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ON SOME EXAMPLES OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS, RELATED TO THE CLASSICAL ORTHOGONAL POLYNOMIALS*

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The problem of obtaining a new class of positive linear operators based on the generating function for Jacobi polynomials and their approximation properties is studied. Among other things we consider the operators of the form:

$$L_{n,z}(f;\omega)\equiv(1+2\omega u)\left(\frac{1-z^2}{1-u^2}\right)^n\sum_{k=0}^{\infty}f(h/n)P_k^{(n,n)}(z)(-2\omega)^k$$

for $0\leqq 2\omega<|z|-\sqrt{z^2-1}$ $(z<-1)$, where $u\equiv(2z+2\omega)/(1+R)$, $R\equiv(1+4\omega z+4\omega^2)^{1/2}$. $P_k^{(n,n)}(\cdot)$ denotes the k-th polynomial of Jacobi with the parameters $\alpha=\beta=n$, $n\in N$.

In the case of Laguerre polynomials a similar problem was considered by E. Cheney and A. Sharma in 1963. This way they obtained an interesting extension of W. Meyer-König and K. Zeller operator.

We use our method of the Lagrange series announced in the Proceedings of the International Conference on Approximation Theory of Function, Kaluga, July 24-28, 1975.

This report is a continuation of our note [4], where the operators generated by Lagrange series have been considered, and is related to the work [2] of the E. W. Cheney and A. Sharma. The method of Lagrange series enables us to obtain two interesting examples of linear positive operators. They are based on the Jacobi and Laguerre polynomials, and the latter resembles one by Cheney and Sharma. Our method of study differs from those of [2]. The key for this is: the formula giving a representation of the operators in terms of much simple operators (see Lemma 1, [4]: cf. also Lemma 1.2 below), and connections between Szász-Mirakyan's and Bernstein's operators, and Jacobi, Laguerre polynomials. These connections will be shown in Sections 2 and 3.

The main theorems of the paper are in Section 4. In Section 1 we begin by recalling some results from [4] (in somewhat extended form). Next we define the operators. We emphasize their connections with Lagrange series, however a more direct procedure is possible. The results of the report have an introductory character.

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1. Introduction. Let us denote by Ω any domain contained, with closure $\bar{\Omega}$, in the set of holomorphicity of a given function σ, such that $\sigma(v) \neq 0$ on $\bar{\Omega}$ and the boundary $\partial\Omega$ of Ω represents a regular closed contour.

For the function σ and the domain Ω we define a function, on the complex plane Z, by

$$r(z) \equiv \min \{ |(v-z)/\sigma(v)| : v \in \partial\Omega \}. \tag{1.1}$$

Now we present a theorem that follows directly from Lagrange's theorem ([7], cf. also [6] ex. 207).

Theorem 1.1. *A map u defined by the equation*

$$u - z = \omega\sigma(u) \tag{1.2}$$

on the set of (ω, z) where $|\omega| < r(z)$ and $z \in \Omega$ is a function with range in Ω. Moreover, if g is a function holomorphic on $\bar{\Omega}$, then for (ω, z) from the set mentioned above

$$D_\omega (g\circ u)(\omega, z) = g'(u)\frac{\sigma(u)}{1-\omega\sigma'(u)} = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} D_z^k (g'(z)\sigma^{k+1}(z)), \tag{1.3}$$

where D_z^k denotes the operator of the k-th derivative at the point z, and $u \equiv u(\omega, z)$ (here the symbol u denotes very often both the function and its value $u(\omega, z)$ at (ω, z)).

The term "Lagrange series" we reserve for expansions of the above kind whenever Ω is prescribed.

Similar to [4], we begin considering the expression

$$Q(f;\ \omega, z) = (D(g\circ u)(\omega, z))^{-1} \sum_{k=0}^{\infty} f(k/\lambda)\frac{\omega^k}{k!} D^k(g'(z)\sigma^{k+1}(z)), \tag{1.4}$$

where $f(\cdot)$, λ denote a function and a positive real number, respectively (in [4], (1.4) is denoted by $H_{g,1}(f;\ \omega)$, for z fixed).

Lemma 1.2. *Let $g^{(i)}(v) \neq 0$ for $i = 0, 1, \ldots, s$ and $z \in \Omega$. Then, for $e_t(x) \equiv x^t$ where $t = 0, 1, \ldots, s$, the formula (1.4) defines continuous functions $Q(e_t; \cdot)$ on the set of all pairs (ω, z) with $|\omega| < r(z)$ and $z \in \Omega$ Moreover,*

$$Q(e_t;\ \omega, z) = \sum_{k=1}^{t} \lambda^{-t+k} A_{k,t}(\omega, z) \prod_{i=1}^{k} \chi_i(\omega, z),\ t = 1, 2, \ldots, s\ (Q(e_0; \omega, z) \equiv 1) \tag{1.5}$$

where

$$A_{t,k}(\omega, z) \equiv \sum_{i \geq 1,k}^{t} a_i(t)\,\omega^{i-k}\, C_{k+1,\,i+1}(u)/C_{k+1,\,k+1}(u), \tag{1.6}$$

$$\chi_i(\omega, z) \equiv \lambda^{-1}.(D^i g'(u)/(D^{i-1} g'(u)).\,\omega u'_\omega(\omega, z) \tag{1.7}$$

and

$$a_i(t) \equiv \frac{1}{i!} \sum_{l=0}^{i-1} (-1)^l \binom{i}{l} (i-l)^t,$$

$$C_{k,i}(u) \equiv \sum_{i=0}^{k-1} (-1)^l \binom{k}{l} u^l(\omega, z) D_\omega^i (u^{k-l}(\omega, z)); \tag{1.8}$$

$$1 \leq k \leq i, \quad u \equiv u(\omega, z).$$

A proof of this lemma runs analogous to that of Lemma 1 from [4], therefore we omit it here. It should be noted that the above Lemma, with z fixed in Ω, is a consequence from the mentioned Lemma 1.

