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REMARKS TO A THEOREM OF B. S. KAŠIN ON CONVERGENCE SYSTEMS

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Summary. The aim of this paper is to give some light on some latest results and problems concerning almost everywhere convergence of linear operator-sequences connecting a result of Banach and that of Saks, further generalizing the latest one, to L^2-case.

1. During generalization of L. Carleson's theorem [3] on almost everywhere convergence of the trigonometric Fourier series of square integrable functions to L^p-case ($1<p\leqq\infty$), R. Hunt proved [4] among others the following inequality:

(1)
$$\|\sup_{n\geqq 0}|S_n(f)|\|_{L^2(0,2\pi)}\leqq C\|f\|_{L^2(0,2\pi)},$$

where $f\in L^2(0,2\pi)$ arbitrary, $S_n(f)$ denotes the n-th partial sum of the trigonometric Fourier series of f and C is absolute constant.

Later on E. M. Nikišin proved [11] a general theorem of similar nature, generalizing some previous results of E. M. Stein [15] on general translation-invariant operator sequences (instead of S_n)*: If $\{f_n\}$ is a sequence of almost everywhere finite Lebesgue measurable functions defined on the interval (0,1), such that for every $a=\{a_n\}\in l_2$ the series $\Sigma a_n f_n$ converges almost everywhere (a. e.) on (0,1) (such a system $\{f_n\}$ is briefly called convergence system), then for every $\varepsilon>0$ and $0<q<2$ there exist a measurable set $E_{\varepsilon,q}\subset(0,1)$ and a constant $C_{\varepsilon,q}$ depending only on ε and q, for which mes $E_{\varepsilon,q}>1-\varepsilon$ and

(2)
$$\|G(a)\|_{L^q(E_{\varepsilon,q})}\leqq C_{\varepsilon,q}\|a\|_{l_2},\ (G(a)=\sup_{n\geqq 1}|\sum_{k=1}^{n}a_k f_k(t)|)$$

for every $a\in l_2$. (This statement is a special case of theorem 8 in [11].) On p. 176 of [11] E. Nikišin asked whether (2) is true or not for $q=2$. In [7] S. Kašin gave a negative answer constructing an orthonormal convergence system on (0,1), such that for every measurable set $E\subset(0,1)$ with mes $E>0$ there exists $a_E\in l_2$ for which

(3)
$$G(a_E)|_E\notin L^2(E).$$

* For applications of such inequalities we refer to [11] and [15].

It seems that Kašin proved much more than the negative answer to Nikišin's problem.

The aim of this lecture is to elucidate the connection between Nikišin's problem and Kašin's result (further in connection with this to generalize the resonance theorem of S. Saks [8, p. 36]*) to L^2-case and to give the connection of these problems with the basic paper of S. Banach [2].) This is contained in the following statement:

If $\{f_n\}$ is a convergence system on (0,1), for which Nikišin's estimate (2) does not hold true at $q=2$, then there is a unique decomposition (0,1) $=E_1\cup^* E_2$ into disjoint measurable sets, with mes $E_1>0$ such that for every $E\subset E_1$, mes $E>0$, there exists a set $S_E\subset l_2$ complement of which is a set of first category for which $a\in S_E$ implies $G(a)|_E\notin L^2(E)$, further on E_2 Nikišin's estimate (2) is fulfilled for the system $\{f_n|E_2\}$ at $q=2$.

However, Kašin's counter-example is not only a convergence system but it is also an orthonormal system, but using a fundamental result (Lemma 4 in our paper) of B. Maurey [9, 10] E. Nikišin proved [12]: every convergence system is an almost orthonormal system.

(Definition: The system $\{f_n\}$ is called almost orthonormal on (0,1) if for every $\varepsilon>0$ there exist a Lebesgue measurable set E_ε with $E_\varepsilon\subset(0,1)$, mes $E_\varepsilon>1-\varepsilon$ and a constant M_ε depending only on ε, further an orthonormal system $\{\psi_n(\varepsilon, x)\}$ on (0,1) such that $f_n(x)=M_\varepsilon\psi_n(\varepsilon, x)$, if $x\in E_\varepsilon$, $n=1,2,\ldots$).

