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ON THE APPLICATION OF INTERVAL ARITHMETIC IN NUMERICAL ANALYSIS AND APPROXIMATION THEORY

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Summary. The aim of this paper is to point out some shortcomings of the interval arithmetic. We propose an extension of the standard interval arithmetic by two new operations between intervals. Several examples illustrate the possible application of the extended interval arithmetic.

Interval arithmetic can be briefly described as follows.

Consider the set $I(R)$ of all intervals $a=[\check{a}, \hat{a}]$, $\check{a}\leqq\hat{a}$, on the real line, and introduce in $I(R)$ the operations addition and multiplication by means of

$$[\check{a}, \hat{a}]+[\check{b}, \hat{b}]=[\check{a}+\check{b}, \hat{a}+\hat{b}],$$

$$[\check{a}, \hat{a}][\check{b}, \hat{b}]=[\min\{\check{a}\check{b}, \check{a}\hat{b}, \hat{a}\check{b}, \hat{a}\hat{b}\}, \max\{\check{a}\check{b}, \check{a}\check{b}, \hat{a}\check{b}, \hat{a}\hat{b}\}].$$

The intervals of the form $[a, a]$ are denoted by a. The product $[-1, -1]a=-1\cdot a$ is denoted by $-a$. The composition of the two operations $a+(-b)$ is called arithmetic subtraction of intervals.

Associate to each $b=[\check{b}, \hat{b}]\not\ni 0$ the element $b^{-1}=[\hat{b}^{-1}, \check{b}^{-1}]$. Then the composition ab^{-1} is called arithmetic division of intervals.

These are the basic definitions in the interval arithmetic (IA). Although IA is very simple, it can be successfully applied to solving certain problems in numerical analysis. This was noticed some two decades ago by T. Sunaga [9].

The application of IA in numerical analysis rapidly developed after the book of R. Moore [6]. Now there exist more than 500 publications on interval mathematics, i. e. on interval arithmetic and its applications. Recently an international symposium on interval mathematics took place in Karlsruhe [3]. It is realized that IA can be applied in various branches of mathematics and in particular in approximation theory (see the paper of Bl. Sendov in this volume, p.131—143).

A typical problem which can be (sometimes) solved by means of IA is that of computing the range of a rational function of many variables varying in given intervals [6, 7]. For brevity we shall further refer to this problem as Moore's problem. For example, if we want to compute the range F of $f(\xi, \eta, \zeta)=(2\xi-\eta)/\zeta$, when ξ, η and ζ vary corresp. in the intervals $x=[4, 6]$, $y=[2, 4]$, $z=[1, 4]$, we can use interval arithmetic and obtain that $F=\{f(\xi, \eta, \zeta): \xi \in x, \eta \in y, \zeta \in z\}$ is equal to

$$(1) \qquad F=(2x+(-y))\cdot z^{-1}=\{2[4, 6]+(-[2, 4])\}[1, 4]^{-1}$$
$$=([8, 12]+[-4, -2])[1/4, 1]=[4,10][1/4, 1]=[1,10].$$

What we did was to replace the variables in the expression $(2\xi-\eta)/\zeta$ by the corresponding intervals and the operations between numbers by the corresponding interval arithmetic operations.

Consider next briefly the concept of interval function. In IA interval functions are usually defined as correspondences of the form $F: T\rightarrow I(R)$, where $T\subset R$ is some set of reals and $I(R)$ is the set of all intervals on the real line R. We often need to extend the domain of F to $I(T)=\{S: S \in I(R), S\subset T\}$. This is usually done (for continuous F) by means of the so-called natural interval extension of F (see [6], p. 18): $F(x)=\cup\{F(\xi):_{\xi \in x}\}$.

Interval functions can be used to solve various interesting problems. As an example, consider the following problem [8]:

Given are the pairs of intervals (x_i, y_i), $x_i, y_i \in I(R)$, $i=0, 1, \ldots, n$, with $x_i\cap x_j=\emptyset$, $i, j=0, 1, \ldots, n$, $i\neq j$. Construct an interval polynomial $p_n(x) =\Sigma_{i=0}^{n} a_i x^i$, $a_i \in I(R)$, such that

$$(2) \qquad \{p_n(x): p_n(x) \text{ interpolates } (\xi_i, \eta_i)\in(x_i, y_i), i=0, 1, \ldots, n\}\subset P_n(x).$$

It is easily seen that the interval function

$$P_n(x)=\sum_{i=0}^{n} y_i \prod_{\substack{j=0\\ j\neq i}}^{n} \{(x+(-x_j))(x_i+(-x_j))^{-1}\}, \quad x\in R,$$

gives one possible solution of the problem (2).