We now go to consider two particular cases of (1.3). It is well known that for $|\omega|<1$ and $z\in R^1$

$$(1.9)\qquad L_\alpha(\omega, z)\equiv(1-\omega)^{-\alpha-1}\exp\left(-\frac{z\omega}{1-\omega}\right)=\sum_{k=0}^{\infty}L_k^\alpha(z)\,\omega^k;\quad \alpha>-1,$$

where $L_k^\alpha(z)$ denotes the k-th Laguerre polynomial with a parameter α (e.g. [8], eq. 5.1.9). Also, from [8]:

$$(1.10)\quad j_{\alpha,\beta}(\omega,z)\equiv 2^{\alpha+\beta}R^{-1}(1-\omega+R)^{-\alpha}(1+\omega+R)^{-\beta}=\sum_{k=0}^{\infty}J_k^{(\alpha,\beta)}(z)\,\omega^k;\ \alpha,\beta>-1,$$

where $R\equiv(1-2\omega z+\omega^2)^{1/2}$ and $J_k^{\alpha,\beta}(z)$ denotes the k-th Jacobi polynomial with parameters α, β. The latter expansion is valid for $|z|\leq 1, |\omega|<1$ (by Theorem 7.32.1 from [8]). Moreover, it holds true for $|z|>1$ and $|\omega|<|z|-\sqrt{z^2-1}$ (for instance, by formula VI. 41 in [8]). On the other hand, these expansions are known as special cases of the expansion (1.3) (e. g. [6], ex. 207). Indeed, taking $g'(z)=z^{\alpha-1}e^{-z}(\equiv\varrho_{\alpha-1}(z))$, $\sigma(z)\equiv z$, or $g'(z)=(1-z)^{\alpha-1}(1+z)^{\beta-1}$ $(\equiv\sigma_{\alpha-1,\,\beta-1}(z))$, $\sigma(z)\equiv(z^2-1)/2$, in (1.3) yields (1.9), or (1.10), respectively. Moreover, the following proposition holds.

P r o p o s i t i o n 1.3 A. *Let (ω, z) be a given pair of real numbers such that $|\omega|<1$, $z<0$. Then there exists a domain Ω (of the type mentioned in Theorem 1.1) that $z\in\Omega$ and $|\omega|<r(z)<1$.*

B. *Let (ω, z) be a given pair of real numbers such that $|z|>1, |\omega|<|z|-\sqrt{z^2-1}$. Then there exists a domain Ω (of the type mentioned in Theorem* 1.1) *that $z\in\Omega$, $|\omega|<r(z)<|z|-\sqrt{z^2-1}$.*

A sketch of the proof will be presented in Section 5 of this report.

R e m a r k. In the proof of the above proposition an explicit form of Ω is given. Hence, according to our terminology for (1.3), the expressions (1.9), (1.10) become the Lagrange series. We also note that the explicit form of Ω enables us to give, it seems, a new proof of form for sets where (1.9), (1.10) hold true.

In what follows we assume that α, β are natural numbers. It is clear, in view of the above-mentioned dependence of (1.9), (1.10) and Lagrange series (1.3), that for the following particular cases of (1.4):

$$(1.11)\qquad L_\alpha(f;\ \omega, z)\equiv l_\alpha^{-1}(\omega, z)\sum_{k=0}^{\infty}f(k/\lambda_{\alpha-1})\,L_k^\alpha(z)\,\omega^k,$$

where $|\omega|<1$, $z<0$, $\lambda_{\alpha-1}>0$;

$$(1.11)'\qquad J^{\pm}_{\alpha,\beta}(f;\ \omega, z)\equiv j^{-1}_{\alpha,\beta}(\omega,z)\sum_{k=0}^{\infty}f(k/\lambda_{\alpha-1,\beta-1})\,J_k^{(\alpha,\beta)}(z)\,\omega^k,$$

where $|\omega|<|z|-\sqrt{z^2-1}$, $(\pm1)\cdot z>1$, the representations of the type (1.5) hold (if necessary with α, β sufficiently large).

Thus, one obtains the continuity of the functions $L_\alpha(f;\cdot)$, $J_{\alpha,\beta}(f;\cdot)$, on the described above sets of (ω, z), for all functions $f\in C_N$ (i. e. continuous on the real line R^1 and such that $f(x)=O(x^s)$ as $x\to+\infty$ for some positive integer s, in general dependent on f). We shall deal mainly with the following sets of (ω, z):

$$W_L \equiv \{(\omega, z) : \omega \in [0, 1), z < 0\};$$

$$W_J^{\pm} \equiv \{(\omega, z) : 0 \le (\pm 1) \cdot \omega < |z| - \sqrt{z^2 - 1}, (\pm 1) \cdot z > 1\},$$

however, much wider sets are possible. This interesting problem is incomplete now. In the case under consideration $L_a(f; \cdot) \in C(W_L)$, $J^{\pm}_{a,\beta}(f; \cdot) \in C(W_J^{\pm})$ and the linear operators L_a, $J^{\pm}_{a,\beta} : C_N \to C(W_L)$, $C(W_J^{\pm})$ are positive.

