КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 317—325

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 317—325

ANALYTIC APPROXIMATION ON CLOSED SUBSETS OF OPEN RIEMANN SURFACES

P. M. Gauthier*

Dedicated to the memory of Alice Roth

Summary. Let F be a (relatively) closed subset of an open Riemann surface R. Denote by $H(F)$ and $M(F)$ respectively the holomorphic and meromorphic functions on (a neighbourhood of) F. In the case where R is a plane domain, the problem of uniform approximation of functions in $H(F)$ by functions in $M(R)$ or $H(R)$ was completely solved by Alice Roth (1938, 1973) and N. U. Arakeljan (1964, 1968). W. Hengartner and I showed that Roth's conditions, although still necessary, are no longer sufficient for the general situation where R is an arbitrary open Riemann surface. In this paper we shall show that Roth's conditions are, however, sufficient for "weighted approximation".

1. Introduction. Let R^* denote the one point compactification of R.

Theorem. *Let F be closed in an open Riemann surface R, and let G be a neighbourhood of F. Then, there exists a positive continuous function ω on F such that for each $f \in H(G)$ there is a $g \in M(R)$ with*

$$(1) \qquad |f(z)-g(z)|<\omega(z), \quad z \in F.$$

If, moreover, $R^* \setminus F$ is connected and locally connected, then g may be taken to be holomorphic.

This weighted approximation is in general weaker than uniform approximation, as the weight ω may grow very quickly. If, however, F has no interior, we can do better than uniform approximation. Denote by $C(F)$ the continuous functions on F.

Theorem A. *Let F be a closed nowhere dense subset of an open Riemann surface R. In order that for each $f \in C(F)$ and each positive continuous ε on F, there exists a $g \in H(R)$, with*

$$(2) \qquad |f(z)-g(z)|<\varepsilon(z), \quad z \in F;$$

it is necessary and sufficient that $R^* \setminus F$ be connected and locally connected.

This theorem was proved for R a plane domain by N. Arakeljan [1, 2]. On surfaces the necessity was proved in [6] and the sufficiency by S. Scheinberg [14].

* Research supported by N. R. C. of Canada and Ministère de l'Éducation du Québec.

This sort of approximation, which gets better near the boundary, is known as Carleman approximation. Even without the restriction that $F^0=\Phi$, necessary and sufficient conditions are known (necessity [5], sufficiency [10]), in order for a closed subset of a plane domain to be a set of Carleman approximation by holomophic functions. On Riemann surfaces the problem is still open.

For meromorphic Carleman approximation the problem is also open. However, we shall make use of the following special case. By a polygon on R we understand an open set P whose boundary ∂P projects by ϱ to (not necessarily bounded) polygonal curves. For the definition of ϱ, see the first paragraph of Section 2. P is locally finite if $K\cap\partial P$ had only finitely many segments, for each compact set K.

L e m m a. *Let P be a locally finite polygon on an open Riemann surface R. Then for each $f\in C(\partial P)$ and each positive continuous ε on ∂P, there is a $g\in M(R)$ with*

$$|f(z)-g(z)|<\varepsilon(z),\ z\in\partial P. \tag{3}$$

We shall not prove the Lemma, as the proof is similar to but easier than the proof of sufficiency in Theorem A.

In the next section we shall prove our Theorem. In the final section we shall invoke a recent theorem of S. S c h e i n b e r g [14] on uniform approximation, in order to construct proper maps of open Riemann surfaces into $\mathbb{C}^2$.

2. The proof. Using Behnke-Stein techniques, R. G u n n i n g and R. N a r a s i m h a n [7] have shown that every open Riemann surface R can be visualized in a very concrete way. Indeed, they showed that R can be spread (without ramification) above the finite plane $\mathbb{C}$. To be precise, they proved that R admits a locally injective holomorphic function ϱ. Thus $R\xrightarrow[\varrho]{}\mathbb{C}$ is the spread.

