КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 395—398

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 395—398

EXTREMAL FUNCTIONS IN RELATION TO SZASZ-MIRAKJAN OPERATORS

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Summary. In this paper the degree of approximation of functions, which are differentiable on $[0, \infty)$ and possess an uniformly continuous derivative, by means of generalized Szász-Mirakjan operators T_n $(n=1, 2, \dots)$, is considered.

The operators T_n are defined by

$$T_n(f, x) = \sum_{k=0}^{\infty} t_{n,k}(x) f(k/\{\gamma(n)+\alpha(n)\})$$

with $t_{n,k}(x) = e^{-\gamma(n)x}\{\gamma(n)x\}^k/k!$, $\gamma(n)$ and $\alpha(n)$ satisfying $\gamma(n)>0$, $\alpha(n)\geq 0$, for all n, $\lim_{n\to\infty} \gamma(n)=+\infty$ and $\lim_{n\to\infty} \alpha(n)/\gamma(n)=0$. The difference $T_n(f;x)-f(x)$ is estimated for all f in terms of $\omega_1(f;\delta)$, i. e. the modulus of continuity of f' with $\delta>0$. In case $\alpha(n)=0$ some best local results are derived, involving the extremal function

$$\tilde{f}(t) = \frac{1}{2}|t-x| + \sum_{j=1}^{\infty}(|t-x|-j\delta)_+.$$

In case $\alpha(n)\neq 0$ an elementary estimate is given.

1. In 1975 F. Schurer and F. Steutel [2] have given the exact degree of approximation with Bernstein polynomials for functions in $C^1[0, 1]$.

Let f be a real function in the space $C^1[0, 1]$ and let $B_n(f; x)$ denote its n-th order Bernstein polynomial. Schurer and Steutel estimated the difference $|B_n(f; x)-f(x)|$ in terms of $\omega_1(f; \delta)$, i. e. the modulus of continuity of f' with argument $\delta>0$, and they proved that for every $x\in[0, 1]$

$$\sup_{f\in C^1[0,1]} \frac{|B_n(f;x)-f(x)|}{\omega_1(f;\delta)} = B_n(\tilde{f}; x);\ n=1, 2, \dots, \tag{1}$$

where $\tilde{f}$, which depends on x and δ, is defined for all real t by

$$\tilde{f}(t) = \frac{1}{2}|t-x| + \sum_{j=1}^{\infty}(|t-x|-j\delta)_+, \tag{2}$$

where $a_+ = \max(a, 0)$ for all real a.

The functions $\tilde{f}$ are called extremal functions (see Fig. 1).

In 1976 F. Schurer, F. Steutel and P. Sikkema [3] proved that in the special case $\delta=1/n$ one has

(3) $$\sup_{f\in C^1[0,\ 1]} \frac{|B_n(f;\ x)-f(x)|}{\omega_1(f;\ 1/n)}=\frac{x(1-x)}{2}+\frac{\alpha(1-\alpha)}{2n}$$

with $x\in[0,\ 1]$ $n\in\mathbb{N}$ and $\alpha=nx-[nx]$, $[nx]$ denoting the largest integer not exceeding nx.

In this paper we consider generalized operators of Szász-Mirakjan T_n $(n=1,\ 2,\ldots)$ and we shall estimate the difference $|T_n(f;\ x)-f(x)|$, where f is a well-chosen function and $x\in[0,\ \infty)$, in terms of the modulus of continuity of f'.

For a certain class of operators that we consider, including the ordinary Szász-Mirakjan operators, we shall derive results that are the analogues of (2) and (3) and in which the extremal functions will play a fundamental role.

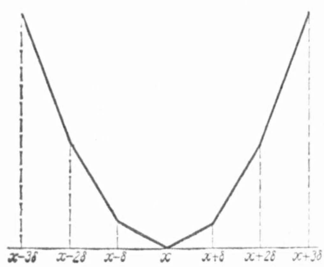

Fig. 1. The extremal function $\tilde{f}(t)$

2. In this section we state two definitions.

Definition 1. *U is the class of real functions f, that are defined on the interval* $[0,\ \infty)$ *and possess the following properties:*

(i) *f is differentiable on* $[0,\ \infty)$.

(ii) *f' is uniformly continuous on* $[0,\ \infty)$.

