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MONOSPLINES OF TWO VARIABLES AND OPTIMAL CUBATURE FORMULAS

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The connection between monosplines of two variables of least deviation from zero in $L_q(0,1\,;\,0,1)$ and monosplines of one variable is considered. This result is used for constructing the optimal formula

$$(1)\qquad \int_0^1\int_0^1 f(x,y)\,dx\,dy=\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i^{(r)}}\sum_{l\in J_k^{(s)}} A_{ik}^{jl} f^{(j,l)}(x_i,y_k)+R(f),$$

where $J_i^{(r)}\leq\{0,1,\dots,r-1\}$, $J_k^{(s)}\leq\{0,1,\dots s-1\}$, for the set $\{f(x,y):\ f^{(j,l)}(x,y)\ (j\leqq r)\ l\leqq s)$ are piecewise continuous on square $D=[0,1]\times[0,1]$,

$$\Big\|\int_0^1 f^{(r,0)}(\,.\,,y)\,dy\Big\|_{L_q(0,1)}\leqq P,\ \Big\|\int_0^1 f^{(0,s)}(x,\cdot)\,dx\Big\|_{L_q(0,1)}\leqq Q,\ \|f^{(r,s)}(\,.\,,.)\|_{L_q(D)}\leqq M\}.$$

The optimal formula (1) has coefficients and nodes $A_{ik}^{jl}=A_{ir}^{j}A_{ks}^{l}$, $(x_i,y_k)=(x_i^{(r)},y_k^{(s)})$, where $\{A_{i\nu}^{j}\}$, $\{x_i^{(\nu)}\}$ are coefficients and nodes of the optimal formula

$$\int_0^1 f(x)dx=\sum_{i=1}^{m_\nu}\sum_{j\in J_{i\nu}^{(\nu)}} A_{i\nu}^{(j)} f^{(j)}(x_i)+R_\nu(f)$$

for the set $\{f(x):f^{(j)}(x)(j\leqq\nu)$ are piecewise continuous, $\|f^{(\nu)}(\cdot)\|_{L_q(0,1)}\leqq 1\}$, $m_\nu=m\ (\nu=r)$, $m_\nu=n\ (\nu=s)$.

The first problems of constructing the optimal quadrature formulas were proposed and solved by A. Sard [1] in the case of fixed nodes and by S. M. Nikolsky [2] in the case of arbitrary nodes. By now the theory of constructing the optimal quadrature formulas has been developed greatly. The theory of optimal cubature formulas is not developed to such a degree. The main results of this theory are obtained for the sets of functions with derivatives of some orders belonging to L_2 [2-5]. The peculiarity of Hilbert metric was essentially exploited in obtaining these results. The field connected with constructing asymptotically optimal cubature formulas was elaborated by S. L. Sobolev and his pupils [6]. It is well known that the construction of optimal integration formulas is closely connected with splines of minimal norm [2—5, 7—11].

In this paper we introduce sets of functions of two variables, which are the natural generalisation of corresponding sets of functions of one variable, and consider the problem of constructing the optimal cubature formulas (for these sets) with rectangular net of nodes. These results are based on the theorem about monosplines of two variables of least deviation from zero in L_q.

Let us introduce notations: $r, s, M, P, Q, m, n, 1\leq q\leq\infty$ be given; $W^{\nu}L_q$ is the set of functions $f(x)$ on $[0, 1]$ with absolutely continuous derivatives of order $\nu-1$ and ν-th derivatives satisfying $\|f^{(\nu)}(\cdot)\|_{L_q(0,1)}\leq 1$; $W^{r,s}L_q$ is the set of functions $f(x,y)$ on square $0\leq x, y\leq 1$ with piecewise continuous derivatives $f^{(j,l)}(x, y)=\partial^{j+l}f_{(x,y)}/\partial x^j\,\partial y^l$ $(j=0,\ldots,r;\ l=0,\ldots,s)$ and satisfying conditions

$$\|\int_0^1 f^{(r,0)}(\cdot, y)\,dy\|_{L_q(0,1)}\leq P,\quad \|\int_0^1 f^{(0,s)}(x,\cdot)\,dx\|_{L_q(0,1)}\leq Q,$$

