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ZAMANSKY-TYPE THEOREMS FOR APPROXIMATION WITH EXPONENTIAL ORDERS

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Summary. For classical rates of convergence Zamansky's theorem is sharp, i. e. if the best approximation of f is $E_n(f)=O(n^{-\tau})$, $n\to\infty$, then the polynomials p_n of best approximation satisfy $\|p_n^{(r)}\|=O(n^{r-\tau})$, $n\to\infty$ $(r>\tau)$, and conversely. There is an analogous result for exponential orders, i. e. for rates of convergence $O(1/\varphi(n))$, $n\to\infty$, where $\varphi(n)/\varphi(2n)\to 0$, $n\to\infty$, thus for orders $\varphi(x)$ like $\exp(\alpha\{\log(1+x)\}^{\beta})$, $\beta>1$ or $\exp(\alpha x^{\beta})$, $0<\beta<1$, $\alpha>0$, etc. But here the converse only holds under a certain additional condition upon the orders φ involved. If this condition is not satisfied, the problem arises to characterize the classes of functions f for which a Zamansky-type inequality with a slower rate of increase of the (generalized or usual) derivatives of the p_n holds. An account of partial solutions of this problem is given as well as of the corresponding problem with generalized de La Vallée Poussin sums as approximants. In particular, the role of lacunarity conditions upon the Fourier coefficients of f is investigated.

In [3], the fundamental theorems of Jackson, Bernstein, Zamansky and Steckin on the rate of best approximation have been extended to exponential orders, i. e. to rates of convergence $O(1/\varphi(n))$, $n\to\infty$, where $\varphi(n)$ increases more rapidly than any power n^{τ}, $\tau>0$. One result of [3] was that the validity of an inverse Zamansky-type theorem now depends on a certain additional condition upon the two exponential orders involved, namely the order of best approximation and the order governing the generalized differentiation operator. As was already indicated in [3], this condition cannot be omitted and, for classical rates of convergence, i. e. $O(1/\varphi(n))$, $n\to\infty$ with $\varphi(2n)=O(\varphi(n))$, $n\to\infty$, it is trivially satisfied. This suggests that there should exist sharper versions of Zamansky's theorem, and it is the purpose of this note to establish some examples of this type. In particular it will be shown that the additional condition mentioned above is sharp. Moreover, quantitative results are obtained which give the best possible version of Zamansky's theorem in $L^2_{2\pi}$ and a lower bound for the best version in $C_{2\pi}$, $L^p_{2\pi}$, $1\le p<\infty$. If the polynomials of best approximation are replaced by an appropriate linear process, further improvements are shown to be possible.

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1. Preliminaries. The following notations will be used: $\mathsf{P}=\{0, 1, 2\dots\}$, $\mathsf{N}=\{1, 2, \dots\}$, Z the set of all integers, $C^k(0, \infty)$ the space of continuous functions f defined on $(0, \infty)$ such that $f^{(k)}$ exists and is continuous there, $k\in\mathsf{N}$. Constants like c, c_1, c_2, x_0 are always meant to be positive and may have different values at each occurrence. By exponential orders of approximation we denote functions φ in estimates of a rate of convergence like $O(1/\varphi(n))$, $n\to\infty$, selecting for φ the elements of the class Φ below.

$$\Phi_0=\{\varphi(x);\varphi\colon [0, \infty)\to[1, \infty),\ \varphi(0)=1,\ \varphi\in C^1(0, \infty),$$
$$\varphi'(x)>0\quad \forall x>0,\quad \lim_{x\to\infty}\varphi(x)=+\infty\}$$

$$(1.1)\qquad \Phi=\{\varphi\in\Phi_0;\ \varphi(x)=e^{g(x)},\ g\in C^3(0, \infty),\ \exists x_0>0 \text{ with } g''(x)<0,$$
$$g'''(x)\geq 0\quad \forall x\geq x_0,\quad\text{and}\quad \lim_{x\to\infty}|g''(x)|(g'(x))^{-2}<1\}.$$

Thus the orders $\varphi\in\Phi$ range from “classical” ones like $\varphi(x)=(1+x)^{\beta}$, $\beta>1$ to “exponential” ones like $\varphi(x)=\exp(\alpha x^{\tau})$, $\alpha>0$, $0<\tau<1$. The above definition of Φ is that of [2]; for technical reasons it is slightly more restricted than that in [3]. (In fact, the condition that $g'''(x)\geq 0$ $\forall x\geq x_0$ is used in this paper only for the verification of a Jackson type inequality via [2, I]; see the remark at the end of Sec. 2).

Given $\varphi(x)=e^{g(x)}\in\Phi$, we associate to φ a function $k(x)$ on $(0, \infty)$ by

$$(1.2)\qquad k(x)=1/g'(x)$$

and denote by $k\leftrightarrow\varphi$ a pair of functions related by (1.2). If ψ is another element of Φ with $h\leftrightarrow\psi$, we say that ψ belongs to the class K_φ if $h(x)$ and $k(x)$ have the same rate of growth for $x\to\infty$, thus

$$(1.3)\qquad K_\varphi=\{\psi\in\Phi;\ h(x)\approx k(x),\quad x\to\infty,\quad h\leftrightarrow\psi,\quad k\leftrightarrow\varphi\},$$

where $h\approx k$ stands for the existence of constants $c_1, c_2, x_0>0$ such that $c_1<h(x)/k(x)<c_2$ for each $x\geq x_0$. (For historical comments cf. also [3]).

The following properties of orders $\varphi\in\Phi$ will be needed.