Remark. The operators similar to L_a were first introduced in [2]. Denoting this operators by P_n we find the correspondence

$$(1.12) \qquad P_n(h(\tfrac{\cdot}{1-\cdot}); \omega) = L_n(h(\cdot); \omega, z),$$

where $z \le 0$ and fixed, $0 \le \omega < 1$. In [2] it is proved, by the other method, that for $f \in C[0, 1]$, $z/n \to 0$ as $n \to +\infty$, $P_n(f; \cdot)$ converges to f almost uniformly on $[0, 1)$. The case $z = 0$ was the subject of study by Meyer-König and Zeller in [5].

2. Laguerre polynomials and Szász-Mirakyan operators. This section is devoted to the following topics: connection between Laguerre polynomials and Szász-Mirakyan operators, and the explicit form of the formulas (1.5), (1.7) for $g'(v) = \varrho_{a-1}(v)$, $\sigma(v) = v$.

Lemma 2.1. *Let $v \in R^1$ and $0 \le t \le a$ (t — an integer). Then*

$$(2.1) \qquad D^t \varrho_a(v) = a^t v^{a-t} e^{-v} \cdot S_a[\prod_{i=0}^{t-1} (\cdot + \frac{a-i}{a}); \frac{-v}{a}],$$

where S_a denotes Szász-Mirakyan's operator. It is defined, for a given function f, by

$$(2.2) \qquad S_a(f(\cdot); x) \equiv e^{-ax} \sum_{k=0}^{\infty} f(\frac{k}{a}) \frac{a^k}{k!} x^k.$$

Proof. Indeed, (2.1) is a simple consequence from (2.2) and

$$D^t \varrho_a(v) = \sum_{k=0}^{\infty} (-1)^k ((k+a) \ldots (k+a-t+1)/k!) v^{k+a-t}.$$

Corollary. *Let $a \ge t \ge 0$, $v \in R^1$. There holds the formula*

$$(2.3) \qquad L_t^{a-t}(v) = \frac{a^t}{t!} \cdot S_a[\prod_{i=0}^{t-1} (\cdot + \frac{a-i}{a}); \frac{-v}{a}].$$

Proof. Using the Rodrigues formula for the Laguerre polynomials: $L_t^{a-t}(v) = D^t(v^a e^{-v})/(t!\, v^{a-t} e^{-v})$ and Lemma 2.1 we get (2.3).

Proposition 2.2. *For any real $a > 0$ and integer $t \ge 0$ we have*

$$(2.4) \qquad \lim_{a \to +\infty} \| S_a[\prod_{l=0}^{t-1} (\cdot + \frac{a-l}{a}); \frac{\cdot}{a}] - 1 \|_{[0,a]} = 0,$$

where $\|\cdot\|_{[0,a]}$ denotes the supremum norm on $C[0, a]$.

Proof. On account of the well-known property of Szász-Mirakyan operators (see [2], for instance) one obtains that

$$\lim_{\alpha\to+\infty} S_\alpha[\prod_{l=0}^{t-1}(\cdot+\frac{\alpha-l}{\alpha});\ w]=(1+w)^t;\ t\geq 0$$

almost uniformly with regard to $w\geq 0$. Moreover,

$$\| S_\alpha[\prod_{l=0}^{t-1}(\cdot+\frac{\alpha-l}{\alpha});\ \frac{\cdot}{\alpha}]-1\|_{[0,a]}\leq\| S_\alpha[\prod_{l=0}^{t-1}(\cdot+\frac{\alpha-l}{\alpha});\cdot]-(1+\cdot)^t\|_{[0,a]}$$

$$+\|(1+\frac{\cdot}{\alpha})^t-1\|_{[0,a]}.$$

Hence the desired result follows.

On the basis of the above corollary and Proposition 2.2 the following property of the Laguerre polynomials can be deduced.

Corollary. *For any integer $t\geq 0$ the following relation holds*

$$\lim_{\alpha\to+\infty}\alpha^{-t}L_t^\alpha(v)=1/t!, \tag{2.5}$$

almost uniformly with regard to $v\leq 0$ (and more general: for such $v\leq 0$ that $v/\alpha\to 0$). Moreover, for the same $t\geq 0$

$$\lim_{\alpha\to\infty}\alpha^{-t}L_t^\alpha(\alpha w)=(1-w)^t/t! \tag{2.6}$$

almost uniformly with regard to $w\leq 0$.

We remark that the case of point convergence in (2.6) is the subject of example 30 in [8]. It is interesting to point out that because of (2.4) the approximation properties of Szász-Mirakyan operators S_α can be treated as a simple consequence of the property (2.5) for the Laguerre polynomials.

The following result, totally analogous to Lemma 1.2, gives a representation of the operators L_α (for the polynomials e_t) in terms of Szász-Mirakyan operators.

Lemma 2.3. *Let us consider the operators $L_{\alpha+1}: C_N\to C(W_L)$ and let $0\leq t<\alpha+1$ (t, α — integer). Then*

$$L_{\alpha+1}(e_t;\ \omega,z)=\sum_{k=1}^{t}\lambda_\alpha^{-t+k}\cdot\prod_{i=1}^{k}\chi_{\alpha,i}(\omega,\ z)\cdot\sum_{i\geq 1,k}^{t}a_i(t)\cdot(\frac{\omega}{1-\omega})^{i-k}\cdot\binom{i}{k}\cdot\frac{(i+1)!}{(k+1)!}, \tag{2.7}$$