2. The statement above is an immediate consequence of a more general theorem. To state it we need some preliminaries. Let $S(0,1)$ be the class of almost everywhere finite Lebesgue measurable real functions, defined on the interval (0,1), endowed with the complete metrizable topology of convergence in measure, further B be an arbitrary Banach space. A mapping $G:B\to S(0,1)$ we call (after [8, p. 432], where $R=S(\{\cdot\})$ is considered instead of $S(0,1)$) *convex* if: $G(x)\geqq 0$, $G(x+y)\leqq G(x)+G(y)$, $G(\lambda x)=|\lambda|G(x)$ are fulfilled a. e. on (0,1) for every $x, y\in B$, $\lambda\in R$ (real), e. g. if $L:B\to S(0,1)$ is linear operator then $|L|$ is convex. The most important example for us is the operator G defined in (2). Our main result is:

Theorem 1: *Let $G:L^2(\Omega, A, \mu)\to S(0,1)$ be a convex, continuous mapping, suppose (Ω, A, μ) is σ-finite and $L^2(\Omega, A, \mu)$ is separable. Then there exists a unique decomposition $(0,1)=E_1\cup^* E_2$ into disjoint measurable sets such that for every $\varepsilon>0$ there exists a set $E_\varepsilon\subset E_2$ with* mes $E_\varepsilon>$ mes $E_2-\varepsilon$ *such that: $G:L^2(\Omega, A, \mu)\to L^2(E_\varepsilon)$ is continuous, further for every $E\subset E_1$, with* mes $E>0$, *for each $f\in L^2(\Omega, A, \mu)$ except at most a set of first category depending on E, $G(f)|_E\notin L^2(E)$. There are cases when* mes $E_1=0$ *or* mes $E_2=0$ *further we cannot replace $\varepsilon=0$ (in general).*

Remark 1. This statement is exactly the L^2-variant of a well-known (and as Theorem 2 shows, useful) theorem of S. Saks (cf. [8], p. 36) which states: if $L_n:B\to S(0,1)$ are continuous linear operators on the Banach space B, then there exists a unique decomposition $(0,1)=E_1\cup^* E_2$ into disjoint measurable sets such that: for every $x\in B$ $\sup\{|L_n(x)(t)|:n\}<\infty$ a. e. on E_2; further for every $x\in B$ except at most a set of first category $\sup\{|L_n(x)(t)|:n\}=\infty$ a. e. on E_1. According to a theorem of Banach (cf. [2]) the convex operator $G(x)(t)=\sup\{|L_n(x)(t)|:n\}:B\to S(0,1)$ is continuous if it is defined on B, so our remark is true.

*) The paper of S. Saks is not available in Hungary.

To prove Theorem 1, we need some lemmas. The essence is contained in:

Lemma 1. *Let* $\{f_i : i \in I\} = S(0,1)$ *be an arbitrary set. Then there is unique decomposition* $(0,1) = E_1 \cup^* E_2$ *such that for every* $\varepsilon > 0$ *there exists* $E_\varepsilon \subset E_2$ *with* $\text{mes}\, E_\varepsilon > \text{mes}\, E_2 - \varepsilon$ *for which* $f_i | E_\varepsilon \in L^2(E_\varepsilon)$ *for every* $i \in I$ *further for every* $E \subset E_1$ *with* $\text{mes}\, E > 0$ *there exists an* $i_E \in I$ *with* $f_{i_E}|_E \notin L^2(E)$.

Proof. Define

$$\varkappa_1 = \{E \subset (0,1) \text{ measurable} : E \text{ has properties of } E_1\},$$
$$\varkappa_2 = \{E \subset (0,1) \text{ measurable} : E \text{ has properties of } E_2\}.$$

It is obvious that if $E_j \in \varkappa_1$ (resp. $\varkappa_2$) for $j = 1, 2, \ldots$, then $\bigcap_1^\infty E_j \in \varkappa_1$ (resp. $\varkappa_2$), further $\varkappa_1 \cap \varkappa_2 = \emptyset$. Choose $\{E_j^{(1)}\} \subset \varkappa_1$ and $\{E_j^{(2)}\} \subset \varkappa_2$ so that:

$$\text{mes } E_j^{(1)} \to \sup_{E \in \varkappa_1} \text{mes}\, E \text{ and } \text{mes}\, E_j^{(2)} \to \sup_{E \in \varkappa_2} \text{mes}\, E \text{ as } j \to \infty.$$

It is easy to see that $E_1 = \bigcup_1^\infty E_j^{(1)}$ and $E_2 = \bigcap_1^\infty E_j^{(2)}$ are satisfactory. The uniqueness is trivial. Lemma 1 is proved.