Lemma 1. *Let φ, $\varphi^*\in\Phi$ with $\varphi(x)=e^{g(x)}$, $\varphi^*(x)=e^{g^*(x)}$ and $k\leftrightarrow\varphi$, $k^*\leftrightarrow\varphi^*$* (cf. (1.2)). *Then*

a) $0\leq\lim_{x\to\infty} k(x)/x<1$,

b) $x-k(x)\to\infty;\quad x\to\infty$,

c) $2\int_{x_0}^{\infty}(\varphi(x))^{-2}(k(x))^{-1}dx=(\varphi(x_0))^{-2};\quad x_0>0$,

d) $\dfrac{k^*(x+1)}{k(x+1)}\,\dfrac{\varphi^*(x+1)}{\varphi(x+1)}\approx\dfrac{k^*(x)}{k(x)}\,\dfrac{\varphi^*(x)}{\varphi(x)};\quad x\to\infty.$

e) *If also $\varphi^*(x)/\varphi(x)\in\Phi$ then* $0<\lim_{x\to\infty}\varphi^*(x)/\varphi^*(x+1)$
$\leq\lim_{x\to\infty}\varphi^*(x)\varphi(x+1)\{\varphi(x)\varphi^*(x+1)\}^{-1}<\infty.$

Proof. a) By (1.1), (1.2), $k(x)$ is a positive function and increases for large x, thus either $\lim_{x\to\infty}k(x)$ is finite, then assertion a) is trivial, or $k(x)$ tends to infinity. In the latter case an application of l'Hospital's rule yields a), observing (1.1) and, for large x, (1.4) $$k'(x)=|g''(x)|(g'(x))^{-2}.$$

b) is a consequence of $\lim_{x\to\infty}(x-k(x))'>0$, which holds in view of (1.1), (1.4), and c) follows by integrating over $(d/dx)(\varphi(x))^{-2} = -2\{(\varphi(x))^2k(x)\}^{-1}$. For d), the mean value theorem is used

$$\frac{k^*(x+1)}{k(x+1)}\ \frac{\varphi^*(x+1)}{\varphi(x+1)}\{\frac{k^*(x)}{k(x)}\frac{\varphi^*(x)}{\varphi(x)}\}^{-1}=\exp\{\log k^*(x+1)-\log k(x+1)$$
$$+g^*(x+1)-g(x+1)-[\log k^*(x)-\log k(x)+g^*(x)-g(x)]\}$$
$$=\exp\{\frac{k^{*\prime}(\xi_x)}{k^*(\xi_x)}-\frac{k'(\xi_x)}{k(\xi_x)}+g^{*\prime}(\xi_x)-g'(\xi_x)\}$$
$$=\exp\{g^{*\prime}(\xi_x)(k^{*\prime}(\xi_x)+1)-g'(\xi_x)(k'(\xi_x)+1)\},$$

where $x<\xi_x<x+1$, using (1.2) in the last identity. By (1.1), $g^{*\prime}(x)$ and $g'(x)$ are positive and eventually decreasing, and also $k^{*\prime}(x)$, $k'(x)$ tend to a finite number as $x\to\infty$ (cf. (1.4)). (We say that $g'(x)$ eventually decreases if there exists $x_0\geqq 0$ such that $g'(x)$ is decreasing for $x\geqq x_0$). Hence d) follows by letting x tend to infinity.

e) Since $\varphi^*(x)/\varphi(x)=\exp\{g^*(x)-g(x)\}\in\Phi$, the following limits exist and are non-negative by (1.1)

$$\alpha=\lim_{x\to\infty} g'(x),\quad \beta=\lim_{x\to\infty} g^{*\prime}(x),\quad \beta-\alpha=\lim_{x\to\infty}(g^{*\prime}(x)-g'(x)).$$

Thus $0<\lim_{x\to\infty}\exp\{-g^{*\prime}(x)\}=e^{-\beta}\leqq e^{\alpha-\beta}=\lim_{x\to\infty}\exp\{g'(x)-g^{*\prime}(x)\}<\infty$. As above there exist ξ_x, η_x with $x<\xi_x\ \ \eta_x<x+1$, so that $\varphi^*(x)\varphi(x+1)\{\varphi(x)\times\varphi^*(x+1)\}^{-1}=\exp\{g'(\xi_x)-g^{*\prime}(\xi_x)\}$,

$$\varphi^*(x)/\varphi^*(x+1)=\exp\{-g^{*\prime}(\eta_x)\}.$$

This implies e) since $0<\lim_{x\to\infty}\varphi^*(x)/\varphi^*(x+1)=e^{-\beta}\leqq e^{\alpha-\beta}=\lim_{x\to\infty}\varphi^*(x)\times\varphi(x+1)\{\varphi(x)\varphi^*(x+1)\}^{-1}<\infty$.

Lemma 2. *Let φ, $\varphi^*\in\Phi$, $k\leftrightarrow\varphi$, $k^*\leftrightarrow\varphi^*$, and $\varphi^*/\varphi\in K_{\varphi^*}$. Then one has*

a) i) $\limsup_{x\to\infty} k^*(x)/k(x)<1$,

ii) $\varphi\in K_{\varphi^*}$ *iff* $\liminf_{x\to\infty} k^*(x)/k(x)>0$,

iii) $\varphi\notin K_{\varphi^*}$ *iff* $\liminf_{x\to\infty} k^*(x)/k(x)=0$;

b) i) *the function* $F(x)=(\varphi^*(x)/\varphi(x))^2(k^*(x)/k(x))$ *is eventually strictly increasing,*

ii) *for any x_0, large enough and fixed, one has*

$$\int_{x_0}^{x}(\frac{\varphi^*(t)}{\varphi(t)})^2\frac{dt}{k(t)}\approx F(x),\quad x\to\infty.$$

Proof. a) i) is an immediate consequence of the assumption that $\varphi^*/\varphi\in\Phi$, and a) ii), iii) are obvious variants of (1.3), observing that $k^*(x)<k(x)$, $x>0$.
Concerning b) i), one has (cf. (1.1), (1.4))

(1.5) $$F'(x)=(\frac{\varphi^*(x)}{\varphi(x)})^2 2(g^{*\prime}(x)-g'(x))\frac{k^*(x)}{k(x)}\{1+G(x)/2(1-k^*(x)/k(x))\},$$

where $G(x)=k^{*\prime}(x)-k'(x)k^*(x)/k(x)$. The factors outside the curly brackets are positive since $\varphi^*/\varphi\in\Phi$ implies $g^{*\prime}(x)-g'(x)>0$ and

$$(1.6)\qquad 0<k^*(x)/k(x)<1;\quad x>0.$$

By a) i) there exists a constant c such that, for large x,

$$1/2<[2(1-k^*(x)/k(x))]^{-1}<c,$$

whence the expression in curly brackets will be eventually positive if it is shown that

$$(1.7)\qquad \lim_{x\to\infty} G(x)\geq 0.$$

From (1.1), (1.4) it follows that $0\leq\lim_{x\to\infty} k'(x)$, $\lim_{x\to\infty} k^{*\prime}(x)<1$. Thus, if $\lim_{x\to\infty}k'(x)=0$, (1.7) is trivial in view of (1.6). If $\lim_{x\to\infty}k'(x)>0$, it follows that $\lim_{x\to\infty}k(x)=+\infty$, and, assuming that also $\lim_{x\to\infty}k^*(x)=+\infty$, l'Hospital's rule implies $\lim_{x\to\infty}\{(k^*(x)/k(x))-k^{*\prime}(x)/k'(x)\}=0$. If $\lim_{x\to\infty}k^*(x)<\infty$, trivially $\lim_{x\to\infty} k^*(x)/k(x)=0$. Thus in both cases (1.7) holds, hence $F'(x)>0$ for large x.