where $a_i(t)$ is defined in Lemma 1.2, and

$$\chi_{\alpha,i}(\omega,\ z)=\frac{\alpha}{\lambda_\alpha}\cdot\frac{\omega}{1-\omega}\cdot S_\alpha[\prod_{l=0}^{i-1}(\cdot+\frac{\alpha-l}{\alpha});\ \frac{-u}{\alpha}]/(S_\alpha[\prod_{l=0}^{i-2}(\cdot+\frac{\alpha-l}{\alpha});\ \frac{-u}{\alpha}]), \tag{2.8}$$

where $u=u(\omega,\ z)=z/(1-\omega)$, $i=1,\ 2,\dots,t$. Moreover,

$$L_{\alpha+1}(e_t;\ \omega,z)=\lambda_\alpha^{-t}\sum_{k=0}^{t}L_k^{\alpha-k}(\frac{z}{1-\omega})\sum_{i\geq 1,k}^{t}a_i(t)\ (\frac{\omega}{1-\omega})^i\, i!\binom{i+1}{k+1} \tag{2.9}$$

for all $(\omega,\ z)\in W_L$ and this representation becomes true also for $z=0$ and $\omega\in[0,1)$. If $t=0$ then $L_{\alpha+1}(e_0;\ \omega,\ z)=1$.

Proof. In the case under consideration we obtain, by (1.1), that $u\equiv u(\omega,\ z)=z/(1-\omega)$. Hence, (1.8) implies the formula

$$C_{k,i}(u)=i!\binom{i-1}{k-1}z^k(1-\omega)^{-k-i},\ k=1,2,\dots,i.$$

Thus (2.7) follows by (1.5). To prove that (2.8) holds we apply Lemma 2.1 to (1.7). Finally, making use of (2.3) to (2.8), we conclude, by (2.7), that the representation (2.9) for $(\omega, z) \in W_L$ holds. The case $z=0, |\omega|<1$ must be carried out independenty. It follows by the formula $L_k^{\alpha+1}(0)=\binom{k+\alpha+1}{k}$ and the identity $k^t=\Sigma_{i=1}^t a_i(t)\, k(k-1)\dots(k-i+1)$, where $a_i(t)$ is defined in Lemma 1.2, that

$$L_{\alpha+1}(e_t;\ \omega,\ 0)=\lambda_\alpha^{-t}(1-\omega)^{\alpha+2}\cdot\sum_{k=0}^{\infty}\sum_{i=1}^{t} a_i(t)\, k\dots(k-i+1)\binom{k+i+\alpha+1}{k}\omega^k$$

$$=\lambda_\alpha^{-t}(1-\omega)^{\alpha+2}\sum_{i=1}^{t} a_i(t)\, i!\cdot\frac{(\alpha+i+1)\dots(\alpha+2)}{i!}\,\omega^i\sum_{k=0}^{\infty}\binom{k+i+\alpha+1}{k}\omega^k.$$

Now, because of (1.9) and the well-known identity

$$\binom{\alpha+i+1}{i}=\sum_{k=0}^{i}\binom{\alpha}{i}\binom{i+1}{k+1},$$

after changing the order of summation, one can obtain the desired result:

$$L_{\alpha+1}(e_t,\ \omega,\ z)=\lambda_\alpha^{-t}\sum_{k=0}^{t}\binom{\alpha}{k}\sum_{i\geq 1,k}^{t} a_i(t)\, i!\left(\frac{\omega}{1-\omega}\right)^i\binom{i+1}{k+1},$$

that is, a particular case of (2.9) $(z=0)$.

3. Jacobi and Bernstein polynomials. In this section we shall consider the topics analogous to those of the preceding section. Now, it will be assumed that $g'(v)=\varrho_{\alpha,\beta}(v)$ (see section 1), $\sigma(v)=(v^2-1)/2$. In this report we are not interested in precise constraints on the integer parameters α, β, t that appear below.

Lemma 3.1. *Let t be any positive integer. For all real $v\neq+1$*

(3.1) $$D^t\varrho_{\alpha,\beta}(v)=\varrho_{\alpha,\beta}(\alpha/(v+1))^t B_\alpha\Big[\sum_{l=0}^{t-1}\Big(\cdot+\frac{\beta-l}{\alpha}\Big);\ \frac{v+1}{v-1}\Big],$$

where $(v+1)/(v-1)\in[0,\ 1)$ whenever $v\leq-1$, and for all $v\neq-1$

(3.2) $$D^t\varrho_{\alpha,\beta}(v)=\varrho_{\alpha,\beta}(\beta/(v-1))^t B_\beta\Big[\prod_{l=0}^{t-1}\Big(\cdot+\frac{\alpha-l}{\beta}\Big);\ \frac{v-1}{v+1}\Big],$$

where $(v-1)/(v+1)\in[0,\ 1)$ whenever $v\geq+1$. $B_s[f(\cdot);\ w]$ denotes the s-th Bernstein's polynomial for a function f:

$$B_s[f(\cdot);\ w]\equiv\sum_{k=0}^{s} f\Big(\frac{k}{s}\Big)\binom{s}{k} w^k(1-w)^{s-k}.$$

Proof. It suffice to prove (3.1) because a proof for (3.2) runs analogously. We have

$$D^t\varrho_{\alpha,\beta}(v)=(-1)^\alpha D^t[(1+v)-2]^\alpha(1+v)^\beta=\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k}2^{\alpha-k}D^t(1+v)^{\beta+k}$$

$$=(1+v)^{\beta-t}(1-v)^\alpha\alpha^t\sum_{k=0}^{\alpha}\prod_{l=0}^{t-1}\Big(\frac{k}{\alpha}+\frac{\beta-l}{\alpha}\Big)\binom{\alpha}{k}\Big(\frac{v+1}{v-1}\Big)^k\cdot\Big(\frac{2}{1-v}\Big)^{\alpha-k}$$

and the desired result follows.