Remark 2. The same argument gives: If $F_1 = \{f_i^{(1)} : i \in I\}$, $F_2 = \{f_i^{(2)} : i \in I\}, \ldots$ are families of functions in $S(0,1)$, then there exists a unique decomposition $(0,1) = E_1 \cup^* E_2$ into disjoint measurable sets so that for every $i \in I : \sup_n |f_i^{(n)}(t)| < \infty$ a. e. on E_2 and for every $E \subset E_1$ with $\text{mes}\, E > 0$ there exists an $i_E \in I$ for which $\sup_n |f_{i_E}^{(n)}(t)| = \infty$ a. e. on E.

Using this, we can give a new proof for Saks' theorem, mentioned in Remark 1, which is much simpler than the original one given in [8, p. 36]. Namely, let $I = \{x \in B : \|x\|_B \leq 1\}$ and $f_x^{(n)} = L_n(x)$. If there is a set $B' \subset I$ of second category, such that for every $x \in B'$: $\text{mes}\, \{t \in E_1 : \sup \{|L_n(x)(t)| : n\} < \infty\} > 0$, then one can see, using only the first step of Saks' proof, that there exists a set $E \subset E_1$ for which $\sup \{|L_n(x)(t)| : n\} < \infty$ a. e. on E, for every $x \in B$ and $\text{mes}\, E > 0$, which is contradictory to the property of E_1. (At this step we use the completeness of B as Baire's category theorem, but only once. Saks' proof uses it four times.)

This new proof was the starting point of the generalization, because the old one does not seems to work for L^2-case.

Lemma 2. *Let* $G : B \to S(0,1)$ *be a convex continuous operator on the separable Banach space* B. *Then there exists a sequence* $L_n : B \to S(0,1)$, $n = 1, 2, \ldots$ *of continuous linear operators such that, for every* $x \in B$ $G(x)(t) = \sup_n |L_n(x)(t)|$ *a. e. on* $(0,1)$.

Proof. Let $\{x_n\} \subset B$ be a countable dense set in B. Define $L_n(\lambda x_n) = \lambda G(x_n)$ and extend L_n by Hahn-Banach's theorem to the whole B so that $|L_n(x)| \leq G(x)$ be fulfilled for every $x \in B$, a. e. on $(0,1)$. (The proof of Hahn-Banach's theorem works, because as it is probably well known, $S(0,1)$ is an ordered-complete lattice, i. e. every non-empty ordered-upper bounded set has smallest upper bound. We do not know literature on this, so we sketch a short proof for it. Suppose $\emptyset \neq \tilde{S} \subset S(0,1)$ and $f \leq g$ for every $f \in \tilde{S}$ for some $g \in S(0,1)$ Let $\alpha = \sup\{\int_0^1 \text{arctg}\, f : f \in \tilde{S}\}$. One can choose $\{f_n\} \subset \tilde{S}$ such that $f_1 \leq f_2 \leq \cdots$ and $\int_0^1 \text{arctg}\, f_n \to \alpha$. It is easy to see using Lebesgue's dominated convergence theorem (cf. B. Sz.-Nagy [14]) that $\tilde{g}=\lim f_n$ is the smallest upper bound of $\tilde{S}$, i. e. $\tilde{g}=\sup \tilde{S}$). Now to prove $G(x)=\sup\{|L_n(x)|:n\}$ it is enough to remark that obviously $\sup\{|L_n(x)|:n\}\leq G(x)$ for every $x\in B$, so both G and $\sup\{|L_n|:n\}$ are continuous, further they are equal on the dense set $\{x_n\}$. Lemma 2 is proved.

Remark 3. a) The method of the proof of Lemma 2 shows that the simple notion of convex operator coincides with that of superlinear ones introduced by E. M. Nikišin [11].

b) We give a new proof for Lemma 1 and the ordered completeness of S which shows their deeper reason.