For the proof of b) ii) we use that

$$(1.8)\qquad F'(t)[(\frac{\varphi^*(t)}{\varphi(t)})^2\frac{1}{k(t)}]^{-1}\approx 1;\quad t\to\infty,$$

which follows from the fact that in (1.5) the expression in curly brackets tends to a positive limit as $x\to\infty$, and from the assumption $\varphi^*/\varphi\in K_{\varphi^*}$, i. e. $2(g^{*\prime}(x)-g'(x))k^*(x)\approx 1$ as $x\to\infty$. Integrating (1.8), we have, for suitably large x_0,

$$\int_{x_0}^{x}(\frac{\varphi^*(t)}{\varphi(t)})^2\frac{dt}{k(t)}\approx F(x)-F(x_0);\quad x\to\infty,$$

whence the assertion follows in view of $F(x)-F(x_0)\approx F(x)$, $x\to\infty$, which is a consequence of b) i).

2. A set of functions for which Zamansky's inequality cannot be improved. We briefly summarize the results of [3] concerning the extension to exponential orders of Zamansky's inequality and its converse. Let X be a Banach space, $\{M_n\}_{n\in\mathsf{P}}$ a sequence of linear subspaces of X, and $E_n[f]=\inf\{\|f-p\|_X;\ p\in M_n\}$ the error of best approximation of an $f\in X$ by elements of M_n in X-norm. We assume that

(W) $\lim_{n\to\infty} E_n[f]=0$ $\forall f\in X$ (Weierstrass property),

(E) for each $f\in X$, $n\in\mathsf{P}$, there exists $p_n^0=p_n^0(f)$ with $\|f-p_n^0\|_X=E_n[f]$,

(M) $M_n\subset M_{n+1}$ $\forall n\in\mathsf{P}$.

Moreover, let Y be a linear subspace of X, equipped with a seminorm $|\cdot|_Y$, satisfying

(S_Y) $M_n\subset Y$ $\forall n\in\mathsf{P}$,

as well as Jackson- and Bernstein-type inequalities of order φ^* for some $\varphi^*\in\Phi$:

$$(J_Y)\quad E_n[f]\leq c(\varphi^*(n))^{-1}|f|_Y;\quad n\in\mathsf{P},\ f\in Y,$$
$$(B_Y)\quad |p_n|_Y\leq c\varphi^*(n)\,\|p_n\|_X;\quad n\in\mathsf{P},\ p_n\in M_n,$$

where c only depends on φ^*.

If X, Y, $\{M_n\}$, φ^* are such that (W), (E), (M), (S_Y), (J_Y), (B_Y) hold, we shall say that X, Y, $\{M_n\}$, φ^* satisfy the "basic assumptions" (BA). In [3], the following extensions of Zamansky's theorem and its converse were proved.

Proposition 1. *Let* X, Y, $\{M_n\}$, φ^* *satisfy* (M), (S_Y) *and* (B_Y). *If, for an* $f\in X$, *there exists a sequence* $\{p_n\}_{n\in\mathrm{P}}$ *with* $p_n\in M_n$ *for each* $n\in\mathrm{P}$, *such that* $\|p_n-f\|_X=O(1/\varphi(n))$, $n\to\infty$, *where* $\varphi\in\Phi$ *and* $\varphi^*/\varphi\in K_{\varphi^*}$ *then* $|p_n|_Y=O(\varphi^*(n)/\varphi(n))$; $n\to\infty$.

Proposition 2. *Let* X, Y, $\{M_n\}$, φ^* *satisfy* (W), (E), (S_Y) *and* (J_Y). *If, for an* $f\in X$ *and some* $\varphi\in\Phi$ *with* $\varphi\in K_{\varphi^*}$,

$$|p_n^0(f)|_Y=O(\varphi^*(n)/\varphi(n));\quad n\to\infty, \tag{2.1}$$

then

$$E_n[f]=O(1/\varphi(n));\quad n\to\infty. \tag{2.2}$$

Hence, if X, Y, $\{M_n\}$, φ^* *satisfy* (BA), *then for each* $\varphi\in\Phi$ *with* $\varphi^*/\varphi\in K_{\varphi^*}$ *and* $\varphi\in K_{\varphi^*}$ *properties* (2.1) *and* (2.2) *are equivalent.*

In order to reformulate this in a slightly more precise way, let us introduce the following notation. For two sequences $\{a_n\}_{n\in\mathrm{P}}$, $\{b_n\}_{n\in\mathrm{P}}$ of positive numbers, the symbol $a_n\sim b_n$, $n\to\infty$, stands for

$$0<\limsup_{n\to\infty}|a_n/b_n|<\infty. \tag{2.3}$$

Thus, for a $\varphi\in\Phi$, the class

$$\mathcal{B}_\varphi=\{f\in X;\quad E_n[f]\sim 1/\varphi(n),\quad n\to\infty\} \tag{2.4}$$

consists of those $f\in X$ for which $1/\varphi(n)$ is the exact order of best approximation.