$$\|f^{(r,s)}(\cdot,\cdot)\|_{L_q(0,1;\,0,1)}\leq M;$$

$$\widetilde{W}^{r,s}L_q=\{f(x,y): f\in W^{r,s}L_q,\ f^{(i,0)}(1,y)\equiv f^{(i,0)}(0,y),\ i=0,\ldots,r-1;$$

$$f^{(0,j)}(x,1)\equiv f^{(0,j)}(x,0),\ j=0,\ldots,s-1\}.$$

Denote by $K_1^*(x)$ and $K_2^*(y)$ the monosplines of least deviation from zero in $L_p(0,1)$ $(p^{-1}+q^{-1}=1)$ from all monosplines of the form

$$K_1(x)=x^r/r!-\sum_{i=1}^{m}\sum_{j\in J_i}B_{ij}(x-x_i)_+^{r-j-1}/(r-j-1)!;\ J_i\subseteq\{0,1,\ldots,r-1\},$$

$$K_2(y)=y^s/s!-\sum_{k=1}^{n}\sum_{l\in L_k}C_{kl}(y-y_k)_+^{s-l-1}/(s-l-1)!;\ L_k\subseteq\{0,1,\ldots,s-1\},$$

respectively, satisfying conditions

(1) $$K_1^{(j)}(1)=0;\ j=0,\ldots,r-1,\ K_2^{(l)}(1)=0;\ l=0,\ldots,s-1.$$

T h e o r e m 1. *The monospline $K^*(x,y)=K_1^*(x)K_2^*(y)$ is of least deviation from zero in the $L_p(0,1;\,0,1)$ metric from the set of all monosplines*

$$K(x,y)=\frac{x^r y^s}{r!\,s!}-\sum_{i=1}^{m}\sum_{j\in J_i}b_{ij}(-1)^j y^s(x-x_i)_+^{r-j-1}/s!\,(r-j-1)!$$

(2) $$-\sum_{k=1}^{n}\sum_{l\in L_k}c_{kl}(-1)^l x^r(y-y_k)_+^{s-l-1}/r!\,(s-l-1)!$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k}A_{ik}^{jl}(-1)^{j+l}(x-x_i)_+^{s-j-1}(y-y_k)_+^{s-l-1}/(r-j-1)!\,(s-l-1)!$$

satisfying conditions

(3) $$K^{(j,0)}(1,y)\equiv K^{(0,l)}(x,1)\equiv 0;\ j=0,\ldots,r-1;\ l=0,\ldots,s-1.$$

P r o o f will be obtained by introducting bases in the spaces of monosplines $K_1(x)$, $K_2(y)$, $K(x,y)$ satisfying (1), (3), expanding $K_1(x)$, $K_2(y)$,

$K(x, y)$ into the functions of these bases and using the reasons of the paper [11] with a similar result.

The formula with remainder $R(f)$ is called optimal for the set H of functions f, if its parameters are picked out of the condition of minimum of the quantity $R[H]=\sup\{|R(f)|: f\in H\}$ which is called exact bound for the remainder of the formula. We denote by B_{ij}^* the coefficients and by δ_1^* the exact bound for the remainder of the optimal for the set $W^r L_q$ formula

$$\int_0^1 f(x)dx=\sum_{i=1}^{m}\sum_{j\in J_i} B_{ij} f^{(j)}(x_i)+R_1(f) \tag{4}$$

with fixed nodes $0\leq x_1<x_2<\cdots<x_m\leq 1$. Denote by C_{kl}^* the coefficients and by δ_2^* the exact bound for the remainder of the optimal for the set $W^s L_q$ formula

$$\int_0^1 f(y)dy=\sum_{k=1}^{n}\sum_{l\in L_k} C_{kl} f^{(l)}(y_k)+R_2(f) \tag{5}$$

with fixed nodes $0\leq y_1<y_2<\cdots<y_n\leq 1$.

Below we use the connection between quadrature formulas and corresponding monosplines [7—10].

