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SOME FINITE DIFFERENCE SCHEMES FOR A SINGULAR PERTURBATION PROBLEM

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Summary. We consider the two-point boundary value problem

$$(P_\varepsilon)\begin{cases} -\varepsilon u''(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x), \ x\in\Omega, \\ u(0)=u(1)=0, \end{cases}$$

where Ω is the open unit interval, $0<\varepsilon\leqq\varepsilon_0$, $a_0(x)\geqq 0$ and $a_1(x)\geqq\alpha>0$ for given constants ε_0, α and all $x\in\overline{\Omega}$. We introduce a class of finite difference approximations (P^h_ε) to (P_ε) and we define what is meant by convergence uniformly in ε of the solution u^h_ε of (P^h_ε) to the solution u_ε of (P_ε). Applying (P^h_ε) to a test problem we derive a necessary condition for the convergence uniformly in ε of u^h_ε to u_ε. We observe that many of the finite difference approximations recently proposed for solving (P_ε) belong to the class (P^h_ε) but do not fulfil this necessary condition.

Let Ω be the open unit interval and consider the two-point boundary value problem

$$(P_\varepsilon)\begin{cases} L_\varepsilon u(x)\equiv-\varepsilon u''(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x), \ x\in\Omega, \\ u(0)=u(1)=0, \end{cases}$$

where $0<\varepsilon\leqq\varepsilon_0$, $a_0(x)\geqq 0$, $a_1(x)\geqq\alpha>0$ for given constants ε_0, α and all $x\in\overline{\Omega}$. We assume also that a_0, a_1 and f are smooth functions.

We are interested here in finite difference approximations to (P_ε) of the form

$$(P^h_\varepsilon)\begin{cases} L^h_\varepsilon u^h(x_i)\equiv-\varepsilon D_+D_-u^h(x_i)/\theta^h_i+\gamma^h_i D_0 u^h(x_i)+\delta^h_i u^h(x_i)=f^h(x_i), \ x_i\in\Omega_h, \\ u_h(0)=u^h(1)=0, \end{cases}$$

where $0<h\leqq h_0$, $Nh=1$, $\Omega_h=\{x_i : x_i=ih, 1\leqq i\leqq N-1\}$, h_0 is a given constant, the quantities θ^h_i, γ^h_i, δ^h_i depend in general on ε, $hD_+=E^+-I$, $hD_-=I-E^-$, $2hD_0=-E+E^-$ and $E^{+/-}w(x)=w(x\pm h)$. We put $\varrho=h/\varepsilon$ and $R=\{(h,\varepsilon): 0<h\leqq h_0, 0<\varepsilon\leqq\varepsilon_0\}$. Let $\|\cdot\|$ denote a norm for the space of functions to which f belongs.

Definition. *The solution u^h_ε of (P^h_ε) converges to the solution u_ε of (P_ε) uniformly in ε with order h^p, $p>0$, if*

(1) $$|u_\varepsilon^h(x_i)-u_\varepsilon(x_i)| \leq C\|f\| h^p, \ \forall x_i \in \Omega_h, \ \forall (h, \varepsilon) \in R,$$

where C is a constant depending only on h_0 and ε_0.

We consider now several difference operators L_ε^h; these are fully determined when the triples

(2) $$(1/\theta_i^h, \gamma_i^h, \delta_i^h), \qquad i=1, \ldots, N-1$$

are given.

We begin with the operator

(3) $$-\varepsilon(1+\varrho\sigma) D_+D_- + a_1(x_i) D_0 + a_0(x_i),$$

which corresponds to the triples $(1+\varrho\sigma, a_1(x_i), a_0(x_i))$, $i=1, \ldots, N-1$, where σ is a parameter. The choice $\sigma=0$ gives immediately the central difference operator $-\varepsilon D_+D_- + a_1(x_i) D_0 + a_0(x_i)$, while the choices $\sigma=A/2$ and $\sigma=\varrho/12$ lead, respectively, to the operators in equations (14) and (25) of I. Christie et al. [3], where these operators are derived via a finite element technique.