From the proof of Prop. 1 and 2 in [3] it is clear that O may be replaced by o in (2.1), (2.2), hence one has:
If X, Y, $\{M_n\}$, φ^* satisfy (BA), then for each $\varphi\in\Phi$ with $\varphi^*/\varphi\in K_{\varphi^*}$ and $\varphi\in K_{\varphi^*}$

$$f\in\mathcal{B}_\varphi\Leftrightarrow|p_n^0(f)|_Y\sim\varphi^*(n)/\varphi(n),\quad n\to\infty. \tag{2.5}$$

Thus the rate of growth φ^*/φ in Zamansky's theorem cannot be improved, as long as one considers the entire set of functions $f\in\mathcal{B}_\varphi$ and only orders φ, φ^* for which $\varphi\in K_{\varphi^*}$. The latter condition is essential; indeed, if $\varphi\notin K_{\varphi^*}$, there exist examples which show that the equivalence (2.5) is no longer valid (cf. the remark following Cor. 2). In this case it may be expected that Prop. 1, which still holds, can be improved. But the question is whether $|p_n^0(f)|_Y=o(\varphi^*(n)/\varphi(n))$, $n\to\infty$ can be shown for all $f\in\mathcal{B}_\varphi$ or only on some proper subset of $\mathcal{B}_\varphi$. For the proof that the latter is true we use a Lemma on step sequences.

Lemma 3. *Given* $\varphi\in\Phi$, *let* $\{e_n\}_{n\in\mathrm{P}}$ *be a sequence of positive numbers with the properties*

i) $e_n\sim 1/\varphi(n)$; $n\to\infty$,

ii) *there exists a sequence* $\{n_k\}_{k\in\mathrm{P}}\subset\mathrm{N}$ *such that*

$$\inf\{n_{k+1}/n_k: k\in\mathrm{P}\}=\lambda>1 \text{ and } e_{n_k}=e_{n_k+1}=\ldots=e_{n_{k+1}-1};\quad k\in\mathrm{P}.$$

Then there exist constants $c>0$, $j_0\in\mathrm{P}$ *and a subsequence* $\{n_{k_j}\}_{j\in\mathrm{P}}$ *satisfying*

$$\varphi(n_{k_j}-1)e_{n_{k_j}-1}-\varphi(n_{k_j})e_{n_{k_j}}>c;\quad j\geqq j_0.$$

Proof. Since $\varphi\in\Phi$ is strictly increasing to infinity, sequences $\{e_n\}$ with properties i), ii) clearly exist. Given such a sequence, and setting

(2.6)
$$S=\limsup_{n\to\infty}\varphi(n)e_n,$$

one has $S>0$, by i), and, for an arbitrary ε with $0<\varepsilon<\min(1/2, (\lambda-1)/(1+2\lambda))$, there exists a sequence $\{n_j\}_{j\in\mathsf{P}}\subset\mathsf{P}$ such that

(2.7)
$$(1-\varepsilon)S<\varphi(n_j)e_{n_j};\quad j\in\mathsf{P}.$$

Here λ is the number given in ii). By ii), for each of these n_j there exists an n_{k_j} such that $n_{k_j-1}\leqq n_j\leqq n_{k_j}-1$ and $e_{n_j}=e_{n_{k_j}-1}$. Using the monotonicity of φ and (2.7), this implies

(2.8)
$$\varphi(n_{k_j}-1)e_{n_{k_j}-1}\geqq\varphi(n_j)e_{n_j}>(1-\varepsilon)S.$$

On the other hand, by ii) and (2.6) one has, for sufficiently large j,

$$\varphi(n_{k_j+1}-1)e_{n_{k_j}}=\varphi(n_{k_j+1}-1)e_{n_{k_j+1}-1}<(1+\varepsilon)S.$$

As $\varphi(x)$ and $\varphi'(x)$ are increasing for large x, and using ii) again, it follows that

$$\begin{aligned}\varphi(n_{k_j})e_{n_{k_j}}&=\varphi(n_{k_j+1}-1)e_{n_{k_j}}-\Big(\int_{n_{k_j}}^{n_{k_j+1}-1}\varphi'(x)dx\Big)e_{n_{k_j}}\\&<(1+\varepsilon)S-\varphi'(n_{k_j})\{n_{k_j+1}-n_{k_j}-1\}e_{n_{k_j}}\\&\leqq(1+\varepsilon)S-\varphi(n_{k_j})e_{n_{k_j}}g'(n_{k_j})\{(\lambda-1)n_{k_j}-1\}.\end{aligned}$$

By the choice of ε and the fact that $n_{k_j}\to\infty$ as $j\to\infty$ we have $(\lambda-1)-3\varepsilon/(1-2\varepsilon)>1/n_{k_j}$ for large j, and hence

$$\begin{aligned}\varphi(n_{k_j})e_{n_{k_j}}&<(1+\varepsilon)S-\varphi(n_{k_j})e_{n_{k_j}}g'(n_{k_j})n_{k_j}3\varepsilon/(1-2\varepsilon)\\&<(1+\varepsilon)S-\varphi(n_{k_j})e_{n_{k_j}}\,3\varepsilon/(1-2\varepsilon),\end{aligned}$$

where in the last inequality Lemma 1 a) and (1.2) were used. Thus $\varphi(n_{k_j})e_{n_{k_j}}<(1-2\varepsilon)S$ which, together with (2.8), yields

$$\varphi(n_{k_j}-1)e_{n_{k_j}-1}-\varphi(n_{k_j})e_{n_{k_j}}>\varepsilon S>0$$

for j large enough. This proves Lemma 3.

Theorem 1. *Let* $X, Y, \{M_n\}, \varphi^*$ *satisfy* (BA). *For each* $\varphi\in\Phi$ *with* $\varphi^*/\varphi\in K_{\varphi^*}$ *there exists a function* $f\in\mathfrak{B}_\varphi$ *such that* $|p_n^0(f)|_Y\sim\varphi^*(n)/\varphi(n)$; $n\to\infty$.

Proof. Let $\{e_n\}_{n\in\mathsf{P}}$ be a non-increasing sequence with properties i), ii) of Lemma 3. In view of Bernstein's theorem (cf. e. g. [5; p. 94]) there exists an $f \in X$ with $E_n[f]=e_n$ for each $n \in \mathsf{P}$, thus $f \in \mathfrak{B}_\varphi$ by i). Using Prop. 1, it suffices to find a sequence $\{m_j\}_{j\in\mathsf{P}} \subset \mathsf{N}$ such that (cf. (2.3))

$$\varphi^*(m_j)/\varphi(m_j)=O(|p^0_{m_j}(f)|_Y); \quad j\to\infty. \tag{2.9}$$

Choosing $m_j=n_{k_j}$, where $\{n_{k_j}\}_{j\in\mathsf{P}}$ is the sequence obtained in Lemma 3, the definition of $E_n[f]$ and (J_Y) yield

$$0\le E_{m_j-1}[f]-E_{m_j}[f]\le E_{m_j-1}[p^0_{m_j}(f)]\le c\{\varphi^*(m_j-1)\}^{-1}|p^0_{m_j}(f)|_Y,$$

and thus, using Lemma 1 e) and Lemma 3 with $e_n=E_n[f]$,

$$|p^0_{m_j}(f)|_Y\ge c\frac{\varphi^*(m_j)}{\varphi(m_j)}\Big\{\frac{\varphi^*(m_j-1)}{\varphi(m_j-1)}\,\frac{\varphi(m_j)}{\varphi^*(m_j)}\,\varphi(m_j-1)E_{m_j-1}[f]$$

$$-\frac{\varphi^*(m_j-1)}{\varphi^*(m_j)}\,\varphi(m_j)E_{m_j}[f]\Big\}\ge c\varphi^*(m_j)/\varphi(m_j)$$

for j large enough. Hence (2.9) holds, and the proof is complete.

