([6]). Let X be a normed linear space and $R \subset X$ a (nonvoid) subset. R is called "approximately compact" iff every minimizing sequence in R contains a norm-converging subsequence with limit in R.

3.7 Theorem. *For* $1 \leqq p \leqq \infty$ *and* $R \subset X_p$ *assumptions* (A1)-(A4) *may hold.*

i) *Then* $x \in X_p$ *possesses the best approximation in* R *if and only if there exists an admissible minimizing sequence for* x *in* R.

ii) *For* $1 \leqq p < \infty$ *every admissible minimizing sequence for* x *in* R *contains a* $\|\cdot\|_p$*-converging subsequence with limit in* R.

Proof. i) Let $r_0 \in R$ be the approximation to x. Then the sequence $\{r_k\}$, $r_k = r_0$, $k \in \mathbb{N}$ is trivially an admissible minimizing sequence. Let, on the other hand, $\{r_k\}$ be an admissible minimizing sequence. $\{r_k\}$ is $\|\cdot\|_p$-bounded and from theorem 3.3 we conclude that some subsequence $\{s_k\} \subset \{r_k\}$ converges Φ_∞ to an element $r_0 \in R$. From lemma 3.4 we have

$$d(x, R) \leqq \| x - r_0 \| \leqq \lim_{k\to\infty} \| x - s_k \| = d(x, R),$$

which implies that r_0 is the best approximation to x in R.

ii) From the proof of i) we know that $\| x - r_0 \|_p = \lim_{k\to\infty} \| x - r_k \|$ and the statement is an immediate consequence of lemma 3.5. q. e. d.

From this result we can easily derive a global existence theorem which applies to a broad class of rational approximation problems:

3.8 Theorem. *For* $1 \leqq p \leqq \infty$ *and* $R \subset X_p$ *assumptions* (A1)-(A4) *may hold.* In addition suppose that R is an admissible set (see definition 3.2 iii). Then

i) $1 \leqq p \leqq \infty$: R is proximinal;

ii) $1 \leqq p < \infty$: R is approximatively compact.

Proof. Because all bounded sequences in R are admissible, we can apply theorem 3.7. q. e. d.

Note that in view of example 3.1 we cannot conclude from assumptions (A1)-(A4) R to be an admissible set in general. But, for instance, in the case when V is a finite dimensional subspace of $C(I)$, this property is true, due to the local compactness of V. Thus, theorem 3.8 includes the results of [3] for generalized rational approximations.

In the next section we will apply the results given above to the approximation by ratios of exponentials.

4. **Approximation by Ratios of Exponentials.** In computations of the magnetic field in neutron transport problems, ratios of exponentials (class $R^{n, m}_{\underline{n}, \underline{m}}$) occur as approximating functions. To get existence theorems, we may apply the results of section 3, but as we already have seen in example 3.1, we cannot obtain at once global existence. Therefore, we could look for proximinal extensions of $R^{n, m}_{\underline{n}, \underline{m}}$ which are (because of the practical background) contained in $C(I)$. To do this, one usually proceeds as follows: Introduce in X_p a topology, in general weaker than the norm-topology but strong enough to guarantee the inequality of lemma 3.4, and consider the closure $\bar{R}$ of R with respect to this topology. If we are able to choose this topology in such a way that $\bar{R}$ is boundedly compact, $\bar{R}$ will clearly be an existence set. In our case, we observe that we cannot introduce a stronger topology than it was done in section 2, to get bounded compactness. But

on the other hand, the ∞-closure of $R^{n,m}_{\underline{n},\underline{m}}$ is not proximinal which can be seen by example 4.2. Thus, we cannot succeed using the method described above. Therefore we look for suitable restrictions of $R^{n,m}_{\underline{n},\underline{m}}$ in order to be able to apply theorem 3.8. Since in view of the example 3.1 $R^{n,m}_{\underline{n},\underline{m}}$ is not closed in X_p for $1 \leqq p < \infty$ and hence is not proximinal, we are mainly concerned with the case $p=\infty$.

First we quote the convergence theorem for exponentials lying in $V_{n,m}$, which is given in [7].

4.1. Theorem. *Let $J \subset \mathbb{R}$ be a connected (not necessarily closed) set and $\{u_k\} \subset V_{n,m}$ a sequence which is uniformly $\|\cdot\|_\infty$-bounded on every compact subset of J (that means:*

$$\underset{\substack{K \subset J \\ \text{K. comp.}}}{\forall} \quad \underset{M \in \mathbb{R}^+}{\exists} \quad \underset{k \in \mathbb{N}}{\forall} \quad \|u_k\|_{\infty,K} = \max_{t \in K} |u_k(t)| \leqq M).$$

Then a subsequence of $\{u_k\}$ converges Φ_∞ to some element belonging to $V_{n,m}$.