We wish to reconstruct the Cauchy kernel of Behnke-Stein on R, something resembling $(q-p)^{-1}$. Conceptually, we prefer to think of p and q as both lying on R, however, for proofs it may be preferable to think of two copies R_p and R_q of R spread respectively above the z and ζ planes:

$$\varrho\times\varrho: R_p\times R_q \longrightarrow \mathbb{C}_z\times\mathbb{C}_\zeta$$
$$(p,q)\longrightarrow(z,\zeta).$$

We construct an open cover of $R\times R$. If $(p,q)\in R\times R$, let D_p and D_q be discs about p and q respectively which lie schlicht over $\mathbb{C}$. Set $U(p,q)=D_p\times D_q$. Consider the Cousin data which is $(\zeta-z)^{-1}$ on $U(p,q)$. Since $R\times R$ is Stein, the first Cousin problem can be solved. Hence there is a meromorphic function Φ on $R\times R$ whose singularities are on the diagonal. In the neighbourhood of a diagonal point, we have, in local coordinates (forever more given by $\varrho\times\varrho$), that

$$\Phi(\zeta,z)-(\zeta-z)^{-1} \tag{4}$$

is holomorphic. $\Phi(\zeta,z)$ means $\Phi(p,q)$, where $\varrho(p)=\zeta$ and $\varrho(q)=z$. We shall persist in this abusive notation, since it is invariant under change of charts within the atlas given by $\varrho\times\varrho$. We call the function Φ a Cauchy kernel on R since

(5) $$\mathrm{Res}_z\,\Phi(\cdot, z)=1.$$

Let us now prove our Theorem. We shall try to keep the proof concise, since we shall in fact lift Roth's proof [12], replacing $(\zeta-z)^{-1}$ in the case $R=\mathbb{C}$ by a Cauchy kernel $\Phi(\zeta, z)$ for general R. Recall that R is spread over $\mathbb{C}$.

Suppose then $f\in H(G)$. We may construct a locally finite closed polygon P for which $F\subset P^0\subset P\subset G$. If in addition $R^*\setminus F$ is connected and locally connected, we may assume the same is true of $R^*\setminus P$. One way to see this is to look at the universal covering surface. For an alternate construction see [14].

Let $\{\varepsilon_\nu\}$ be any sequence of positive constants. Then by our Lemma, if we write $\partial P=\sum_1^\infty p_\nu$, where p_ν are straight line segments, then there is a $g_1\in M(R)$ with

(6) $$|g_1(z)-f(z)|<\varepsilon_\nu,\ z\in p_\nu,\ \nu=1, 2, \ldots$$

Moreover, if $R^*\setminus F$ is connected and locally connected, then by Theorem A, we may take $g_1\in H(R)$. We shall choose the $\{\varepsilon_\nu\}$ later. For $\nu=1, 2\ldots$, the Cauchy integral

(7) $$I_\nu(z)=(2\Pi i)^{-1}\int_{p_\nu}\{g_1(\zeta)-f(\zeta)\}\,\Phi(z, \zeta)d\zeta,$$

is holomorphic on $R\setminus p_\nu$, and if U is a precompact domain in $R\setminus p_\nu$, then we have the estimate

(8) $$|I_\nu|_U\leqq(2\Pi)^{-1}\cdot\varepsilon_\nu\cdot|\Phi|_{U\times p_\nu}\cdot\lambda_\nu,$$

where λ_ν is the length of p_ν.

Thus, if U is precompact in $R\setminus\partial P$, we have for $z\in U$

(9) $$\sum_{\nu=1}^\infty|I_\nu(z)|\leqq\sum_{\nu=1}^\infty(2\,\Pi)^{-1}\cdot\varepsilon_\nu\cdot|\Phi|_{U\times p_\nu}\cdot\lambda_\nu.$$

It will be convenient to write

(10) $$R\setminus\partial P=\bigcup_1^\infty U_\nu,\ \bar{U}_\nu\subset U^0_{\nu+1},$$

where each $\bar{U}_\nu$ is compact.