In cases where the polynomials of best approximation can be explicitly determined the assertion of Theorem 1 can be made more precise, namely, the functions in question may be chosen such that their Fourier series have gaps.

We say that the Fourier series of $f \in C_{2\pi}$ has Bernstein gaps if the series is of the form

$$\sum_{j=0}^{\infty} a_j \cos n_j x, \tag{2.10}$$

where $a_j>0$ $\forall j \in \mathsf{P}$ and $n_{j+1}/n_j=2p_j+1$ for certain $p_j \in \mathsf{N}$, $j \in \mathsf{P}$ (cf. [9, p. 77]). Moreover, e.g. for an $f \in L^2_{2\pi}$, the Fourier series of f is said to have Hadamard gaps if it has the form

$$\sum_{j\in\mathsf{Z}} c_j e^{ijx} \tag{2.11}$$

with $c_j=0$ for all $j \in \mathsf{Z}$ save perhaps those belonging to a set $E=\{\pm n_k;\ k \in \mathsf{N}\}$, where the $n_k \in \mathsf{N}$ form a Hadamard sequence: $\inf_{k\in\mathsf{N}}\{n_{k+1}/n_k\}>1$ (cf. [4, p. 211]).

Denoting by Π_n the set of trigonometric polynomials of degree $\le n$ one has

Corollary 1. *Let $\varphi, \varphi^* \in \Phi$ and $\varphi^*/\varphi \in K_{\varphi^*}$.*
a) If Y is such that $C_{2\pi}$, Y, $\{\Pi_n\}$, φ^* satisfy (BA) *and $f \in C_{2\pi}$ has a Fourier series with Bernstein gaps, or*
b) if $L^2_{2\pi}$, Y, $\{\Pi_n\}$, φ^* satisfy (BA) *and $f \in L^2_{2\pi}$ has a Fourier series with Hadamard gaps, then $f \in \mathfrak{B}_\varphi$ implies $|p^0_n(f)|_Y\sim\varphi^*(n)/\varphi(n)$; $n\to\infty$.*

Proof. In both cases one has $p^0_n(f)=S_n f$, where S_n denotes the n-th partial sum of the Fourier series (cf. [9, p. 77] in case a)). Therefore the sequence $\{e_n\}$, defined by $e_n=E_n[f]$ $\forall n \in \mathsf{P}$, satisfies conditions i), ii) of Lemma 3, and the assertion follows as in the proof of Theorem 1.

As examples of subspaces Y one may take e. g. Y_C and Y_2, respectively, where, denoting by $X_{2\pi}$ one of the spaces $C_{2\pi}$, $L^p_{2\pi}$, $1 \leq p < \infty$, we set

$$(2.12) \qquad Y_{X_{2\pi}} = \{f \in X_{2\pi};\ \exists h \in X_{2\pi} \text{ such that } \varphi^*(|j|) f^\frown(j) = h^\frown(j) \forall j \in \mathbb{Z}\}$$

and write $Y_{L^p_{2\pi}} = Y_p$, $Y_{C_{2\pi}} = Y_C$. Here $f^\frown(j)$ denotes the j-th Fourier coefficient of f, and the associated seminorm is $|f|_{Y_{X_{2\pi}}} = \|h\|_{X_{2\pi}}$. For the verification of (B_{Y_C}), (J_{Y_C}) we refer to [6; 2, I].

3. Best possible version of Zamansky's inequality in $L^2_{2\pi}$. In the following two theorems we give a quantitative result on the best possible Zamansky-type inequality in $L^2_{2\pi}$, i. e. the slowest possible rate of increase of $|p^0_n(f)|_Y$ for $f \in \mathcal{B}_\varphi$. Whereas the negative result of Theorem 1 and Cor. 1 holds in particular for those f for which the sequence $\{E_n[f]\}_{n \in \mathbb{P}}$ is piecewise constant, now the functions f with a rather "smooth" sequence $\{E_n[f]\}$ seem to be significant.

Theorem 2. *Let* $f \in L^2_{2\pi}$, $M_n = \Pi_n$, *and* $Y = Y_2$ *be given by* (2.12). *If* φ, $\varphi^* \in \Phi$, $\varphi^*/\varphi \in K_{\varphi^*}$, $\varphi \leftrightarrow k$, $\varphi^* \leftrightarrow k^*$, *and* $E_n[f] = c/\varphi(n)$ $\forall n \in \mathbb{P}$ *for some constant* $c > 0$, *then*

$$(3.1) \qquad |p^0_n(f)|_Y \sim \left(\frac{k^*(n)}{k(n)}\right)^{1/2} \frac{\varphi^*(n)}{\varphi(n)}; \quad n \to \infty.$$

Proof. Clearly (BA) are satisfied and functions f with the required property exist. By Parseval's equation and (2.12)

$$|p^0_n(f)|^2_Y = |S_n f|^2_Y = \sum_{|j| \leq n} (\varphi^*(|j|) f^\frown(j))^2$$

$$= O(1) + \sum_{j=n_0+1}^{n} (\varphi^*(|j|))^2 \{(E_{j-1}[f])^2 - (E_j[f])^2\}; \quad n \to \infty,$$

where $n_0 \in \mathbb{N}$ is some fixed number. The hypothesis and Lemma 1 c) yield

$$|p^0_n(f)|^2 = O(1) + 2c^2 \sum_{j=n_0+1}^{n} (\varphi^*(j))^2 \int_{j-1}^{j} (\varphi(x))^{-2} (k(x))^{-1} dx,$$

and, by Lemma 1 e) and Lemma 2 b) ii)

$$\sum_{j=n_0+1}^{n} (\varphi^*(j))^2 \int_{j-1}^{j} (\varphi(x))^{-2}(k(x))^{-1} dx \approx \int_{n_0}^{n} (\varphi^*(x)/\varphi(x))^2 (k(x))^{-1} dx$$

$$\approx (\varphi^*(n)/\varphi(n))^2 (k^*(n)/k(n)), \quad n \to \infty.$$

This gives the assertion in view of Lemma 2 b) i).