Denote by α the equivalence classes of Lebesgue measurables sub sets of $(0,1)$, i. e. let $E_1\sim E_2$ if $\rho(E_1, E_2)=\text{mes}\{(E_1\setminus E_2)\cup(E_2\setminus E_1)\}=0$. α is a lattice in the natural ordering and this lattice is ordered-complete, as it follows from the same property of $S(0,1)$. One can prove easily also the converse of this. The metrical space (α, ρ) is complete, and it is easily seen that this completeness is equivalent to the completeness of $L^p(0,1)$ spaces (or $S(0,1)$). Finally if $\{A_i:i\in I\}\subset\alpha$ is an arbitrary set, then for every $\varepsilon>0$ there exists a finite index set $J(\subset I)$ for which $\text{mes}\{\sup_{i\in I} A_i \setminus \sup_{i\in J} A_i\}<\varepsilon$.

This property is equivalent to the statement of Lemma 1. We leave the proof to the reader.

Lemma 3. *Suppose $G:B\to S(0,1)$ is continuous convex operator on the separable Banach space B and $G(B)\subset L^2(0,1)$. Then $G:B\to L^2(0,1)$ is continuous.*

Proof. It is an adaptation of a Theorem of Banach (cf. [2], Theorem 1). According to Lemma 2: $G=\sup\{|L_n|:n\}$, where $L_n: B\to S(0,1)$ $(n=1,2,\ldots)$ are continuous linear operators. From the convexity of G, $|G(x)-G(y)|\leq G(x-y)$ follows and so it is enough to prove the continuity of G at $O_B\in B$. Now suppose indirectly, G is not coninuous from G into $L^2(0,1)$, at $O_B\in B$. Then there exist sequences $\{\alpha_n\}\subset R$, $\{x_n\}\subset B$ such that

1. $\|x_n\|_B=1$ $(n=1,2,\ldots)$;
2. $\lim\alpha_n=+\infty$;
3. $\|G(x_n)\|_{L^2(0,1)}>\alpha_n$ $(n=1,2,\ldots)$.

Denote $G_N(x)(t)=\sup\{|L_n(x)(t)|:1\leq n\leq N\}$. Obviously $G_N(x)(t)\nearrow G(x)(t)$ for every $x\in B$, a. e. on $(0,1)$, as $N\to\infty$. Hence by Lebesgue's dominated convergence theorem (cf. [14]):

$$\|G_N(x)\|_{L^2(0,1)}\to\|G(x)\|_{L^2(0,1)} \text{ as } N\to\infty, \text{ for every } x\in B. \tag{4}$$

This makes possible to define the following quantities:

$$N_x=\min\{N:\|G(x)\|_{L^2(0,1)}\leq\|G_N(x)\|_{L^2(0,1)}+1\},$$

$$\gamma(n)=\sup\{r\in R:\|x\|_B\leq r \text{ implies } \|G_n(x)\|_{L^2(0,1)}\leq 1\}.$$

Pick a sequence $\{n_k\}$ of natural numbers, for which

(a) $\quad \sum_{k=1}^{\infty} x_{n_k}\alpha_{n_k}^{-1/2}$ converges in B,

$$\left.\begin{array}{ll}\text{b)} & \| G(\alpha_{n_k}^{-1/2}\sum_{s=1}^{k-1}\alpha_{n_s}^{-1/2}x_{n_s})\|_{L^2(0,1)}<1/2 \\ \text{(c)} & \|\sum_{s=k+1}^{\infty}\alpha_{n_s}^{-1/2}x_{n_s}\|_B<\gamma(N_{x_{n_k}})\end{array}\right\} k=1,2,\ldots$$

are fulfilled. We can choose such a sequence $\{n_k\}$ by induction. Condition c) will be fulfilled if we choose n_k in the k-th step so that $\alpha_{n_k}^{-1/2}<2^{-1}\min\{\alpha_{n_{k-1}}^{-1/2}, \gamma(N_{x_{n_{k-1}}})\}$. The following inequality follows easily from the convexity of the operators G, G_N $(N=1,2,\ldots)$ and from the definition of G_N:

$$G_{N_{x_{n_k}}}\left(\sum_{s=1}^{\infty}\alpha_{n_s}^{-1/2}x_{n_s}\right)\geq G_{N_{x_{n_k}}}\left(\alpha_{n_k}^{-1/2}x_{n_k}\right)-G\left(\sum_{s=1}^{k-1}\alpha_{n_s}^{-1/2}x_{n_s}\right)$$
$$-G_{N_{x_{n_k}}}\left(\sum_{s=k+1}^{\infty}\alpha_{n_s}^{-1/2}x_{n_s}\right).$$