Corollary. *Let α, $\beta>t\geqq 0$ are integer. For all real $v\neq+1$*

$$J_t^{(\alpha-t,\beta-t)}(v)=\frac{1}{t!}(\alpha(v-1)/2)^t B_\alpha\Big[\sum_{l=0}^{t-1}(\cdot+\frac{\beta-l}{\alpha});\ \frac{v+1}{v-1}\Big] \tag{3.3}$$

and for all real $v\neq-1$

$$J_t^{(\alpha-t,\beta-t)}(v)=\frac{1}{t!}(\beta(v+1)/2)^t B_\beta\Big[\prod_{l=0}^{t-1}(\cdot+\frac{\alpha-l}{\beta});\ \frac{v-1}{v+1}\Big]. \tag{3.4}$$

Proof. Using the Rodrigues formula for the Jacobi polynomials

$$J_t^{(\alpha-t,\,\beta-t)}(v)=((-1)^t/(2^t\,t!\,\varrho_{\alpha-t,\,\beta-t}(v))\cdot D^t\varrho_{\alpha,\beta}(v)$$

and (3.1) the result (3.3) follows at once. A proof for (3.4) runs analogously.

Proposition 3.2. *Let (α_n), (β_n) denote two sequences of natural numbers. It is assumed that their limits exist in $[2, +\infty]$. Let us define*

$$p\equiv\lim_{n\to+\infty}\beta_n/\alpha_n \text{ and } q\equiv\lim_{n\to\infty}\alpha_n/\beta_n. \tag{3.5}$$

(To simplify our notation we will often write α, β instead α_n, β_n.) For all integers $t\geqq 0$

a) *if $\beta=O(\alpha)$ as $\alpha\to+\infty$ then*

$$(p+\frac{v+1}{v-1})^t=\lim_{\alpha\to\infty}B_\alpha\Big[\prod_{l=0}^{t-1}(\cdot+\frac{\beta-l}{\alpha});\ \frac{v+1}{v-1}\Big]$$

uniformly with regard to $v\in(-\infty, -1]$;

b) *if $\alpha=O(\beta)$ as $\beta\to+\infty$ then*

$$(q+\frac{v-1}{v+1})^t=\lim_{\beta\to\infty}B_\beta\Big[\prod_{l=0}^{t-1}(\cdot+\frac{\alpha-l}{\beta});\ \frac{v-1}{v+1}\Big]$$

uniformly with regard to $v\in[1, +\infty)$.

Proof. This proposition follows at once from the expression

$$B_\alpha\Big[\prod_{l=0}^{t-1}(\cdot+\frac{\beta-l}{\alpha});\ w\Big]=B_\alpha[e_t;\ w]$$

$$+\sum_{i=1}^{t}B_\alpha[e_{t-i};\ w]\sum_{0\leqq l_1<\ldots<l_i\leqq t-1}\frac{\beta-l_1}{\alpha}\cdots\frac{\beta-l_i}{\alpha}$$

and the well-known property of Bernstein polynomials: for all $f\in C[0, 1]$ $\lim_{s\to+\infty}B_s[f;\ w]=f(w)$ uniformly on $[0, 1]$.

As in Section 2, on the basis of the above Proposition and (3.3), (3.4) we obtain the following corollary.

Corollary. *For the sequences (α), (β) and p, q defined as in Proposition 3.2*

a) *if $\beta=O(\alpha)$ as $\alpha\to+\infty$ then for all integers $\beta>t\geqq 0$*

$$\lim_{\alpha\to+\infty}t!\,(2/\alpha(v-1))^t J_t^{(\alpha-t,\beta-t)}(v)=(p+(v+1)/(v-1))^t \tag{3.6}$$

uniformly with regard to $v\in(-\infty, -1]$;

b) *if* $a=O(\beta)$ *as* $\beta\to+\infty$ *then for all integers* $a>t\geqq 0$

$$\lim_{\beta\to+\infty} t!\,(2/\beta(v+1))^t J_t^{(a-t,\,\beta-t)}(v)=(q+(v-1)/(v+1))^t$$

uniformly with regard to $v\in[1,\,+\infty)$.

Remark. These properties of the Jacobi polynomials in the prescribed sense extend the following one (cf. [8], ex. 26): for any fixed β, t, x (now a, β, and x — reals), a, $\beta>-1$,

$$\lim_{a\to+\infty} a^{-t} J_t^{(a,\beta)}(x)=((x+1)/2)^t/t!$$

Now we are going to state the result, for the operators $J^{\pm}_{a,\beta}$ defined in Section 1 (cf. (1.11) and below), which is analogous to Lemma 1.2.