Although we dealt with a very special class of functions in Theorem 2, it can be shown that in $\mathcal{B}_\varphi$ there are no functions for which a further improvement of Zamansky's inequality is possible, i. e. the following inverse Zamansky type theorem holds.

Theorem 3. *If* $L^2_{2\pi}$, Y, Π_n, φ^*, φ, k^*, k *are as in Theorem 2 and* $f \in L^2_{2\pi}$, *satisfies* (3.1), *then* $E_n[f] = O(1/\varphi(n))$, $n \to \infty$.

Proof. By (3.1), (2.12), and Lemma 1 c) one has

$$(E_n[f])^2=\sum_{|j|>n}(\varphi^*(|j|))^{-2}\{\varphi^*(|j|)f^{\wedge}(j)\}^2=\sum_{j=n+1}^{\infty}(\varphi^*(j))^{-2}\{|P_j^0(f)|_Y^2-|P_{j-1}^0(f)|_Y^2\}$$

$$=\sum_{j=n+1}^{\infty}|P_j^0(f)|_Y^2\{(\varphi^*(j))^{-2}-(\varphi^*(j+1))^{-2}\}$$

$$=c\sum_{j=n+1}^{\infty}\Big(\frac{\varphi^*(j)}{\varphi(j)}\Big)^2\frac{k^*(j)}{k(j)}\int_j^{j+1}\{(\varphi^*(x))^2k^*(x)\}^{-1}dx.$$

In view of Lemma 2 b) i), $F(j)\int_j^{j+1}(\varphi^*(x))^{-2}(k^*(x))^{-1}dx\leq c\int_j^{j+1}F(x)(\varphi^*(x))^{-2}\times(k^*(x))^{-1}dx$ for j large enough, thus

$$E_n[f]\leq c\{\int_{n+1}^{\infty}(\varphi(x))^{-2}(k(x))^{-1}dx\}^{1/2}=c/\varphi(n+1)<c/\varphi(n)$$

for large n, by Lemma 1 c).

Remarks. i) One reason for the restriction to $L^2_{2\pi}$ in Theorem 3 is the fact that the proof of Prop. 2 (Lemma 9 and Theorem 2 in [3]) can no longer be used here. Moreover, it seems reasonable to expect that Theorem 3 cannot be extend to all $L^p_{2\pi}$, $p\neq 2$, without altering condition (3.1).

ii) Returning to the remarks following (2.5), Theorem 2 also shows (cf. Lemma 2a)) that the condition $\varphi\in K_{\varphi^*}$ is best possible in (2.5) in the sense that it cannot be replaced by a weaker condition.

iii) To give an idea of the size of the factor of improvement in (3.1), i. e. the factor $(k^*(n)/k(n))^{1/2}$, one may take for example $\varphi(x)=(1+x)^2$, $\varphi^*(x)=\exp\{x^\tau\}$, $0<\tau<1$. In this case we have $\varphi\notin K_{\varphi^*}$ and $(k^*(n)/k(n))^{1/2}=2n((1+n)\tau n^\tau)^{-1}\approx n^{-\tau}$, $n\to\infty$.

We summarize the results for $L^2_{2\pi}$ in the following

Corollary 2. *Let $L^2_{2\pi}$, Y, $\{\Pi_n\}$, φ^*, φ, k^*, k be as in Theorem 2. The slowest possible and the most rapid rate of increase of $|P_n^0(f)|_Y$ in Zamansky's inequality for functions $f\in\mathfrak{B}_\varphi$ are*

$$(k^*(n)/k(n))^{1/2}(\varphi^*(n)/\varphi(n)),\quad \varphi^*(n)/\varphi(n);\quad n\to\infty,$$

respectively. Both orders are attained in $\mathfrak{B}_\varphi$.

The proof of this follows by Prop. 1, Cor. 1 b), Theorem 2 and observing that the proof of Theorem 3 also yields the implication

$$|P_n^0(f)|_Y=o(\{k^*(n)/k(n)\}^{1/2}(\varphi^*(n)/\varphi(n))\Longrightarrow E_n[f]=o(1/\varphi(n));\ n\to\infty.$$

We further note that, in case $\varphi\notin K_{\varphi^*}$, one might as well investigate the following problem. Given a fixed rate of increase in Zamansky's inequality, for which $f\in X$ is this order attained? In view of the proof of Theorem 3 such functions f do not need to satisfy $E_n[f]=O(1/\varphi(n))$, $n\to\infty$. For example, in the situation of Cor. 2, take

$$f(x)=\sum_{k\in Z}(1+|k|)^{-2}e^{ikx},\quad \varphi(x)=(1+x)^{7/4},\quad \varphi^*(x)=(1+x)^{7/4}e^{\sqrt{x}}.$$

Then one has $\varphi^*/\varphi\in K_{\varphi^*}$, $\varphi\notin K_{\varphi^*}$, $|p_n^0(f)|_Y\approx\varphi^*(n)/\varphi(n)$, $n\to\infty$, however, $\varphi(n)E_n[f]\approx(n+1)^{1/4}\to\infty$ as $n\to\infty$. This example also justifies the remark following (2.5).

4. A lower estimate in general spaces X. For general spaces $X_{2\pi}$ the situation is more complicated. For example, in $C_{2\pi}$, the functions with Bernstein gaps, which allow an explicit representation of their polynomials of best approximation, unfortunately are useless for establishing an improvement of Zamansky's inequality (Cor. 1). However, it can easily be shown that in general the rate of increase in this inequality cannot be slower than $(k^*(n)/k(n))(\varphi^*(n)/\varphi(n))$, $n\to\infty$.