Note that an inclusion relation is used in (2). Of course, it will be nice if we can solve the above problem with an equality relation.

It would be also nice if we can solve Moore's problem exactly for a larger class of rational function. However, the simple approach used for constructing the interval function (1) fails to work when some variables appear more than once in the expression of the rational function. For example, the ranges of $f(\xi)=1-2\xi+\xi^2$, $\xi \in x$, or $g(\xi, \eta)=(\xi+\eta)/(\xi-\eta)$, $\xi \in x$, $\eta \in y$, cannot be computed exactly by means of IA. The IA produces only inclusion relations which are usually not sharp.

These examples show that the IA does not satisfy in its present form. It is often realized that IA suffers from certain shortcomings restricting its application to very narrow limits or causing it to produce results which are not sharp enough.

Let us first start with the observation that IA cannot generally help in finding out the exact ranges of rational functions. A possible reason for this may be the imperfect choice of the basic operations in IA. Note that in IA basically only two operations are used. Indeed, as we made clear in the very beginning of the paper, the arithmetic subtraction is a composition of these two operations (addition and multiplication) and its properties are similar to the properties of the addition (for instance, the length of the difference is equal to the length of the sum). The situation is not much different with the arithmetic division (division of a and b is multiplication of a and b^{-1}).

These arguments lead us to the idea of a possible extension of IA. It is to be noted that such an idea is not new. Attempts to extend IA for achieving sharper results for a wider class of Moore's problems have already been done [1, 2].

We suggest a simple approach to extending the IA by two non-standard operations for subtraction and division:

$$a-b=(\breve{a}-\breve{b})\vee(\widehat{a}-\widehat{b});$$

$$a/b=\begin{cases}(\breve{a}/\breve{b})\vee(\widehat{a}/\widehat{b}), & \text{if } ab>0,\ b\not\ni 0,\\ (\breve{a}/\widehat{b})\vee(\widehat{a}/\breve{b}), & \text{if } ab<0,\ b\not\ni 0,\\ (1/\widehat{b})a, & \text{if } a\ni 0,\ b\not\ni 0.\end{cases}$$

Here $\vee$ is the operation used in [9]: $a\vee b=[\min\{\breve{a},\breve{b}\},\max\{\widehat{a},\widehat{b}\}]$; $\widehat{b}$ is defined by $\widehat{b}=\{\widehat{b}$, if $b>0$; $\breve{b}$, if $b<0\}$.

Note the following relations $a-a=0$, $b/b=1$, $0-a=-a$, $1/b=b^{-1}$. Note also that $a-b=a+(-b)$ and $a/b=ab^{-1}$ iff $\mu(a)\mu(b)=0$, where $\mu(a)$ denotes the length of a, $\mu(a)=\widehat{a}-\breve{a}$.

We can consider now the algebraic structure consisting of: the set $I(R)$ together with the standard operations for addition and multiplication and the non-standard operations for subtraction and division. We shall call this structure extended interval arithmetic (EIA) (for more detail see [4]). Let us give some examples, showing that EIA offers better possibilities to solving Moore's problems exactly.

As R. Moore pointed out [6, p. 26] the set of solutions of the equation $\alpha\xi+\beta=0$, when $\alpha\in a$, $\beta\in b$, $a, b\in I(R)$, can be easily computed by means of IA; if $\xi\in x$, then we obtain, $x=-ba^{-1}$.

Consider now the problem of finding the set of solutions of two simultaneous linear equations

$$\alpha_{11}\xi+\alpha_{12}\eta+\beta_1=0,\quad \alpha_{21}\xi+\alpha_{22}\eta+\beta_2=0,$$

when $\alpha_{ij}\in a_{ij}\subset I(R)$, $\beta_k\subset b_k I(R)$, $i, j, k=1, 2$.