Theorem 2. *The optimal for the set $W^{r,s} L_q$ formula*

$$\int_0^1\int_0^1 f(x, y)dxdy=\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k} A_{ik}^{jl} f^{(j,l)}(x_i, y_k)+R(f) \tag{6}$$

with fixed nodes $0\leq x_1<\cdots<x_m\leq 1$, $0\leq y_1<\cdots<y_n\leq 1$ has coefficients

$$A_{ik}^{jl}=B_{ij}^* C_{kl}^* (i=1,\ldots, m;\ k=1,\ldots, n;\ j\in J_i;\ l\in L_k) \tag{7}$$

and exact remainder bound

$$R[W^{r,s} L_q]=P\delta_1^*+Q\delta_2^*+M\delta_1^*\delta_2^*. \tag{8}$$

Proof. We consider the case $x_1, y_1>0$, $x_m, y_n<1$. The other cases may be obtained by limit passage.

1. As it follows from [12], the formulas (6) with finite value of quantity $R[W^{r,s} L_q]$ are to be exact for functions

$$f_1=\sum_{\alpha=0}^{r-1} a_\alpha(y)(\alpha+1)(1-x)^\alpha,\quad f_2=\sum_{\beta=0}^{s-1} b_\beta(x)(\beta+1)(1-y)^\beta$$

satisfying conditions $\Sigma_{\alpha=0}^{r-1} a_\alpha^{(s)}(y)\equiv 0$, $\Sigma_{\beta=0}^{s-1} b_\beta^{(r)}(x)\equiv 0$, where $a_\alpha(y)$, $b_\beta(x)$ are arbitrary functions with absolutely continuous derivatives of orders $s-1$, $r-1$, respectively. Because of this fact we conclude that the coefficients of formula (6) with finite value of quantity $R[W^{r,s} L_q]$ satisfy conditions*

$$\sum_{\substack{i=1\\ 0\in J_i}}^{m} A_{ik}^{0l}=\sum_{i=1}^{m}\sum_{j\in J_i} A_{ik}^{jl}(-1)^j(\alpha+1)!\,(1-x_i)^{\alpha-j}/(\alpha-j)!; \tag{9}$$

$$k=1,\ldots, n;\ l\in L_k;\ \alpha=1,\ldots, r-1,$$

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* We suppose $u^{m-j}/(m-j)!=0$, if $m<j$.

(10) $$\sum_{\substack{k=1\\0\in L_k}}^{n} A_{ik}^{j0}=\sum_{k=1}^{n}\sum_{l\in L_k} A_{ik}^{jl}(-1)^l(\beta+1)!\,(1-y_k)^{\beta-l}/(\beta-l)!;$$
$$i=1,\ldots,m;\ j\in J_i;\ \beta=1,\ldots,s-1.$$

2. Let us consider an arbitrary formula (6) with coefficients satisfying (9)-(10) and a function $K(x,y)$ in the form (2) with coefficients

(11) $$c_{kl}=\sum_{\substack{i=1\\0\in J_i}}^{m} A_{ik}^{0l},\ b_{ij}=\sum_{\substack{k=1\\0\in L_k}}^{n} A_{ik}^{j0};\ k=1,\ldots,n;\ l\in L_k;\ i=1,\ldots,m;\ j\in J_i.$$

As it follows from (9)-(11), the function $K(x,y)$ satisfies the conditions (3).

3. Let $f(x,y)\in W^{r,s}L_q$, $x_0=y_0=1-x_{m+1}=1-y_{n+1}=0$. Integrating by parts from

$$\int_0^1\int_0^1 f(x,y)\,dxdy=\sum_{i=0}^{m}\sum_{k=0}^{n}\int_{x_i}^{x_{i+1}}\int_{y_k}^{y_{k+1}} f(x,y)K^{(r,s)}(x,y)\,dxdy,$$

we obtain the formula (6) with remainder

$$R(f)=(-1)^r\int_0^1\int_0^1 f^{(r,0)}(x,y)K^{(0,s)}(x,y)\,dx\,dy$$
$$+(-1)^s\int_0^1\int_0^1 f^{(0,s)}(x,y)K^{(r,0)}(x,y)\,dxdy-(-1)^{r+s}\int_0^1\int_0^1 f^{(r,s)}(x,y)K(x,y)\,dxdy.$$