Theorem 4. *Let* X, Y, $\{M_n\}$, φ^* *satisfy* (BA). *If* $\varphi\in\Phi$, $\varphi^*/\varphi\in K_{\varphi^*}$, $k\leftrightarrow\varphi$, $k^*\leftrightarrow\varphi^*$ *are such that*

$$\limsup_{x\to\infty} k^{*\prime}(x)k(x)/k^*(x)<1, \tag{4.1}$$

then, for each $f\in X$,

$$|p_n^0(f)|_Y=o(\{k^*(n)/k(n)\}\{\varphi^*(n)/\varphi(n)\});\ n\to\infty \tag{4.2}$$

implies $f\notin\mathcal{B}_\varphi$.

Proof. The proof of Prop. 2 (Lemma 9 and Theorem 2 in [3]) immediately yields that (4.2) implies $E_n[f]=o((k^*(n))^{-1}\int_n^\infty k^*(x)/(k(x)\varphi(x))dx$; $n\to\infty$. To show that the integral is $O(k^*(n)/\varphi(n))$ as $n\to\infty$, partial integration gives, if n is large enough,

$$\int_n^\infty \frac{k^*(x)}{k(x)}\frac{1}{\varphi(x)}\{1-k^{*\prime}(x)\frac{k(x)}{k^*(x)}\}dx\leq\frac{k^*(n)}{\varphi(n)},$$

and by (4.1) it follows that $E_n[f]=o(1/\varphi(n))$, $n\to\infty$.

We remark that condition (4.1) is only a slight restriction. It is satisfied if $k^*(x)/k(x)$ is eventually decreasing, which is the case for all the usual examples of pairs φ, φ^* with $\varphi^*/\varphi\in K_{\varphi^*}$, e. g. for those mentioned previously, or

$$\varphi(x)=\exp\{a(\log(1+x))^\beta\},\ a>0,\ \beta>1,$$

$$\varphi(x)=\exp\{a(\log(1+x))\log\log(e+x)\},\ a>0.$$

5. Zamansky's inequality for linear processes. We use the notations $X_{2\pi}$, $Y_{X_{2\pi}}$ as introduced at the end of Sec. 2. By a linear process on $X_{2\pi}$ we denote a sequence $\{T_n\}_{n\in\mathbb{P}}$ of bounded linear operators T_n on $X_{2\pi}$ into itself which are polynomial operators, i. e. which map $X_{2\pi}$ into Π_n. (Thus $\{T_n\}$ is not supposed to form an approximation process, i. e. the sequence of operator norms may be unbounded.) Such a process clearly satisfies a Bernstein type inequality

$$|T_nf|_{Y_{X_{2\pi}}}\leq c\varphi^*(n)\|T_nf\|_{X_{2\pi}};\ f\in X_{2\pi},\ n\in\mathbb{P}, \tag{5.1}$$

for any $\varphi^*\in\Phi$, which implies that the analogue of Prop. 1 remains valid: If $\varphi\in\Phi$, $\varphi^*/\varphi\in K_{\varphi^*}$, then

$$\|T_nf-f\|_{X_{2\pi}}=O(1/\varphi(n))\Longrightarrow|T_nf|_{Y_{X_{2\pi}}}=O(\varphi^*(n)/\varphi(n)),\ n\to\infty. \tag{5.2}$$

For a complete discussion of possible improvements of this Zamansky type inequality and for a comparison of the results with those obtained above for the polynomials of best approximation one would have to distinguish a variety of cases, starting from a discussion of the conditions under which a process $\{T_n\}$ is "asymptotically optimal on $\mathfrak{B}_\varphi$", i. e. the conditions upon $\{T_n\}$, φ, and the space $X_{2\pi}$ under which $f\in\mathfrak{B}_\varphi$ implies $\|T_nf-f\|_{X_{2\pi}}=O(1/\varphi(n))$, $n\to\infty$. For a study of the latter aspect in case $X_{2\pi}=C_{2\pi}$ we refer to [2], and for the case when φ and φ^* are classical orders see [1].

Instead of going into details we confine ourselves to the discussion of two particular processes in order to illustrate two aspects:

i) Now the functions with gaps in their Fourier series behave particularly well, in contrast to the extremely poor behaviour they showed in Cor. 1.

ii) Moreover, linear processes may satisfy better Zamansky type inequalities than the $p_n^0(f)$. In particular, for the generalized de La Vallée Poussin means, which form the key process in questions of exponential approximation (cf. [2]), the best possible Zamansky inequality is at least of order $O(\{k^*(n)/k(n)\}\{\varphi^*(n)/\varphi(n)\})$ in all spaces $X_{2\pi}$ (whereas for the $p_n^0(f)$ it was at most of this order), and there can be constructed further processes which yield even better orders.

Given $\varphi\in\Phi$ with $k\leftrightarrow\varphi$, the generalized de La Vallée Poussin sums $V_{n,[k(n)]}f$ of an $f\in X_{2\pi}$ are defined by (cf. also [7])

$$V_{n,[k(n)]}f=([k(n)]+1)^{-1}\sum_{j=n-[k(n)]}^{n}S_jf;\quad n\in\mathbb{P}. \tag{5.3}$$

Here $[x]$ denotes the integer part of $x\in\mathbb{R}$. (In case $\varphi(x)=(1+x)^2$ the usual delayed means $V_{n,[(n+1)/2]}$ are obtained.)

Theorem 5. *Let φ, $\varphi^*\in\Phi$, $\varphi^*/\varphi\in K_{\varphi^*}$, $k\leftrightarrow\varphi$, $k^*\leftrightarrow\varphi^*$. For each $f\in X_{2\pi}$ for which*

$$\|S_nf-f\|_{X_{2\pi}}=O(1/\varphi(n));\quad n\to\infty, \tag{5.4}$$

he operators (5.3) *satisfy the Zamansky type inequality*

$$|V_{n,[k(n)]}f|_Y=O\left(\frac{k^*(n)}{k(n)}\frac{\varphi^*(n)}{\varphi(n)}\right);\quad n\to\infty, \tag{5.5}$$

twhere $Y=Y_{X_{2\pi}}$ is given by (2.12).