EIA extends our possibilities for solving this problem. In order to illustrate this consider the system [6, p. 27]

$$\xi+2\eta=1,\quad \xi+\alpha\eta=0,\quad \alpha\in[10, 12].$$

The range of ξ is $x=\{\alpha/(\alpha-2):\ \alpha\in[10, 12]\}=[6/5, 5/4]$. IA produces

$$x_{(\mathrm{IA})}=[10, 12]([10, 12]-2)^{-1}=[10, 12][1/10, 1/8]=[1, 3/2],$$

whereas EIA gives exactly

$$x_{(\mathrm{EIA})}=[10, 12]/([10, 12]-2)=[10, 12]/[8, 10]=[6/5, 5/4].$$

As another example consider the rational function $\varphi(\xi)=(\alpha\xi+\beta)/(\gamma\xi+\delta)$, $\gamma\xi+\delta\neq0$, for $\xi\in x$. Assume that $\varphi(\xi)$ has a constant sign when $\xi\in x$, and denote $\operatorname{sgn}\varphi(\xi)=\sigma$. It is easily seen then, that the range of φ is

$$\{\varphi(\xi):\ \xi\in x\}=\begin{cases}(\alpha x+\beta)(\gamma x+\delta)^{-1}, & \text{if } \operatorname{sgn}(\alpha\gamma)=\sigma,\\ (\alpha x+\beta)/(\gamma x+\delta), & \text{if } \operatorname{sgn}(\alpha\gamma)\neq\sigma.\end{cases}$$

We now turn to a discussion of the concept of interval polynomial. The interval polynomials are defined over R usually by $P(x)=a_0+a_1x+\dots+a_nx^n$, $a_i\in I(R)$, $x\in R$. We suggest the following generalization of the concept of interval polynomial:

Definition. A generalized interval polynomial is a function of the form

$$P(x)=\sum_{i=0}^{n} a_i Q_{n,i}(x),$$

where $a_i \in I(R)$ and $Q_{n,i}(x)$ are (real-valued) polynomials of degree $\leq n$.

Generalized interval polynomials can be used for treating various interpolating and approximation problems.

We shall illustrate this by means of two examples.

1. Let $f: D \to I(R)$, $D \subset R$, be an interval function with domain D and $\{x_i\}_{i=0}^{n}$ are $n+1$ different points such that $x_i \in D$, $i=0, \ldots, 1$. Then the generalized interval polynomial

$$L_n(x)=\sum_{i=0}^{n} f(x_i) l_{n,i}(x),\quad l_{n,i}=\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

interpolates the interval function f in the sense that

$$L_n(x_i)=f(x_i), \text{ for } i=0, 1, \ldots, n.$$

2. As a second example consider the generalized interval polynomial defined in [0, 1]:

$$B_n(f;\, x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

where f is an interval function with domain [0,1].

It is easily seen that the interval operator $B_n(f)$ satisfies:

1. $B_n(af;\, x)=aB_n(f;\, x)$, $a \in R$,
2. $B_n(f+g;\, x)=B_n(f;\, x)+B_n(g;\, x)$,
3. $f \geq 0 \Rightarrow B_n(f;\, x) \geq 0$, $x \in [0,1]$.

Theorem. *If f is continuous (in the sense of* [6]) *interval function in* [0,1] *and $\varepsilon>0$, then there exists n, such that*

$$\| f(x)-B_n(f;\, x) \|_{[0,1]} < \varepsilon,$$

where $\| g(x) \|_{[0,1]} = \max_{0\leq x\leq 1} \| g(x) \|$, $\| g \| = \max\{ |\check{g}|, |\hat{g}| \}$ *and "$-$" means non-standard subtraction.*

Similarly interval trigonometric polynomials can be considered and their properties can be studied.

Finally we shall note that the non-standard subtraction can be used for the following definition of a derivative of an interval function:

$$f'(x)=\lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

where $\lim_{x\to x_0} g(x)=[\lim_{x\to x_0} \check{g}(x), \lim_{x\to x_0} \hat{g}(x)]$.

Using this concept of derivative a differential calculus for interval functions can be developed. An even more interesting theory is obtained when based on the concept of S-limit, proposed by Prof. Bl. Sendov (see [5]):

$$\mathrm{S}\lim_{x\to x_0} g(x)=[\underline{\lim}_{x\to x_0} \check{g}(x), \overline{\lim}_{x\to x_0} \hat{g}(x)].$$

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