Now choose ε_ν so that

(11) $$(2\,\Pi)^{-1}\cdot\varepsilon_\nu\,|\Phi|_{U_\nu\times\sum_1^\nu p_j}\cdot\lambda_\nu=2^{-\nu}.$$

Then, for $z\in U_n$ we have

(12) $$\sum_{\nu=1}^\infty|I_\nu(z)|\leqq\sum_{\nu=1}^{n-1}|I_\nu(z)|+2^{-(n+1)}\leqq\sum_{\nu=1}^{n-1}(2\,\Pi)^{-1}\varepsilon_\nu\,|\Phi|_{U_n\times p_\nu}\cdot\lambda_\nu+2^{-(n+1)}.$$

Hence, the integral

(13) $$I(z)=\sum_{\nu=1}^\infty I_\nu(z)=\int_{\partial P}\{g_1(\zeta)-f(\zeta)\}\,\Phi(z, \zeta)d\zeta$$

represents a holomorphic function in $R\setminus\partial P$.

For $z \in F$, set $\nu(z) = \inf \{\nu : z \in U_\nu\}$ and let $\omega(z)$ be any positive continuous function such that

$$\omega(z) \geq \sum_{\nu=1}^{\nu(z)-1} (2\Pi)^{-1} \varepsilon_\nu \cdot |\Phi|_{U_{\nu(z)} \times P_\nu} \cdot \lambda_\nu 2^{-(\nu(z)+1)}. \tag{14}$$

Then

$$|I(z)| < \omega(z), \quad z \in F, \tag{15}$$

and ω does not depend on f.

Now set

$$g_2(z) = \begin{cases} I(z), & z \in R \setminus P, \\ I(z) - g_1(z) + f(z), & z \in P^0. \end{cases} \tag{16}$$

Then g_2 is meromorphic on $R \setminus \partial P$. One of the beautiful surprises of Roth's proof is that g_2 will extend across ∂P to give a function meromorphic on all of R.

Fix a point ζ_0 on ∂P. For simplicity we assume ζ_0 is not a vertex. The reader will easily check that the proof needs only a trivial modification in case ζ_0 is a vertex. Let γ be a straight line segment on ∂P of which ζ_0 is the mid-point and such that the disc K of center ζ_0 and subtended by γ meets ∂P only in γ. We may also assume that g_1 is holomorphic on K. We will show that g_2 extends holomorphically to K. This will verify our claim that g_2 has a meromorphic continuation to all of R.

The segment γ divides K into two half-discs. One of them, say K_2, is in P, and the other K_1 is outside of P. K_2 is bounded by γ and a semicircle which we denote by σ. By Cauchy's theorem,

$$\int_\gamma \{g_1(\zeta) - f(\zeta)\} \Phi(z, \zeta)\, d\zeta = -\int_\sigma \{g_1(\zeta) - f(\zeta)\} \Phi(z, \zeta)\, d\zeta,$$

for $z \in K_1^0$.

Thus, by (16), we have for $z \in K_1^0$,

$$\begin{aligned} g_2(z) = [\, I(z) - (2\Pi i)^{-1} \int_\gamma \{g_1(\zeta) - f(\zeta)\} \Phi(z, \zeta) d\zeta] \\ -(2\Pi i)^{-1} \int_\sigma \{g_1(\zeta) - f(\zeta)\} \Phi(z, \zeta) d\zeta. \end{aligned} \tag{17}$$

However, the expression on the right is holomorphic not only on K_1^0 but on all of K^0. Moreover, in K_2^0 by Cauchy's formula.

$$(2\Pi i)^{-1} \int_{\gamma+\sigma} \{g_1(\zeta) - f(\zeta)\} \Phi(z, \zeta)\, d\zeta = g_1(z) - f(z).$$

Thus the right side of (17) is still equal to g_2 in K_2, and g_2 is holomorphic in K as claimed. Hence, g_2 is meromorphic on all of R.

The function $g(z) = g_1(z) + g_2(z)$ has the required properties, for by (16), on F, $g(z) = f(z) + I(z)$, and so by (15), $|f(z) - g(z)| < \omega(z)$, $z \in F$. Also, it is clear from the construction, that g is holomorphic if $R^* \setminus F$ is connected and locally connected. This completes the proof.