Now we show by an example that $R^{1,0}_{2,1}$ is not proximinal in X_∞. By a slight modification of the function to be approximated, it will also be possible to get an example for the general case $(R^{n,m}_{\underline{n},\underline{m}})$.

4.2. Example ($p=\infty$). For $a, b \in \mathbb{R}$, $a \geqq 3$, $1/2 < b < 1$ define $I_1 = [-a, -1]$, $I_2 = [-1, 1+2b]$, $I_3 = [1+2b, a]$ and $I = [-a, a]$. Define

$$x(t) = \begin{cases} 1 & \text{for} \quad t \in I_1, \\ (1-t)/2 & \text{for} \quad t \in I_2, \\ -b & \text{for} \quad t \in I_3, \end{cases}$$

and $r_k(t) = (1+e^{kt})^{-1}$, $k \in \mathbb{N}$. Then we have

i) $\{r_k\}$ is a minimizing sequence for $x(rp\|\cdot\|_\infty)$

ii) $b = d(x, R^{1,0}_{2,1}) < \|x-r\|_\infty$ for all $r \in R^{1,0}_{2,1}$.

Proof. From $r_k(0)=1/2$ and the monotonicity of r_k it is easy to see that $|x(t)-r_k(t)| \leqq 1/2$ for all $t \in [-1, 1]$ and $k \in \mathbb{N}$ and therefore

$$\|x-r_k\|_\infty \geqq \|x-r_{k+1}\|_\infty > b(>1/2) \quad \text{for all} \quad k \in \mathbb{N}.$$

Because of the convergence of $r_k(t)$ to zero on the positive real axis we obtain $\lim_{k\to\infty} \|x-v_k\|_\infty = b$. The functions in $R^{1,0}_{2,1}$ cannot change sign anywhere in $\mathbb{R}$. If $r \in R^{1,0}_{2,1}$ is negative then $\|x-r\|>1$, if r is positive then $\|x-r\|>b$ and for $r=\theta$ we have $\|x-\theta\|=1$. This proves that $\{r_k\}$ is a minimizing sequence for x and moreover that ii) holds. q. e. d.

Remark. If we change x in the interval I_3 to a function oscillating (sufficiently often) about the t-axis with amplitude b, r_k remains a minimizing sequence, now in $R^{n,m}_{\underline{n},\underline{m}}$ $(\underline{n} \geqq 2, \underline{m} \geqq 1)$. That is due to the fact that all functions in $R^{n,m}_{\underline{n},\underline{m}}$ may only have a finite number of changes in sign; thus ii) remains also true for $R^{n,m}_{\underline{n},\underline{m}}$.

We now prove the $\|\cdot\|_\infty$-closedness of $R^{1,0}_{n,m}$.

4.3. Lemma. $R=R^{1,0}_{\underline{n},\underline{m}}=(\mathsf{R}/V_{\underline{n},\underline{m}})\cap C(I)$ $(\underline{n}\geq 2,\ \underline{m}\geq 1)$ *is* $\|\cdot\|_\infty$*-closed*

Proof. Assume $\lim_{k\to\infty}\|x-1/q_k\|_\infty=0$, $q_k\in V_{\underline{n},\underline{m}}(k\in\mathsf{N})$, $x\in X_\infty$. Define $z_x=\{t\in I\mid x(t)=0\}$; because of the continuity of x, Z_x is closed and we have

(*) $\quad t\in Z_x \Leftrightarrow \lim_{k\to\infty}|q_k(t)|=\infty.$

Case 1. $Z_x=\emptyset$. We can easily conclude that there are constants c, C such that for all $k\in\mathsf{N}$ we have $0<c\leq\|q_k\|_\infty\leq C<\infty$ Therefore $\{1/q_k\}$ is an admissible sequence. Thus x belongs to R by theorem 3.3.

Case 2. $Z_x\neq\emptyset$. Choose a compact interval $I_1\subset\mathsf{R}$ such that $I_1\cap Z_x\neq\emptyset$, but int $(I_1)\cap Z_x=\emptyset$ (int (I_1) denotes the interior of I_1). From (*) we see that $\{q_k\}$ satisfies the assumptions of theorem 4.1 in $J=\operatorname{int}(I_1)$ and we conclude that there exists $q_0\in V_{n,m}$ such that

$$x(t)=1/q_0(t) \quad \text{for} \quad t\in I\setminus E,$$

where E contains the endpoints of I_1 and also the related set of the convergence of $\{1/q_k\}$ to $1/q_0$. Take $t_0\in I_1\cap Z_x$ and a sequence $\{t_k\}\subset I_1\setminus E$ converging to t_0. We have

$$0=x(t_0)=\lim_{k\to\infty}x(t_k)=\lim_{k\to\infty}1/q_k(t)=1/q(t_0),$$

which is a contradiction to the continuity of q_0 in R. q. e. d.