Proof. In view of (5.2) and (5.4) we have $|S_jf|_Y=O(\varphi^*(j)/\varphi(j))$, $j\to\infty$, whence, using (5.3), the fact that $\varphi^*(x)/\varphi(x)$ is an increasing function, and Lemma 1 b), it follows that

$$|V_{n,[k(n)]}f|_Y\leq([k(n)]+1)^{-1}\sum_{j=n-[k(n)]}^{n}|S_jf|_Y$$

$$\leq(c/k(n))\int_{n-[k(n)]}^{n+1}\varphi^*(x)/\varphi(x)dx\leq(c/k(n))\int_{n_0}^{n+1}\varphi^*(x)/\varphi(x)dx$$

for some $n_0\in\mathbb{N}$. By partial integration, the latter integral equals

$$\int_{n_0}^{n}(g^{*\prime}(x)-g'(x))^{-1}\{(g^{*\prime}(x)-g'(x))\varphi^*(x)/\varphi(x)\}dx$$

$$<(g^{*\prime}(n+1)-g'(n+1))^{-1}\varphi^*(n+1)/\varphi(n+1)-\int_{n_0}^{n+1}\frac{\varphi^*(x)}{\varphi(x)}\frac{|g^{*\prime\prime}(x)-g''(x)|}{(g^{*\prime}(x)-g'(x))^2}dx,$$

where we have set $\varphi(x)=e^{g(x)}$, $\varphi^*(x)=e^{g^*(x)}$, and suppose that n_0 is large enough. Hence

$$\int_{n_0}^{n+1}\varphi^*(x)/\varphi(x)dx<\int_{n_0}^{n+1}\{1+\frac{|g^{*\prime\prime}(x)-g''(x)|}{(g^{*\prime}(x)-g'(x))^2}\}\frac{\varphi^*(x)}{\varphi(x)}dx<\frac{1}{g^{*\prime}(n+1)-g'(n+1)}\frac{\varphi^*(n+1)}{\varphi(n+1)}$$

and, observing that $\varphi^*/\varphi\in K_{\varphi^*}$ implies $g^{*\prime}(x)-g'(x)\approx g^{*\prime}(x)$, $x\to\infty$, the result is

$$\|V_{n,[k(n)]}f\|_Y=O(\frac{k^*(n+1)}{k(n)}\frac{\varphi^*(n+1)}{\varphi(n+1)});\quad n\to\infty,$$

and this implies (5.5) in view of Lemma 1 d).

Remark. Obviously the analogue of Theorem 5 with the order φ in (5.4) replaced by some other order $\psi\in\Phi$ is also valid, thus if ψ, φ, $\varphi^*\in\Phi$, $\varphi^*/\psi\in K_{\varphi^*}$, $k\leftrightarrow\varphi$, $k^*\leftrightarrow\varphi^*$, then $\|S_nf-f\|_{X_{2\pi}}=O(1/\psi(n))$ implies $\|V_{n,[k(n)]}f\|_Y=O(\{k^*(n)/k(n)\}\{\varphi^*(n)/\psi(n)\})$, $n\to\infty$.

Condition (5.4) is satisfied for all $f\in\mathfrak{B}_\varphi$ if $X_{2\pi}=L^p_{2\pi}$, $1<p<\infty$. If $X_{2\pi}=C_{2\pi}$, (5.4) is easily seen to hold e. g. for those $f\in\mathfrak{B}_\varphi$ whose Fourier spectrum is a Sidon set or for those with non-negative Fourier coefficients. We say that $f\in C_{2\pi}$ has Fourier spectrum E for some set $E\subset Z$ if $f^\wedge(k)=0$ $\forall k\notin E$, and E is called a Sidon set (cf. [8], [4, p. 215, Remark (1)]) if there exists a constant B such that

$$\sum_{k\in E}|f^\wedge(k)|\leqq B\|f\|_{C_{2\pi}}$$

for each $f\in C_{2\pi}$ with spectrum E. This is satisfied in particular by functions whose Fourier series have Bernstein- or Hadamard gaps. Hence we have

Corollary 3. *If φ, φ^*, k, k^* are as in Theorem 5 and $X_{2\pi}=L^p_{2\pi}$ $1<p<\infty$, or $X_{2\pi}=C_{2\pi}$ and the spectrum of f is a Sidon set, or $X_{2\pi}=C_{2\pi}$ and $f^\wedge(k)\geqq 0\ \forall k\in Z$, then $f\in\mathfrak{B}_\varphi$ implies* (5.5).

We do not investigate whether the result (5.5) is best possible since better results can be easily obtained by modifying the process. For example, repeating the averaging procedure of (5.3), one obtains the process

$$(5.6)\qquad W_{n,[k(n)]}f=([k(n)]+1)^{-1}\sum_{j=n-[k(n)]}^{n}V_{n,[k(n)]}f,\ f\in X_{2\pi};n\in\mathrm{P},$$

where again $\varphi\in\Phi$, $k\leftrightarrow\varphi$. Concerning its approximation properties, the process $\{W_{n,[k(n)]}\}$ is similar to $\{V_{n,[k(n)]}\}$ since $f\in\mathfrak{B}_\varphi$ implies $\|V_{n,[k(n)]}f-f\|_{X_{2\pi}}=O(1/\varphi(n))$, $n\to\infty$, (see [2] for $X_{2\pi}=C_{2\pi}$) which, by a straightforward calculation, using [3, Lemma 2], implies that $\|W_{n,[k(n)]}f-f\|_{X_{2\pi}}=O(1/\varphi(n))$, $n\to\infty$.

Theorem 6. *Let φ, φ^*, k, k^*, Y be as in Theorem 5. For each $f \in X_{2\pi}$ condition* (5.4) *implies*

$$\| W_{n,[k(n)]} f \|_Y = O(\{k^*(n)/k(n)\}^2 (\varphi^*(n)/\varphi(n))); \quad n \to \infty.$$

The proof, being similar to that of Theorem 5, is omitted. The results of this section should also be compared with those of Section 2. If $f \in C_{2\pi}$ has a Fourier series with Bernstein gaps and $f \in \mathcal{B}_\varphi$, Cor. 1 a) implies that $\| p_n^0(f) \|_Y = \| S_n f \|_Y \sim \varphi^*(n)/\varphi(n)$, $n \to \infty$, though condition (5.4) is satisfied. Thus the Fourier partial sums already show that the validity of an improved Zamansky-type inequality for a linear process as well as the best possible rate of increase may depend on the process as well as on the structural properties of the functions in $\mathcal{B}_\varphi$.

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