The preceding theorem gives sufficient conditions for weighted approximation. There remains the more important problem of uniform approximation. This is of course equivalent to finding a bounded ω in the preceding theorem; for then it is easy to check that we can replace ω by any scalar multiple and therefore by arbitrarily small constant ε

Let us say that a Cauchy kernel Φ on R is *bounded at infinity* if for any compact set K and any neighbourhood V of K on R, Φ is bounded on both $(R\setminus V)\times K$ and $K\times(R\setminus V)$. Note that $(\zeta-z)^{-1}$ has this property on R. Suppose now, R has a Cauchy kernel bounded at infinity. Then

$$|\Phi|_{F\times\sum_{1}^{\nu}p_j}=C_\nu<+\infty,\quad \nu=1,2,\dots\,. \tag{18}$$

From (8), we have that

$$|I_\nu(z)|\leqq(2\Pi)^{-1}\cdot\varepsilon_\nu\cdot C_\nu\cdot\lambda_\nu,\ z\in F. \tag{19}$$

Thus, for $z\in F$, we have

$$|I(z)|\leqq\sum_{\nu=1}^{\infty}(2\Pi)^{-1}\cdot\varepsilon_\nu\cdot C_\nu\cdot\lambda_\nu,$$

and hence for an appropriate choice of the ε_ν's, we have uniform approximation.

Corollary. *Let F be closed in an open Riemann surface R, and suppose R admits a Cauchy kernel bounded at infinity. Then, for each $f\in H(F)$ and constant $\varepsilon>0$, there is a $g\in M(R)$ with $|f(z)-g(z)|<\varepsilon$, $z\in F$. If moreover $R^*\setminus F$ is connected and locally connected, then g may be taken to be holomorphic.*

We do not know whether the above condition on the Cauchy kernel is also necessary.

As an example of a Riemann surface having a Cauchy kernel bounded at infinity, let R be a precompact domain in a larger open Riemann surface R'. Let Φ' be a Cauchy kernel for R' and let Φ be the restriction of Φ' to R. Then Φ is bounded at infinity. In particular, any compact bordered Riemann surface satisfies the hypothesis of the corollary.

By the same reasoning, the property of having a Cauchy kernel bounded at infinity is hereditary Thus we retrieve Roth's Runge theorem for unbounded sets [12, 13] since every domain in C satisfies the hypothesis of the corollary. Of course, we recall for the last time that we have been mimicking Roth's proof all along.

S. Scheinberg [14] has a theorem on uniform approximation on Riemann surfaces. A particular case of his theorem is the following.

Theorem B (S. Scheinberg [14]). *Suppose F^{0-} has a covering by a locally finite family of pairwise disjoint open sets, each of finite genus. Then, in order that for each f continuous on F and holomorphic on F^0 and each positive constant ε, there exists a $g\in H(R)$, with $|f(z)-g(z)|<\varepsilon$, $z\in F$; it is necessary and sufficient that $R^*\setminus F$ be connected and locally connected.*

As mentioned before, the necessity of the condition on $R^*\setminus F$ was shown earlier [6].

3. Proper maps into $\mathbb{C}^2$. In this section we shall use Scheinberg's Theorem to construct proper maps of arbitrary open Riemann surfaces into $\mathbb{C}^2$. That is, we construct a holomophic map, $g: R\longrightarrow\mathbb{C}^2$, such that, as z tends to the ideal boundary of R, $g(z)$ tends to the ideal boundary of $\mathbb{C}^2$.

The result itself is not new. In fact, Bishop [3] has shown that a Stein manifold of dimension n always admits a proper map into $\mathbb{C}^{n+1}$. However, perhaps our approach will shed some light on the following problem. Can every open Riemann surface be properly embedded into $\mathbb{C}^2$? The answer is yes when R is simply connected, or when R is an annulus. For R simply connected, the proof was given by Nishino (see [8] for an account in English). For R an annulus, the result is Laufer's [9]. For R a punctured disc, see [15]. But let us return to our proper map.

First we outline the construction of g.

a) We cover R with two closed sets F_1 and F_2, with locally-finite, simply-connected components, such that the ideal boundary of R has harmonic measure zero for each of these components.

b) For $j=1,2$; we construct f_j continuous on F_j and holomorphic on F_j^0 with the property that $f_j(z) \to \infty$, as z goes to the ideal boundary of R along F_j.

c) Using Scheinberg's theorem, for $j=1,2$; there is a $g \in H(R)$ with $|f_j(z) - g_j(z)| < 1$, $z \in F_j$. Then, $g=(g_1, g_2)$ is the desired proper map.