Lemma 3.3. *For any integer* $0\leqq t<a$, β *and* $(\omega,\,z)\in W_J^{\mp}$

$$J^{\pm}_{a+1,\,\beta+1}(e_t;\,\omega,\,z)=\sum_{k=1}^{t}\lambda_{a,\beta}^{-t+k}\cdot\prod_{l=1}^{k}\chi^{\pm}_{a,\beta,l}\,(\omega,\,z)\cdot A_{t,k}(\omega,\,z), \tag{3.7}$$

where $A_{t,k}(\omega,\,z)$ *is defined in* (1.6) *with*

$$u\equiv u(\omega,\,z)=(2z-\omega)/(1+R);\; R\equiv(1-2\omega z+\omega^2)^{1/2}. \tag{3.8}$$

Moreover, for $\chi^{\pm}_{a,\beta,i}$ *defined in general by* (1.7) *we have*

(3.9)

$$\chi^{-}_{a,\beta,i}\,(\omega,\,z)=\frac{a}{\lambda_{a,\beta}}\cdot\frac{\omega(u-1)}{2(1-\omega u)}\cdot B_a\,[\sum_{l=0}^{i-1}(\cdot+\frac{\beta-l}{a});\;\frac{u+1}{u-1}]/B_a[\sum_{l=0}^{i-2}(\cdot+\frac{\beta-l}{a});\;\frac{u+1}{u-1}],$$

whenever $(\omega,\,z)\in W_J^-$, *and*

(3·10)

$$\chi^{+}_{a,\beta,i}\,(\omega,\,z)=\frac{\beta}{\lambda_{a,\beta}}\cdot\frac{\omega(u-1)}{2(1-\omega u)}\cdot B_\beta\,[\prod_{l=0}^{i-1}(\cdot+\frac{a-l}{\beta});\;\frac{u-1}{u+1}]/B_\beta[\prod_{l=0}^{i-2}(\cdot+\frac{a-l}{\beta});\;\frac{u-1}{u+1}],$$

whenever $(\omega,\,z)\in W_J^+$. *Also, for* $(\omega,\,z)\in W_J^-$, W_J^+, *there hold*

$(u+1)/(u-1)$, $(u-1)/(u+1)\in[0,\,1)$,

respctively. Everywhere $u=u(\omega,\,z)$ *and is given by* (3.8).

Proof. In view of (1.2) and the assumption; $\sigma(v)=(v^2-1)/2$, the formula (3.8) for the map u holds. Next, in the case under consideration (that is for $g'(v)=\varrho_{a,\beta}(v)$), on account of (1.5) (see also (1.11) and below), it is not difficult to prove that (3.7) holds. To obtain (3.9) and (3.10) we apply (3.1) and (3.2) to (1.7), respectively. Now, it will be observed that for $(\omega,z)\in W_J^-$ $1-2\omega z+\omega^2>0$ and equals to zero whenever $\omega=-(|z|-\sqrt{z^2-1})$, $(z<-1)$. From this follows that $-u\equiv-u(\omega,\,z)=-(2z-\omega)/(1+R)>(|z|+\sqrt{z^2-1})|(1+R)>+1$, for $(\omega,\,z)\in W_J^-$, so that $(u+1)/(u-1)\in[0,\,1)$. To prove that $(u-1)/(u+1)\in[0,1)$ for $(\omega,z)\in W_J^+$ we proceed similarly.

Corollary. *For any integer* $0\leqq t<a$, β *and for* $(\omega,\,z)\in W_J^{\mp}$

$$J^{\pm}_{a+1,\,\beta+1}(e_t;\;\omega,z)=t!\;\lambda_{a,\beta}^{-t}\sum_{k=1}^{t}J_k^{(a-k,\,\beta-k)}\,(u)\cdot A_{t,k}\,(u)\,(\omega/(1-\omega u))^k, \tag{3.11}$$

where $A_{t,k}$ *is defined in* (1.6) *for* u *given by* (3.8).

Proof. Let $(\omega, z) \in W_J^-$. Because of (3.9) and (3.7):

$$J^-_{\alpha+1,\beta+1}(e_t;\ \omega, z)=\lambda^{-t}_{\alpha,\beta}\sum_{k=1}^{t}((\alpha(u-1))/2)^k B_\alpha[\prod_{l=1}^{k-1}(\cdot+\frac{\beta-l}{\alpha});\ \frac{u+1}{u-1}]\cdot(\omega/(1-\omega u))^k\cdot A_{t,k}(\omega,z).$$

Hence, by (3.4), the desired result follows. In the case $(\omega, z)\in W_J^+$ a proof runs analogously using (3.7) and (3.4).

Remark. There are various methods of obtaining (3.11). For example, one can prove (3.11) without using the Bernstein polynomials. But this method is not suitable for our purposes.

4. Main results. In this section we will state our main results on the approximation of functions from C_N by the sequences of the operators L_α, $J^{\pm}_{\alpha,\beta}$ introduced in Section 1. In order to attain it we present the lemma which represents some adapted form of the well-known Korovkin theorem [3].

Lemma 4.1. *Let (Φ_α) be a sequence of positive, linear functionals on C_N. Let $\chi\geqq 0$ be a given real number and suppose that for $t=0, 1, \ldots, s+2$ ($s\geqq 0$ and integer)*

$$\lim_{\alpha\to+\infty}\Phi_\alpha(e_t)=e_t(\chi).$$

Then for each $f\in C_s$

$$\lim_{\alpha\to+\infty}\Phi_\alpha(f)=f(\chi)$$

(C_s denotes the subspace in C_N consisting of those functions for which $f(x)=O(x^s)$ as $x\to+\infty$).

Theorem 4.2. *Suppose that $\mu\equiv\lim_{\alpha\to\infty}\alpha/\lambda_\alpha$ exists in $[0, +\infty)$. Then for any $f\in C_N$*

(4.1) $$\lim_{\alpha\to+\infty}L_\alpha(f;\ \omega, z)=f(\mu\cdot\omega/(1-\omega))$$

almost uniformly with regard to (ω, z) from the set of all pairs (ω, z) that $\omega\in[0, 1)$, $z\leq 0$ (in particular, almost uniformly with regard to $(\omega, z)\in W_L$).

