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PIECEWISE C^1 CUBIC INTERPOLATION AND APPLICATIONS

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Summary. We first consider the interpolation of a continuous function by piecewise cubic polynomials which are globally C^1 on $\Omega=[0,1]$ or $\Omega=[0,1]\times[0,1]$. The scheme uses the Gaussian and boundary points. If h is the length of the partition, $hN=1$, if Q_N is the linear projector which to each f assigns the interpolant $Q_N f$, we show that Q_N is bounded and that $\|(I-Q_N)f\|_\infty \to 0$, as $N\to\infty$. In particular, if $f\in W^{4,\infty}(\Omega)$ we show that $\|I-Q_N)f\|_\infty \leqq ch^4$. We give some numerical demonstrations of this fact.

Next, we introduce a Collocation method, at the Gaussian points, for equations of the second kind $(I-T)u=f$, where T is a compact linear operator $W^{4,\infty}(\Omega)\to W^{4,\infty}(\Omega)$ and $(I-T)^{-1}$ exists and is continuous. The method generates a sequence of approximants u_N. Using the interpolation theory described above, we show that $\|u_N-u\|_\infty\to 0$ with optimum speed. An example — application to 2-dimensional Fredholm integral equations of the second kind is also given.

Introduction. We analyse one- and two-dimensional interpolation by piecewise cubic polynomials which are globally C^1. The points of interpolation, used in this scheme, are the Gaussian and boundary points. Our analysis leads to optimum error estimates in the uniform norm. We also introduce a collocation method for the approximate solution of equations $(I-T)u=f$. We apply the interpolation results to show that the approximants converge to the exact solution, in the uniform norm, with optimum speed. As a special case, we give an application to two-dimensional Fredholm integral equations of the second kind.

1. Interpolation Theory. In this section, we extend some of the results of [1] and prove some of them in a slightly different way. The case of interest is the interpolation of a continuous function by piecewise cubic polynomials which preserve C^1 continuity at the nodes of a uniform partition $\Delta_N\equiv\{0=x_1\leqq\cdots\leqq x_{N+1}=1\}$, $x_{j+1}-x_j\equiv h$, of $I=[0,1]$.

It is well established that the $2(N+1)$-dimensional space H_N, of these interpolants, is generated by the basis $\underset{\sim}{B}_N\equiv[B_1,\ldots,B_{2(N+1)}]^T$, where

$$B_{2j-1}(x)\equiv\Phi\left(\frac{x}{h}-j+1\right),\quad B_{2j}(x)=h\left(\frac{x}{h}-j+1\right);\ j=1,\ldots,N+1 \tag{1.1}$$

and

$$\Phi(t)=\begin{cases}(t+1)^2(1-2t) & -1\le t\le 0,\\ (t-1)^2(1+2t) & 0\le t\le 1,\\ 0 & \text{otherwise},\end{cases}$$

(1.2) $$w(t)=\begin{cases}t(t+1)^2 & -1\le t\le 0\\ t(1-t)^2 & 0\le t\le 1\\ 0 & \text{otherwise}.\end{cases}$$

Associated with each subinterval $[x_j, x_{j+1}]$ of the partition Δ_N, are two Gaussian points $2^{-1}(x_j+x_{j+1})\pm h/2\sqrt{3}$, which, together with the two end points of I, comprise the set $\{\sigma_i\}_{i=1}^{2(N+1)}$, $\sigma_i\le\sigma_{i+1}$, of the interpolation points.

The verification of the following facts is elementary: Given N and a continuous function f, (we write $f\in C^0(I)$), there exists a unique element $Q_N f\in H_N$ which interpolates f at σ_i, $i=1,\ldots,2(N+1)$. Moreover, this element of H_N is given by

(1.3) $$Q_N f=\underset{\sim}{f}(\sigma)^T(G_N^T)^{-1}\underset{\sim}{B}_N,$$

where $\underset{\sim}{f}(\sigma)^T=[f(\sigma_1),\ldots,f(\sigma_{2(N+1)})]$, G_N is the Gramian, i.e., the matrix with entries $B_i(\sigma_j)$, $i,j=1,\ldots,2(N+1)$, and the superscript T denotes transposition.

Thus, a linear projector $Q_N: C^0(I)\to H_N$, $f\to Q_N f$, is well defined on $C^0(I)$ with range H_N. A standard question of the interpolation theory concerns the boundedness of Q_N. We answer it by the following:

L e m m a 1.1 *$\|Q_N\|$ is bounded independent of N.*

P r o o f: Let H_N be the diagonal $2(N+1)$-matrix with entries $[1\ h,\ldots,1\ h]$. Set $K_N=G_N H_N^{-1}$. It is proved in [1], that the row norm $\|K_N^{-1}\|_\infty$, of the matrix K_N^{-1}, is bounded by a constant c, which is independent of N. Therefore, using (1.3), we get that for any $x\in I$ and any $f\in C^0(I)$

$$|Q_N f(x)|\le c\sum_{i=1}^{2(N+1)} a_i\,|B_i(x)|\ \|f\|_\infty,$$

where $a_{2j-1}=1$, $a_{2j}=1/h$, $j=1,\ldots,N+1$. But it is easily seen by (1.1) and (1.2) that the sum on the right of this inequality is bounded by 5/4. Hence $\|Q_N f\|_\infty\le(5c/4)\|f\|_\infty$, which completes the proof.