In order to construct the cover (F_1, F_2) it will be convenient to introduce the Stoïlow-Kérékjártó representation $\widetilde{R}$ of R [11]. The Riemann surface S is topologically equivalent to a surface $\widetilde{R}$ constructed as follows. Select a nonempty compact totally disconnected subset β of the real axis and a sequence Δ_n of disjoint closed discs in the upper half-plane which cluster in $\mathbb{C}^*$ only at points of β. Remove from $\mathbb{C}^*$ all points of $\cup \Delta_n^0$ and the complex conjugates of these points. Identify each point of $\partial\Delta_n$ with its complex conjugate. Also remove all points of β from this surface. Every open Riemann surface R is homeomorphic to a surface $\widetilde{R}$ of the type just constructed. In our discussion we shall switch back and forth between R and $\widetilde{R}$ depending on whether the particular question we are discussing is primarily concerned with the complex structure or the topological structure.

We shall now construct a cover $(\widetilde{F}_1, \widetilde{F}_2)$ of $\widetilde{R}$ which has all of the properties required in a) except for the property concerning harmonic measure. We construct a set S^+ in the upper half-plane on $\widetilde{R}$; and we may visualise S^+ as a root system, or upside down tree. For each Δ_n, S^+ has a component S_n^+ constructed as follows. Initially S_1^+ is a curve starting from a point on $\partial\Delta_1$ and heading in the direction of β while avoiding the other Δ_n. The curve may branch along the way and the branches (or roots) may themselves branch again and again. The root system S_1^+ must grow down in such a way that for each point $b \in \beta$, there is a path in S_1^+ reaching down to b. Moreover, S_1^+ reaches down only to points of β and S_1^+ is simply connected (i. e. two branches never fuse). For each $n>1$, inductively construct S_n^+ as a simple path starting out at a point on $\partial\Delta_n$ and proceeding down to a point of β without including that point and without intersecting any other Δ_j or any preceding S_j^+. For each n, let S_n^- be the set of points conjugate to the points of S_n^+. Finally, if $[\alpha, \beta]$ is the smallest closed interval containing β, let $S_\infty = [\infty, \alpha) \cup (\beta, \infty]$. Now set $S = \bigcup_n S_n^+ \cup S_n^- \cup S_\infty$. Then each component of S on R is simply connected, and the reader can verify that the same is true of $\widetilde{R}\setminus S$. But $\widetilde{R}\setminus S$ is not closed. To rectify this we may take a suitable closed neighbourhood $\widetilde{F}_1$ of S, such that $\widetilde{F}_1$ is a two-dimensional tree which retracts the one-dimensional tree S. Now let $\widetilde{F}_2=(\widetilde{R}\setminus\widetilde{F}_1)^-$. The cover $(\widetilde{F}_1, \widetilde{F}_2)$ of $\widetilde{R}$ has all of the properties in a), except the one concerning harmonic measure.

In order to achieve this additional requirement we shall twist $\widetilde{F}_1$ (and therefore $\widetilde{F}_2$) so that it spirals into each point of β. Thus the probability will be zero, that a particle moving at random on $\widetilde{F}_j^0$, should reach the ideal boundary β, without hitting $\partial\widetilde{F}_j$. But this is the same as saying that the ideal boundary of $\widetilde{R}$ has harmonic measure zero for (each component of) each $\widetilde{F}_j$. However, this argument is not topological, and cannot be made rigourous on $\widetilde{R}$. We must return to R for a convincing argument.