Proof. If possible, let f be function from C_N for which (4.1) does not hold, for some compact subset W of those points that $z\leqq 0$, $\omega\in[0, 1)$. Hence, for some $\varepsilon>0$, a sequence (α_n) (of natural numbers) that tends to infinity, a sequence of points (ω_n, z_n) in W, the following inequality

(4.2) $$|\Phi_n(f)-f(\mu\cdot\omega_n/(1-\omega_n))|>\varepsilon$$

for all n holds. We have denoted: $\Phi_n(f)\equiv L_{\alpha_n+1}(f;\ \omega_n, z_n)$. It may, without loss of generality, be supposed that $\lim_{n\to+\infty}(\omega_n, z_n)=(\omega', z')$ and that $f\in C_s$.

Now, because of the representation (2.9) and property (2.5) one can easily obtain that

$$\lim_{\alpha\to+\infty}\Phi_n(e_t)=e_t(\mu\omega'/(1-\omega'))$$

for $t=0, 1, \ldots, s+2, \ldots$ $(\Phi_n(e_0)\equiv 1)$. Hence, from Lemma 4.1 we obtain that

$$\lim_{\alpha\to+\infty} \Phi_n(f)=f(\mu\omega'/(1-\omega'))$$

for all $f\in C_s$, but this contradicts (4.4), and the theorem is proved.

Remark. The above theorem extends in the prescribed sense the result of Cheney and Sharma on the operators L_α (cf. (1.12) and below). As was already pointed out, the method of proof differs from that of [2]. Indeed, it is based on the properties of Szász-Mirakyan operators (see (2.9), (2.5), (2.3)). On the other hand, for $z=0$ the operator L_α reduces to the one of V. A. Baskakov [1]. Thus the above theorem extends the result of Baskakov on the approximation of functions $f\in C_0$ by the sequence of type $L_{\alpha+1}(f;\ \omega,\ 0)$.

Now, we turn out to the operators $J^{\pm}_{\alpha,\beta}$ defined in (1.11)' and below. At first we note that the following formula holds

$$J^{+}_{\alpha,\beta}(f;\ \omega,\ z)=J^{-}_{\beta,\alpha}(f;\ -\omega,\ -z), \tag{4.3}$$

whenever $\lambda_{\alpha,\beta}=\lambda_{\beta,\alpha}$. Indeed, it is sufficient to note that $-u(\omega,z)=u(-\omega,-z)$ (see (3.8)), $J_k^{(\alpha,\beta)}(u)=(-1)^k J_k^{(\beta,\alpha)}(-u)$ and that $(\omega,\ z)\in W_J^+$ is equivalent to $(-\omega,\ -z)\in W_J^-$. Therefore, it is sufficient to consider the operators $J^{+}_{\alpha,\beta}$ only.

Let (α_n), (β_n) be the sequences considered in Proposition 3.2 and let p, q be defined as in (3.5) and belong to $[0,\ +\infty]$ Denote by α', β' the limits of (α_n), (β_n), assumed to be in $[2,\ +\infty]$. Let $\lambda_{\alpha,\beta}=\lambda_{\beta,\alpha}$.

Theorem 4.3. *For any $f\in C_N$ ($f\in C_s$, respectively) the following relation holds*

$$\lim_{n\to+\infty} J^{+}_{\alpha_n,\beta_n}(f;\ \omega,\ z)=f(\chi(\omega,\ z)) \tag{4.4}$$

almost uniformly with regard to $(\omega,\ z)\in W_J^+$ if either:

a) $\alpha'=+\infty$, $\beta'=+\infty$ (*or* $\beta'<+\infty$; *in this case* $0\leq s<\beta'-1$),
$\chi(\omega,\ z)\equiv\mu^+(1/R-1)$,
$\mu^+\equiv\lim\limits_{n\to\infty}\alpha_n/\lambda_{\alpha_n,\beta_n}$ *exists in* $[0,\ +\infty)$, *whenever* $p=0$ (*that is* $q=+\infty$);

b) $\alpha'=+\infty$, $\beta'=+\infty$

$$\chi(\omega,\ z)\equiv\mu^+(u-1)\ (p+(u+1)/(u-1))/2, \tag{4.5}$$

μ^+ *is such as in* a) (*or equivalently*: $\mu^+=\mu^-/p$ *where* $\mu^-\equiv\lim\limits_{n\to\infty}\beta_n/\lambda_{\alpha_n,\beta_n}$), *whenever* $p\in(0,\ +\infty)$ (*that is* $q=1/p\in(0,\ +\infty)$);

c) $\alpha'=+\infty$ (*or* $\alpha'<+\infty$; *in this case* $0\leq s<\alpha'-1$), $\beta'=+\infty$,

$$\chi(\omega,\ z)\equiv\mu^-(1/R-1), \tag{4.6}$$

μ^- *is such as in* b), *whenever* $p=+\infty$ (*that is* $q=0$). *Everywhere* $u=u(\omega,z)$ *and is given by* (3.18), *and* $R=(1-2\omega z+\omega^2)^{1/2}$.

Remark. The analogous theorem can be formulated for the operators $J^{-}_{\alpha,\beta}$ (say $J^{-}_{\beta,\alpha}$) with W_J^- instead W_J^+.

Proof. This theorem can be proved with making use of the same scheme as in the proof of Theorem 4.2. Therefore, the proof will be sketched only. Let us suppose that for some compact subset $W \subset W_J^+$ and for some $f \in C_N$ (or $f \in C_s$, $0 \leq s < \beta' - 1$, respectively) the relation (4.4) fails. Hence,

$$(4.7) \qquad |\Phi_m(f) - f(\chi(\omega_m, z_m))| > \varepsilon \ (m \to +\infty)$$

for some $\varepsilon > 0$, a sequence $((\omega_m, z_m))$ in W which is convergent to $(\omega', z') \in W$. We denoted

$$\Phi_m(f) \equiv J^+_{\alpha_{n_m}, \beta_{n_m}}(f; \omega_m, z_m),$$

where (α_{n_m}), (β_{n_m}) are subsequences of the sequences (α_n), (β_n), respectively.