(iii) *f is not linear.*

Definition 2. *For* $n=1,\ 2,\ldots$ *the operators* T_n *are defined on U by*

(4) $$T_n(f;\ x)=e^{-\gamma(n)x}\sum_{k=0}^{\infty}\frac{\gamma(n)^k\cdot x^k}{k!}\cdot f\left(\frac{k}{\gamma(n)+\alpha(n)}\right),$$

where $x\in[0,\ \infty)$ *and* $\gamma(n)$ *and* $\alpha(n)$ *possess the following properties:*

(i) *for all* $n:\gamma(n)>0$ *and* $\alpha(n)\geq 0$.

(ii) $\lim\limits_{n\to\infty}\gamma(n)=\infty$.

(iii) $\lim\limits_{n\to\infty}\alpha(n)/\gamma(n)=0$.

Remarks. 1. In the case $\alpha(n)=0$ and $\gamma(n)=n$ the operator T_n equals the ordinary Szász-Mirakjan operator.

2. It is easily proved [1] that all functions $f\in U$ are well-chosen in relation to the operators $T_n(n=1,\ 2,\ldots)$, i. e. the series in the right hand of (4) is convergent. Furthermore, property (ii) in definition 1 assures that for all $f\in U$ the modulus of continuity of f' exists.

3. Theorem I. *Let* $n \in \mathbb{N}$, $x \in [0, \infty)$ *and* $\delta > 0$; *if* $\alpha(n) = 0$ *then* $\sup \{|T_n(f; x) - f(x)|/\omega_1(f; \delta) : f \in U\} = T_n(\tilde{f}; x)$, *where* $\tilde{f}$ *is defined as in* (2).

Short sketch of the proof. The theorem is proved in two steps.

Step 1. For all $x \in [0, \infty)$, $\delta > 0$ and $n \in \mathbb{N}$ we have

$$\sup \{|T_n(f; x) - f(x)|/\omega_1(f; \delta) : f \in U\} \leq T_n(\tilde{f}; x). \tag{5}$$

For the proof of this inequality we refer the reader to [1] and [2, theorem 4.1].

Step 2. We can construct a sequence of functions f_m $(m = 1, 2, \ldots)$ in U such that this sequence converges uniformly on $[0, \infty)$ to $\tilde{f}$ [1, lemma 6]. Then it follows that in (5) equality holds [1, theorem 7]. This theorem has a number of corollaries of which we state two. For the proofs of these corollaries the reader is referred to [1].

Corollary 1. *Let* $n \in \mathbb{N}$, $x \in [0, \infty)$; *if* $\alpha(n) = 0$ *and* $\delta = 1/n$ *then*

$$\sup \{|T_n(f; x) - f(x)|/\omega_1(f; 1/n) : f \in U\} = \frac{1}{2} x + \mu(1 - \mu)/2n,$$

where $\mu = x\gamma(n) - [x\gamma(n)]$, $[x\gamma(n)]$ *denoting the largest integer not exceeding* $x\gamma(n)$.

Corollary 2. *Let* $n \in \mathbb{N}$, $x \in [0, \infty)$ *and* $\delta > 0$; *if* $\alpha(n) = 0$ *then for all* $f \in U$ *the inequality*

$$x\omega_1(f; \delta)/2\delta\gamma(n) \leq |T_n(f; x) - f(x)| \leq \{\delta/8 + x/2\delta\gamma(n)\}\,\omega_1(f; \delta)$$

holds.

Remarks. 1. In the case of the ordinary Szász-Mirakjan operators $\alpha(n) \equiv 0$ for all $n \in \mathbb{N}$, thus theorem I and its corollaries hold with $\gamma(n) \equiv n$.

2. If $\alpha(n) = 0$ then the operator T_n maps linear functions on itself. It follows that the linear functions are of no interest in the problem we are concerned with. That is the reason why functions in U have property (iii) in definition 1.

4. Theorem II. *Let* $n \in \mathbb{N}$, $x \in [0, \infty)$ *and* $\delta > 0$; *if* $\alpha(n) > 0$ *then*

$$\sup \{|T_n(f; x) - f(x)|/\omega_1(f; \delta) : f \in U\} = +\infty.$$

In this case for all $f \in U$ *the inequality*

$$|T_n(f; x) - f(x)| \leq \omega_1(f; \delta) \left\{ \frac{(x^2\alpha^2(n) + x\gamma(n)}{2\alpha(\gamma(n) + \alpha(n))^2} + \frac{\sqrt{x^2\alpha^2(n) + x\gamma(n)}}{\gamma(n) + \alpha(n)} \right\} + \frac{x\,\alpha(n)}{\gamma(n) + \alpha(n)} \cdot |f'(x)|$$

holds.

For a proof of this theorem the reader is referred to [1].

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Received August 23, 1977