Using Minkovski's inequality in $L^2(0,1)$, we get from the last inequality by (b), (c) and 3.:

$$G_{N_{x_{n_k}}}\left(\sum_{s=1}^{\infty}\alpha_{n_s}^{-1/2}x_{n_s}\right)\geq\alpha_{n_k}^{1/2}-\alpha_{n_k}^{1/2}/2-1\to\infty \quad as \quad k\to\infty.$$

Taking into consideration (4), we obtain:

$$\| G(\sum_{s=1}^{\infty}\alpha_{n_s}^{-1/2}x_{n_s})\|_{L^2(0,1)}=\infty,$$

which contradicts the assumption: $G(B)=\{G(x): x\in B\}\subset L^2(0,1)$. Lemma 3 is proved.

Lemma 4. (B. Maurey [9, 10]): *Let $L: L^2(\Omega, A, \mu)\to S(0,1)$ be continuous linear operator and suppose (Ω, A, μ) is σ-finite. Then for every $\varepsilon>0$ there exists a measurable set $E_\varepsilon\subset(0,1)$ with* mes $E_\varepsilon>1-\varepsilon$ *such that L is continuous from $L^2(\Omega, A, \mu)$ into $L^2(E_\varepsilon)$.**

Now we prove Theorem 1. Lemma 1 gives the decomposition $(0,1)=E_1\cup{}^*E_2$ and Lemma 3 shows that $G: L^2(\Omega, A, \mu)\to L^2(E_\varepsilon)$ is continuous. To prove the statement on the category we use Lemma 4 and Lemma 2. By Lemma 2: G is of the form $\sup\{|L_n|: n\}$, where $L_n: L^2(\Omega, A, \mu)\to S(0,1)$ are continuous linear operators. For given $\eta>0$ we can choose $E_\eta\subset(0,1)$ with mes $E_\eta>1-\eta$ such that $L_n: L^2(\Omega, A, \mu)\to L^2(E_\eta)$ are continuous for $n=1,2,\ldots$. (We used Lemma 4.) Let $E\subset E_1$, mes $E>0$. Choose $\eta>0$ above so that mes $(E\cap E_\eta)>0$ be fulfilled. Obviously,

$$(A=)\{x\in B: G(x)\in L^2(E\cap E_\eta)\}=\bigcup_{n=1}^{\infty}\{x\in B: \| G_N(x)\|_{L^2(E\cap E_\eta)}\leq n, N=1,2,\ldots\}.^{**}$$
$$(G_N=\sup\{|L_n|: 1\leq n\leq N\}).$$

The set A is of type F_σ, i. e. the union of many countable closed sets. Suppose, indirect, $\{x\in B: G(x)\in L^2(E)\}$ is a set of second category, then so is A and because it is of type F, there exists an n for which: $\{x\in B: \| G_N(x)\|_{L^2(E\cap E_\eta)}\leq n, N=1,2,\ldots\}$ contains a ball, hence $G(x)\in L^2(E\cap E_\eta)$ by Lebesgue's theorem, for every $x\in B$, which contradicts $E\subset E_1$. So indeed $\{x\in B: G(x)\in L^2(E)\}$ is a set of first category in B. The inequality (1) shows

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* Lemma 1 makes possible to give an essentially new simple proof for Lemma 4. (cf. [6]).

** Next $G(x)\in L^2(E)$ means: $G(x)|_E\in L^2(E)$, etc.

the possibility mes $E_1=0$. Kašin's result mentioned at (3) provides a G for which mes $E_2=0$. Previously such a G (for which mes $E_2=0$) was given by the author in [5], namely if $\{0\leqq\varphi_n)\subset L^\infty(0,1)$ is such that for $y>0$ mes $\{\varphi_n(t)>y\}<1/y$, $n=1,2,\ldots$, further for every $E\subset(0,1)$ with mes $E>0:\limsup_{n\to\infty}\int_E\varphi_n=\infty$, then for $G(a)=\sup|a_n\sqrt{\varphi_n}|:l_2\to S(0,1)$ mes $E_2=0$ in the decomposition. This is proved in [5]. The fact that we cannot replace $\varepsilon=0$ in Theorem 1 follows from the same statement for Maurey's result (Lemma 4), namely $|L|$ is convex if L is linear. Theorem 1 is proved.