At first we note that the continuity of $A_{t,k}(\omega, z)$, in the case under consideration (see (1.6), (1.8) and (3.8)), is justified by the following reasons: continuity of $C_{k,i}(u)$ and $C_{k+1,\,k+1}(u) \neq 0$ on W_J^+ (the former is clear by (1.8) and (3.8); the latter is the consequence of the almost evident formulas $C_{k+1,\,k+1}(u) = u'_\omega(\omega, z) \cdot (k+1) \cdot C_{k,k}(u)$, $C_{1,1}(u) = u'_\omega(\omega, z)$).

Now, suppose that the conditions a), b) are satisfied. To show that

$$(4.8) \qquad \lim_{m \to +\infty} \Phi_m(e_t) = e_t(\chi), \ \chi \equiv \chi(\omega, z),$$

for $\chi(\omega, z)$ given by (4.5), we apply (3.11) and (3.6) (with appropriate restriction for t, for example $s + 2 < \beta' - 1$, whenever $\beta' < +\infty$; this restriction is a consequence of the assumptions in Corollary to Proposition 3.2). Finally, one can obtain, by (4.18) and Lemma 4.1, the contradiction with (4.7).

Let us suppose that the condition c) is satisfied. Note that in fact (because of (4.5), and taking into account minor modifications), we have just proved this theorem for the operators $J^-_{\beta, \alpha}$ with c), b) assumptions. Hence,

$$\lim_{n \to +\infty} J^+_{\alpha_n, \beta_n}(f; \omega, z) = \lim_{n \to \infty} J^-_{\beta_n, \alpha_n}(f; -\omega, -z) = f(\chi(-\omega, -z)),$$

where χ is given by (4.6), and the desired result follows by the equality $\chi(\omega, z) = \chi(-\omega, -z)$, $(\omega, z) \in W_J^+$.

Remark. It should be noted that the limit cases $(z \to \pm 1)$ of the operators $J^\pm_{\alpha,\beta}$ lead us to those one of V. A. Baskakov, ([1], cf. remarks to Theorem 4.2).

5. Proof of Proposition 1.3. We shall give sketches of the proofs only.

A. Fix (ω', z'), where $|\omega'| < 1$ and $z' < 0$. It is sufficient to show that the appropriate choice of Ω is the following

$$\Omega \equiv \Omega_{\gamma,\tau} \equiv \{z \in Z : Re z < 0, \ 0 < \tau < |z| < \gamma\}.$$

It is clear that for suitable γ, τ

$$z' \in I_{\gamma,\tau} \equiv (-\gamma, \ -2/(\gamma^{-1} + \tau^{-1})).$$

Moreover, $I_{\gamma,\tau} \subset \Omega$. Now, for $z' \in I_{\gamma,\tau}$, it is possible to show that

$$r(z') \equiv \min\{|(v - z')/v| : v \in \partial\, \Omega_{\gamma,\tau}\} = 1 - |z'|/\gamma,$$

so that, taking γ large enough, we obtain $|\omega'|<r(z')<1$ (of course, $z'\in I_{\gamma,\tau}$ becomes still true).

B. Fix (ω', z') where $|\omega'|<|z'|-\sqrt{z'^2-1}$ and $z'<-1$. In the case $z'>+1$ a proof runs analogously. There exist $\gamma'>\tau'>1$ such that $z'\in(-\gamma', -\tau')\subset\Omega_{\gamma',\tau'}$ $\equiv\{z\in Z: |z^2-1|<\gamma'^2-1,\ Re z<-\gamma'\}$. It should be noted that this becomes still true, whenever $\gamma>\gamma'$, $1<\tau<\tau'$. Now, one can prove that

$$r(z)\equiv\min\{2\cdot|(v-z)/(v^2-1)|: v\in\partial\Omega_{\gamma,\tau}\}=2(\gamma-|z|)/(\gamma^2-1),$$

for all $z\in(-\gamma, -\tau)$ (for such z the only point in $\partial\Omega_{\gamma,\tau}$, where the minimum is attained, is $v=-\gamma$). For the functions defined in the following manner

$$r_\gamma(z)\equiv\begin{cases}0, & \text{for}\quad z\leq-\gamma;\\ 2(\gamma-|z|)/(\gamma^2-1), & \text{for}\quad z\in(-\gamma, -\tau),\end{cases}$$

for each $z<-1$ (say $z=z'$) we have

$$\widetilde{r}(z)\equiv\sup\{r_\gamma(z):\gamma>\gamma'\}=\max=\{2(\gamma-|z|)/(\gamma^2-1):\gamma>|z|\},$$

and this maximum is attained for $\gamma=|z|+\sqrt{z^2-1}$. Hence, $\widetilde{r}(z)=|z|-\sqrt{z^2-1}$. We point out that for any fixed $z<-1$, $r_\gamma(z)$ is the non-decreasing function of the parameter γ for $1<\gamma<|z|+\sqrt{z^2-1}$. Collecting these facts we obtain that

$$|\omega'|<r_{\gamma''}(z')<|z'|-\sqrt{z'^2-1}$$

for suitable γ''. Thus $\Omega=\Omega_{\gamma'',\tau'}$ is the required domain.

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