4. Denoting $K_1(x)=K^{(0,s)}(x,y)$, $K_2(y)=K^{(r,0)}(x,y)$ we obtain with the help of Hölder's inequality an estimate

(12) $$|R(f)|\le P\|K_1\|_{L_p(0,1)}+Q\|K_2\|_{L_p(0,1)}+M\|K\|_{L_p(0,1;\,0,1)}.$$

Let $q>1$,

$$M_1^{(r)}(x)=\|K_1\|_{L_p(0,1)}^{1-p}(-1)^r|K_1(x)|^{p-1}\operatorname{sign}K_1(x),$$
$$M_2^{(s)}(y)=\|K_2\|_{L_p(0,1)}^{1-p}(-1)^s|K_2(y)|^{p-1}\operatorname{sign}K_2(y),$$
$$M^{(r,s)}(x,y)=\|K\|_{L_p(0,1;\,0,1)}^{1-p}(-1)^{r+s+1}|K(x,y)|^{p-1}\operatorname{sign}K(x,y),$$
$$f_0(x,y)=M\Big(M(x,y)-\int_0^1 M(x,y)dx-\int_0^1 M(x,y)dy\Big)+PM_1(x)+QM_2(y).$$

The inequality (12) turns into equality for $f_0(x,y)\in W^{r,s}L_q$. Therefore

(13) $$R[W^{r,s}L_q]=P\|K_1\|_{L_p(0,1)}+Q\|K_2\|_{L_p(0,1)}+M\|K\|_{L_p(0,1;\,0,1)}.$$

This equality is also correct, if $q=1$ and may be obtained from (13) by limit passage $p\to\infty$.

5: The function $K(x,y)=K_1^*(x)K_2^*(y)$ satisfies the conditions (7), (9)—(11). As it follows from Theorem 1 the quantity (13) has the minimal value if $K(x,y)=K_1^*(x)K_2^*(y)$, $K_1(x)=K_1^*(x)$, $K_2(y)=K_2^*(y)$. The theorem is proved.

Let now the nodes of formulas (4)-(5) be arbitrary. Denote by $\overline{B}_{ij}$, $\overline{x}_i$, $\overline{\delta}_1$ the coefficients, nodes and exact bound for the remainder of the optimal for the set $W^r L_q$ formula (4), and by $\overline{C}_{kl}$ $\overline{y}_k$, $\overline{\delta}_2$ the coefficients, nodes and exact bound for the remainder of the optimal for the set $W^s L_q$ formula (5).

Theorem 3. *The optimal for the set $W^{r,s}L_q$ formula* (6) *(with arbitrary nodes) has coefficients and nodes*

$$A_{ik}^{jl}=\overline{B}_{ij}\,\overline{C}_{kl},\ (x_i, y_k)=(\overline{x}_i, \overline{y}_k);\ i=1,\ldots,m;\ k=1,\ldots,n;\ j\in J_i;\ l\in L_k$$

the exact bound for the remainder $R[W^{r,s}L_q]=P\overline{\delta}_1+Q\overline{\delta}_2+M\overline{\delta}_1\overline{\delta}_2$.

Proof follows at once from the estimate (8).

Remark. Similar result remains valid, if some nodes of formulas (1)-(3) are fixed.

Analogous statement may be obtained for the set $\widetilde{W}^{r,s}L_q$. In particular, the following theorem holds.

Theorem 4. *The optimal for the set $\widetilde{W}^{r,s}L_q$ formula*

$$\int_0^1\int_0^1 f(x, y)\,dx\,dy=\sum_{i=1}^{m}\sum_{k=1}^{n} A_{ik}f(x_i, y_k)+R(f)$$

has coefficients, nodes and exact bound for the remainder $A_{ik}=1/mn$, $x_i=i/m$, $y_k=k/n$ $(i=1,\ldots,m;\ k=1,\ldots,n)$,

$$R[\widetilde{W}^{r,s}L_q]=m^{-r}P\,B_{rp}+n^{-s}Q\,B_{sp}+m^{-r}\,n^{-s}\,M\,B_{rp}B_{sp},$$

where $B_{jp}=(1/j!)\min\limits_c\|B_j(\cdot)+c\|_{L_p(0,1)}$, $B_j(x)$ *are Bernoulli polynomials.*

The next result follows from theorem 4 and the properties of the Gregory formula.