Let R_j, $j=1, 2, \ldots$, be a normal exhaustion of R. Thus, each R_j is bounded by finitely many disjoint Jordan curves; $R^*\setminus R_j$ is connected, and $\bar{R}_j\subset R_{j+1}$, $j=1, 2, \ldots$. Let $\widetilde{R}_j$ be the image of R_j on $\widetilde{R}$, for $j=1, 2, \ldots$. For geometric simplicity, we may assume that each $\widetilde{R}_j$ is symmetric with respect to the real axis and misses each $\partial\Delta_n$. We may also assume that each boundary curve of $\widetilde{R}_j$ is convex. Thus it is geometrically clear that we can construct our root system $\widetilde{F}_1$ in such a way that it is monotonic with respect to the exhaustion $\{\widetilde{R}_j\}$. That is, as each root of $\widetilde{F}_1$ grows towards β, it must eventually leave each domain $\widetilde{R}_j$. Monotonicity means that once it leaves $\widetilde{R}_j$, it never returns. Of course, $\widetilde{F}_2$ is then also monotonic with respect to $\{\widetilde{R}_j\}$. A rigourous way of saying this is that each $\widetilde{R}\setminus(\widetilde{R}_j\cup\widetilde{F}_k)$ has no relatively compact components, for $k=1, 2$ and $j=1, 2, \ldots$. In particular, our construction is such that $R^*\setminus F_k$ is connected and locally connected, for $k=1, 2$. Thus F_1 and F_2 satisfy the hypotheses of Scheinberg's theorem. We shall, of course, use this fact in c).

For each j, let C_j be a collar about ∂R_j. That is, C_j consists of finitely many disjoint annuli, one about each component of ∂R_j. We may assume that for each j, $\bar{C}_j\subset R_{j+1}$, and $\bar{C}_j$ is disjoint from each $\partial\Delta_n$. We may also assume that F_1 (and hence F_2) is monotonic with respect to the collars. That is, if a branch enters one side of an annulus, then it leaves by the other side and never returns.

In order to achieve the desired effect on harmonic measure, we shall modify F_1 (and hence F_2) only within the collars. Each collar has an inner boundary $\partial_i C_j$ (in R_j) and an outer boundary $\partial_0 C_j$ (disjoint from R_j). We shall modify F_1 (and hence F_2) within each C_j, by fixing $F_1\cap\partial_i C_j$ and rotating $F_1\cap\partial_0 C_j$ by t_j (to be specified) full turns about $\partial_0 C_j$. Please note that we do not twist the surface itself, but rather we are modifying the definition of F_1 (and hence F_2). We continue to denote the twisted cover by (F_1, F_2). Clearly there is a homeomorphism of the surface, which *does* twist the surface itself and which carries the original cover into the twisted cover. Hence F_1 and F_2 continue to satisfy the hypotheses of Scheinberg's theorem.

Now we specify the number, t_j, of twists on the collar C_j. Let W_ν, $\nu=1, 2, 3, \ldots$, be the components of F_1 and of F_2. For each ν, fix a point

$a_\nu \in W^0_\nu$. No W_ν is compact, and so part of the ideal boundary of each W_ν lies at the ideal boundary of R. Our goal is to so twist F_1 (and hence F_2 and hence each W_ν), that the harmonic measure for W_ν of the ideal boundary of R, evaluated at a_ν, is zero. We may order the W_ν, and choose the points a_ν in such a way that $a_\nu \in R_j \setminus C_j$, $\nu \leq \nu(j)$, where the $\nu(j)$ are non-decreasing. Set $W^j_\nu = W_\nu \cap (R_j \cup C_j)$, $\nu \leq \nu(j)$, and let ω^j_ν be the harmonic measure for W^j_ν of $\partial W^j_\nu \cap \partial_0 C_j$. Since W^j_ν, $\nu \leq \nu(j)$ are finite in number, there is a t_j such that, by twisting C_j in the manner indicated earlier t_j times, we may be certain that

$$\omega^j_\nu(a_\nu) < 1/j, \; \nu \leq \nu(j). \tag{20}$$

This is because, twisting $W^j_\nu \cap C_j$ many times, decreases the probability that a particle moving randomly from a_ν would first meet ∂W^j_ν along $\partial_0 C_j$. If we are uncomfortable with such an argument on a Riemann surface, then we may note that this probability is the product of the probability that the particle will reach $\partial_i C_j \cap W^j_\nu$ and the probability that a particle will go from $\partial_i C_j \cap W^j_\nu$ to $\partial W^j_\nu \cap \partial_0 C_j$. The first probability is a fixed quantity independent of the twist, and our argument certainly is valid for the second probability becoming small since C_j is planar.