3. Next we give an application of Saks' theorem mentioned in Remark 1. Let D be an arbitrary N-dimensional domain.* Denote by $\{u_k\}$ the system of eigenfunctions of the Laplace operator for the first boundary-value problem, i. e.

$$\triangle u+\lambda u=0,\quad u|_{\partial D}=0.$$

As it is well known, the eigenfunctions $\{u_k\}$ form a complete orthonormal system in $L^2(D)$, further they are bounded. Define for $f\in L^1(D)$ its Riesz-Bochner means of order α by

$$\sigma_\mu^\alpha(f,x)=\sum_{\sqrt{\lambda_k}\leqq\mu}(1-\mu^{-2}\lambda_k)^\alpha f_k u_k(x),$$

where

$$f_k=\int_D f(x)\overline{u_k(x)}dx.$$

(For a good survey on convergence problems for eigenfunction expansions, or more generally spectral expansion, we refer the reader to [1].) We shall prove a theorem which at $D=T^N$ (torus) reduces to Theorem 8 of [15].

Theorem 2. *For every* $p\in[1,2N/(2N+1)$, $0\leqq\alpha<N(1/p-1/2)-1/2$ *there exists an* $f\in L^p(D)$ *such that* $\sup_\mu|\sigma_\mu^\alpha(f,x)|=\infty$ *a. e. on* D.

Proof. Nikišin's proof for his Theorem 4 in [13] gives: for every $E\subset D$ with mes $E>0$ there exists $f_E\in L^p(D)$ for which

$$\sup_\mu|\sigma_\mu^\alpha(f,x)|=\infty\ \text{for a. e. } x\in F,$$

where $F\subset E$, mes $F>0$. From this Saks' theorem given in Remark 1 gives Theorem 2. Theorem 2 is proved.

There are also some other results in [1] which are generalizable in a similar manner.

4. At last we remark the following important fact, concerning Theorem 2.

Let $\{L_n\}:B\to S(0,1)$ be a sequence of continuous linear operators on the Banach space such that for every $x\in B'(\subset B)$, $\{L_n(x)\}$ converges a. e. on (0,1), where B' is a dense set in B. Then there exists an $x_0\in B$ with $\sup_n|L_n(x_0)(t)|=\infty$ for a. e.

$t\in(0,1)$ *if and only if for every set* $E\subset(0,1)$ *with* mes $E>0$
(5) *there exists* $x_E\in B$ *for which* $\sup_n|L_n(x_E)(t)|=\infty$** *a. e. on* E.

* Suppose ∂D satisfy some continuity conditions (cf. S. Sobolev [16]).

** One can replace this condition with the following one: for every set $E\subset(0,1)$ with mes $E>0$: $\{L(x)\}$ cannot converge a. e. on E, for every $x\in B$.

P r o o f. This is an immediate consequence of Saks' theorem and of the following theorem of Banach [2]: under the assumptions above: $\{L_n(x)\}$ converges a. e. on (0,1) for every $x \in B$, if and only if, for every $x \in B$: $\sup_n |L_n(x)(t)| < \infty$ for a. e. $t \in (0,1)$.

(That is, at a. e convergence the finiteness of $\sup\{|L(x)|:n\}$ a. e. for every x plays a role analogous to the uniform boundedness of the norms $\|L_n\|$ in norm-convergence). The proof is complete. At last an interesting open problem of P. Erdös and P. Turán:

Given a system of notes $\{x_k^n\}_{k=1}^n$ $(n=1, 2, \ldots)$ in $[-1,1]$ arbitrarily. Is there an $f \in C[-1,1]$ for which its Lagrange-interpolation sequence based on this point set diverges a. e. on $(-1,1)$? I hope the answer is yes. To prove this it is enough to prove (5) for this case. Unfortunately I can prove (5) only for intervals.

The just mentioned thoughts allow to understand the reason of possibility of divergence a. e. of sequences of non-translation-invariant operators. For translation-invariant case see Stein [15].

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