Let $J_{rn}=\{0, 1,\ldots,r-1\}\cup\{n-r+1,\ n-r+2,\ldots,n\}$, B_j — Bernoulli numbers, $\overline{B}_i^r=B_i$; $i\neq r$, $\overline{B}_r^r=B_r-c_{rp}$, $\|B_r(x)-c_{rp}\|_{L_p(0,1)}=\min\limits_c\|B_r(x)-c\|_{L_p(0,1)}$,

$\omega_{n,r}(x)=x\,(x-1/n)\ldots(x-(r-1)/n)$,

$$\lambda_{kn}^r=\lambda_{n-k,n}^r=\sum_{j=1}^{r-1}\frac{\overline{B}_{j+1}^r}{(j+1)!\ n^{j+1}}\left[\frac{\omega_{n,r}(x)}{\left(x-\frac{k}{n}\right)\omega'_{n,r}\left(\frac{k}{n}\right)}\right]^{(j)}\Bigg|_{x=0};\ k=0, 1,\ldots,r-1,$$

$$A_{kn}^r=\begin{cases}1/n, & k\overline{\in}J_{rn},\\ 1/2n+\lambda_{kn}^r, & k\in\{0, n\},\\ 1/n+\lambda_{kn}^r, & k\in J_{rn}\setminus\{0, n\}.\end{cases}$$

Theorem 5. *The formula*

$$\int_0^1\int_0^1 f(x, y)\,dxdy=\sum_{i=0}^{m}\sum_{k=0}^{n} A_{ik}f(x_i, y_k)+R(f)$$

with nodes and coefficients $x_i=i/m$, $y_k=k/n$, $A_{ik}=A_{im}^r A_{kn}^s$; $i=0,\ldots,m$; $k=0,\ldots,n$ *is asymptotically optimal for the set* $W^{r,s}L_q$ *and has the bound for the remainder*

$$R[W^{r,s}L_q]=m^{-r}\,P\,B_{rp}+n^{-s}\,Q\,B_{sp}+o(m^{-r}+n^{-s}).$$

REFERENCES

1. A. Sard. Best Approximate Integration Formulas. *Amer. J. Math.*, **71**, 1949, 80—91.
2. С. М. Никольский. Квадратурные формулы. Москва, 1974.
3. М. Левин. О наилучшихкубатурных формулах. *Известия АН ЭССР*, **19**, 1970, 499—502.
4. М. Левин. Об аппроксимации функций в гильбертовом пространстве и кубатурных формулах. *Известия АН ЭССР*, **25**, 1976, 331—337.
5. G. Coman. Two-Dimensional Monosplines and Optimal Cubature Formulae. *Studia univers. Babes-Bolyai*, **1**, 1973, 41—53.
6. С. Л. Соболев. Введение в теорию кубатурных формул. Москва, 1974.
7. I. J. Schoenberg. On Monosplines of Least Deviation and Best Quadrature Formulae, I. *SIAM J. Numer. Anal.*, **2**, 1965, 144—170.
8. I. J. Schoenberg. On Monosplines of Least Deviation and Best Quadrature Formulae, II. *SIAM J. Numer Anal.*, **3**, 1966, 321—328.
9. S. Karlin. Best Quadrature Formulas and Splines. *J. Approx. Theory*, **4**, 1971, 59—90.
10. M. Levin, J. Girshovich. Optimal Quadrature Formulae for Sets of Functions Satisfying Boundary Conditions. *Известия АН ЭССР*, **24**, 1975, 264—269.
11. М. Левин. Об одной экстремальной задаче. *Известия АН ЭССР*, 18, 1969, 464—468.
12. М. Левин. Одно свойство наилучших квадратурных формул. *Известия АН ЭССР*, **20**, 1971, 90—91.

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