Finally, let ω_ν be the harmonic measure, for W^0_ν, of the ideal boundary of R. Then $\omega_\nu(a_\nu) \leq \omega^j_\nu(a_\nu)$, for all large j, by the Erweiterungs Prinzip. Hence by (20), we have that $\omega_\nu(a_\nu) = 0$, for each ν, which completes the construction a).

We shall now construct f_1 to satisfy b). It will be clear that the construction of f_2 can be done analogously. Let W_ν be a component of F_1. From our construction of F_1, we may assume that each component of ∂W_ν is a simple curve both "ends" of which tend to the ideal boundary of R. Moreover, since these arcs on the boundary are free, the prime end theory applies along ∂W_ν, and so if φ_ν is a conformal map of W^0_ν onto the unit disc U, it follows that ∂W_ν corresponds to an open subset of ∂U. In particular, the ideal boundary of R corresponds to a closed subset E of ∂U. Since harmonic measure is a conformal invariant, it follows from a), that E has measure zero. Following Fatou [4], we may construct a holomophic function h_ν on U continuous at each point of $\partial U \setminus E$, infinite at each point of E, and such that Re $h_\nu \geq \nu$. Now set $f_1 = h_\nu \circ \varphi_\nu$ on W_ν. From the prime end theory, it follows that f_1 is continuous on W_ν, holomorphic on W^0_ν, and that $f_1(z) \to \infty$, as z goes to the ideal boundary of R along W_ν. We define f_1 in a similar fashion on each component W_ν of F_1. Since these components are locally finite, f_1 is also continuous on all of F_1, and of course, holomorphic on F^0_1. Also, since Re $f_1 \geq \nu$ on W_ν, we have that $f_1(z) \to \infty$, as z goes to the ideal boundary of R along F_1. The construction of f_2 on F_2 follows the same pattern. This completes the step b).

Step c) is self evident.

The construction of the proper map g is complete.

REFERENCES

1. Н. У. Аракелян. О равномерном приближении целыми функциями на замкнутых множествах. *Известия АН СССР, сер. мат.*, **28**, 1966, 1187—1206.
2. Н. У. Аракелян. Равномерные и касательные приближения аналитическими функциями. *Известия АН Арм. ССР (мат.)*, **3**, 1968, 273—286.
3. E. Bishop. Mappings of Partially Analytic Spaces. *Amer. J. Math.*, **83**, 1961, 209—242.
4. P. Fatou. Séries trigonométriques et séries de Taylor. *Acta Math.*, **30**, 1906, 335—400.
5. P. M. Gauthier. Tangential Approximation by Entire Functions and Functions Holomorphic in a Disc. *Известия АН Арм. ССР (мат.)*, **4**, 1969, 319—326.
6. P. M. Gauthier, W. Hengartner. Uniform Approximation on Closed Sets by Functions Analytic on a Riemann Surface. Approximation Theory. Dordrecht, 1975, 63—70.
7. R. C. Gunning, R. Narasimhan. Immersion of Open Riemann Surfaces. *Math. Ann.*, **174**, 1967, 103—108.
8. S. Hitotumatu. Some Recent Topics in Several Complex Variables by the Japanese School. Proceedings of the Romanian-Finnish Seminar Bucarest, 1971, 187—192.
9. H. B. Laufer. Imbedding Annuli in C^2. *J. Analyse Math.*, **26**, 1973, 187—215.
10. A. A. Нерсесян. О множествах Карлемана. *Известия АН Арм. ССР, мат*, **6**, 1971, 465—471.
11. I. Richards. On the Classification of Noncompact Surfaces. *Trans. Amer. Math. Soc.*, **106**, 1963, 259—269.
12. Alice Roth. Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen. *Com. Math. Helv.*, **11**, 1938, 77—125.
13. Alice Roth. Meromorphe Approximationen. *Com. Math. Helv.*, **48**, 1973, 151—176.
14. S. Scheinberg. Uniform Approximation by Functions Analytic on a Riemann Surface. *Ann. Math.*, **108**, 1978, 257—298.
15. H. Alexander. Explicit Imbedding of the (Punctured) Disc into C^2. *Notices Amer. Math. Soc.*, **24**, 1977, A-472.

Université de Montréal
Montréal Canada

Received September 14, 1977