We now want to look for proximinal restrictions of $R^{n,m}_{\underline{n},\underline{m}}$. To do this we need some more information about convergence in $\overline{V}_{n,m}$. We quote from [7]

4.4. Lemma. *Let* $\{v_k\}\subset V_{n,m}$ *be a* $\|\cdot\|_\infty$*-bounded sequence converging* Φ_∞ *to* $v_0\in V_{n,m}$. *Suppose* $v_k(t)=\Sigma^s_{j=1}q_{jk}(t)\exp(p_{jk}(t))$, $k\in\mathsf{N}$ *and* $\|p_{jk}\|_\infty\leq c$ *for* $1\leq j\leq s$ *and all* $k\in\mathsf{N}$. *Then* $\{v_k\}$ *converges uniformly on* I *to* v_0.

The proof in [7] uses a theorem of [8] and the fact that the functions of $V_{n,m}$ can be assumed to be solutions of (nice) differential equations (see [2]). Just similar arguments are used in [7] to prove theorem 4.1. An easy consequence of this proof is

4.5. Lemma. *Let* $\{w_k\}\subset V_{n,m}$ *be a* $\|\cdot\|_\infty$*-bounded sequence converging* Φ_∞ *to* $w_0\in V_{n,m}$. *Suppose* $w_k(t)=\Sigma^s_{j=1}q_{jk}(t)\exp(p_{jk}(t))$, $k\in\mathsf{N}$ *and* $\|p_{jk}\|_\infty\to 0$ *for all* $1\leq j\leq s$. *Then* $w_0=\theta$.

4.6 Corollary. *Let* $\{u_k\}\subset V_{n,m}$ *be a* $\|\cdot\|_\infty$*-bounded sequence converging* Φ_∞ *to* $u_0\in V_{n,m}$. *Suppose* $u_k=v_k+w_k\,(k\in\mathsf{N})$ *where the exponents of* $\{v_k\}$ *are bounded and the exponents of* w_k *are unbounded. Then* $\{v_k\}$ *and* $\{w_k\}$ *are* $\|\cdot\|_\infty$*-bounded and* $\lim_{k\to\infty}\|v_k-u_0\|_\infty=0$ (*and* Φ_∞-$\lim w_k=\theta$).

Proof. It is sufficient to show that v_k and w_k are $\|\cdot\|_\infty$-bounded. Assume the countrary; then both sequences are unbounded. Therefore $\{u_k/\|v_k\|\}$ converges uniformly to θ. But, on the other hand, $\{v_k/\|v_k\|\}$ and $\{w_k/\|v_k\|\}$ are both bounded and because of lemma 4.4 and 4.5 we obtain a Φ_∞-limit of $\{u_k/\|v_k\|\}$ not identical to zero. This yields an obvious contradiction. q. e. d.

We now make use of lemma 4.4 to describe proximinal subsets of $R^{n,m}_{\underline{n},\underline{m}}$.

4.7. Definition. i) For $C\in\mathsf{R}^+$ define $V_{n,m}(C)$ by $u=\Sigma^s_{j=1}q_j\exp(p_j)\in V_{n,m}(C)\Leftrightarrow u\in V_{n,m}$ and $\|p_j\|_\infty\leq C$ for all $1\leq j\leq s$.

ii) $R^{n,m}_{\underline{n},\underline{m}}(C):=V_{n,m}/V_{n,m}(C):=\{u/v: u\in V_{n,m},\ v\in V_{\underline{n},\underline{m}}(C)\setminus\{\theta\}\}\cap C(I)$.

We can now use theorem 3.8 to get the following global theorem:

4.8 Theorem. *For $C\in\mathbb{R}^+$ we have*

i) $1\leq p\leq\infty$: $R^{n,m}_{\underline{n},\underline{m}}(C)$ *is proximinal*

ii) $1\leq p<\infty$: $\overline{R}^{n,m}_{\underline{n},\underline{m}}(C)$ *is approximately compact.*

Proof: Because of lemma 4.4, $R^{n,m}_{\underline{n},\underline{m}}(C)$ is an admissible set and we can apply theorem 3.8 to complete the proof. q. e. d.

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