If f is sufficiently smooth, one consequence of Lemma 1.1 is that the error of the interpolation, measured in the uniform norm, tends to zero with optimum speed. Specifically, considering the Sobolev space $W^{4,\infty}(I)$, we have:

P r o p o s i t i o n 1.1. *For any $f\in W^{4,\infty}(I)$, there exists a constant C_f, ndependent of N (and x) such that*

(1.4) $$\|f-Q_N f\|_\infty\le C_f h^4.$$

P r o o f: For any $v\in H_N$, $v=Q_N v$. Hence the triangle inequality gives $\|f-Q_N f\|_\infty\le(1+\|Q_N\|)\inf\{\|f-v\|_\infty : v\in H_N\}$.

It is known (cf. [4]) that the infimum on the right is bounded by $c_f h^4$, where c_f does not depend on N. This completes the proof.

We present a numerical demonstration of Proposition 1.1, applied to the functions $f(x)=e^x$ and $f(x)=x^4$. The rate of convergence is estimated by

$$p = \log\left(\frac{\text{error for } h/2}{\text{error for } h}\right) / \log 2,$$

while (1.4) predicts that $p=4$.

N	$\|f-Q_N f\|_\infty$		p	
	$f(x)=e^x$	$f(x)=x^4$	$f(x)=e^x$	$f(x)=x^4$
3	3.106×10^{-5}	4.155×10^{-4}		
6	2.325×10^{-6}	2.678×10^{-5}	3.74	3.96
12	1.646×10^{-7}	1.674×10^{-6}	3.8	3.99
24	1.096×10^{-8}	1.047×10^{-7}	3.9	4.00
48	7.070×10^{-10}	6.541×10^{-9}	3.95	4.00

These results were obtained at the Purdue CDC 6500 using single precision.

The theory presented so far is extendable to higher dimensions. Of particular interest to some applications is the extension to two dimensions. In this case, the domain of definition of the functions is $I\times I$ and the elements are rectangles of sides $h_1=1/N_1$, $h_2=1/N_2$. The points of interpolation are the Gaussian points inside each rectangle, the four vertices of $I\times I$ and the Gaussian points of each boundary element side. The piecewise polynomial subspace H_N is the tensor product of the two one-dimensional subspaces H_{N_1} and H_{N_2}, one for each direction. The linear projector Q_N is now defined by $Q_N \equiv Q_{N_1}Q_{N_2}=Q_{N_2}Q_{N_1}$. If $h=\max\{h_1, h_2\}$, an extension of Proposition 1.1, which we do not prove here, gives:

Proposition 1.2. (a) $\|Q_N\|$ *is bounded independent of* N, *and* (b) *for any* $f\in W^{4,\infty}[I\times I]$, *there exists a constant* c_f, *independent of* h_1, h_2, *such that* $\|f-Q_N f\|_\infty \leq c_f h^4$.

2. Collocation at the Gaussian Points and Equations of the Second Kind. We introduce a collocation method at the Gaussian points. We then apply the results of Section 1 to show that the approximants converge to the exact solution with optimum speed.

In the sequel Ω stands for either $(0,1)$ or $(0,1)\times(0,1)$. Let T be a compact linear operator, mapping $W^{4,\infty}(\Omega)$ to $W^{4,\infty}(\Omega)$, and assume that $(I-T)^{-1}: W^{4,\infty}(\Omega)\to W^{4,\infty}(\Omega)$ exists, and is continuous. We are interested in solving

$$(I-T)\,u=f, \tag{2.1}$$

where f is given in $W^{4,\infty}(\Omega)$.

Let H_N be the one- or two-dimensional subspace of section 1 and Q_N be the corresponding interpolation operator. Let S_N be the finite set of interpolation points σ, used in section 1. A collocation method, for the approximate solution of (2.1), is to seek $u_N\in H_N$ such that

$$(I-T)u_N(\sigma)=f(\sigma), \text{ all } \sigma\in S_N. \tag{2.2}$$

Equivalently, seek $u_N\in H_N$ such that

$$(I-Q_N T)u_N=Q_N f. \tag{2.3}$$

For existence-uniqueness and error estimation, we prove:

P r o p o s i t i o n 2.1 (i) *For sufficiently large N the collocation system* (2.2) *is uniquely solvable,* (ii) *the error between the approximate solutions u_N and the exact solution u, measured in uniform norm, tends to zero with optimum speed. That is to say, there exists a constant c independent of h such that*

$$(2.4) \qquad \| u_N - u \|_\infty \leq c h^4.$$

P r o o f: By Proposition 1.1 or 2.1, each Q_N is bounded and $Q_N \to I$. Since T is compact, it follows that $\lim_{N\to\infty} \| (I-Q_N)T \| = 0$ (cf. [3], ch. 4). This fact, the continuity of $(I-T)^{-1}$ and Theorem 15.3 of [3] gives $\| u_N - u \|_\infty \leq c\,(\inf\{ \| u-v \|_\infty : v \in H_N\}) \| Q_N \|$. But we proved in section 1 that $\| Q_N \|$ is bounded independent of N (and h), while as in the proof of Proposition 1.1 the infimum on the right is bounded by $c_u h^4$, with c_u independent of N. This completes the proof.

As an application, let $P=(x, y)$, $Q=(s, t)$, $dQ=ds\,dt$, $\Omega=(0,1)\times(0,1)$, and consider the integral operator T defined by

$$Tu(P) = \lambda \int_\Omega k(P, Q) u(Q) dQ.$$

Then (2.3) represents a collocation method for the approximate solution of Fredholm integral equations of the second kind, in two dimensions. In this situation, Proposition 2.1 may be phrased as follows:

P r o p o s i t i o n 2.2. *If λ is not an eigenvalue of the kernel k, and if $k(\cdot\,; Q)$, $k(P;\cdot)$ and f are in $W^{4,\infty}(\Omega)$, then for sufficiently large N the collocation equations are uniquely solvable and the optimum error estimate* (2.4) *holds.*

Finally, for the results and analysis of numerical experiments with this collocation method we refer to [2].

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