We now introduce the more general operator

(4) $$-\varepsilon(1+\varrho\sigma_i) D_+ D_- + a_1(x_i) D_0 + a_0(x_i),$$

having $N-1$ parameters σ_i, $i=1, \ldots, N-1$. This corresponds to the triples $(1+\varrho\sigma_i, a_1(x_i), a_0(x_i))$, $i=1, \ldots, N-1$. It is easy to verify that the choice $\sigma_i=-a_1(x_i)/2$ gives the forward difference operator $-\varepsilon D_+ D_- + a_1(x_i) D_+ + a_0(x_i)$ and that $\sigma_i=a_1(x_i)/2$ gives the backward difference operator $-\varepsilon D_+D_- + a_1(x_i) D_- + a_0(x_i)$. Both of these operators are well-known in this context and have been studied by many authors. The choice $\sigma_i = -\alpha a_1(x_i)/2$ gives the difference operator in equation (2.4.2) of P. W. Hemker [5], which itself is a generalization of the operator in equation (11) of J. C. Heinrich et al. [4], where it is obtained using a finite element approach.

Let us now look at operators L_ε^h corresponding, respectively, to the triples

$$(1-\varrho(a_1(x_i+h/2)+ha_0(x_i+h/2))/2, \ a_1(x_i+h/2)+ha_0(x_i+h/2)/2,$$
$$a_0(x_i+h/2))$$

and

$$(1+\varrho(a_1(x_i-h/2)-ha_0(x_i-h/2)/2)/2, \ a_1(x_i-h/2)-ha_0(x_i-h/2)/2,$$
$$a_0(x_i-h/2)).$$

It is not hard to verify that these operators may also be written in the forms

(5) $$-\varepsilon D_+D_- + a_1(x_i+h/2) D_+ + a_0(x_i+h/2)(E^+ + I)/2$$

and

(6) $$-\varepsilon D_+ D_- + a_1(x_i-h/2) D_- + a_0(x_i-h/2)(I+E^-)/2,$$

respectively. We note that (5) and (6) are, respectively, the forward and backward difference operators appearing in equation (1.12) of L. R. Abrahamsson et al. [2].

Similarly, corresponding to the triples $(1-\varrho(\alpha_i a_1(x_i)+2h\beta_i a_0(x_i))/2, a_1(x_i)+2hd_i a_0(x_i), a_0(x_i))$ we obtain the difference operator

$$-\varepsilon D_+D_-+a_1(x_i)((1+\alpha_i)D_++(1-\alpha_i)D_-)/2+a_0(x_i)\ ((\beta_i+d_i)E^++(1-2\beta_i)I+(\beta_i-d_i)E^-), \tag{7}$$

which is the operator in equation (2.4.14) of P. W. Hemker [5].

The final difference operator we wish to consider is

$$-(a_1(x_i)h/2)\ \coth(a_1(x_i)\varrho/2)D_+D_-+a_1(x_i)D_0+a_0(x_i), \tag{8}$$

which corresponds to the triples

$$(((\tanh a_1(x_i)\varrho/2)/a_1(x_i)\varrho/2)^{-1},\ a_1(x_i),\ a_0(x_i)).$$

We remark that this may be obtained from (4) by choosing $\sigma_i=(a_1(x_i)/2)\coth(a_1(x_i)\varrho/2)-1/\varrho$, as was noticed in a special case in [3] and [4]. However, the difference operator (8) was studied in the general case in the earlier paper of A. M. Ilin [6]. In [7] a difference operator of similar form to (8) was obtained by a finite element approach.

We emphasize that in [6] Ilin proves, in the case when $a_0=0$ and L^h_ε is given by (8), that the solution u^h_ε of (P^h_ε) converges to the solution u_ε of (P_ε) uniformly in ε with order h. This gives rise to the natural question, whether the difference schemes (P^h_ε) corresponding to any of the operators L^h_ε defined in (3) to (7) possess a similar property. To answer this we now derive a simple necessary condition for the convergence, uniformly in ε, of a class of difference schemes (P^h_ε).

Theorem. *Assume that (P^h_ε) is such that the triples $(1/\theta^h_i, \gamma^h_i, \delta^h_i)$, $i=1,\dots,N-1$, corresponding to L^h_ε, all reduce to the form $(1/\theta(\varrho), 1, 0)$, when (P^h_ε) is applied to (P_ε) in the case $a_1=1$, $a_0=0$. Then for u^h_ε to converge to u_ε, uniformly in ε, it is necessary that the quantities θ^h_i, $i=1,\dots,N-1$ all reduce to the quantity $\theta(\varrho)=(2/\varrho)\tanh\varrho/2$, when (P^h_ε) is applied to (P_ε) in the case $a_1=1$, $a_0=0$.*

Proof. We consider the special problem $-\varepsilon u''+u'=1$, $u(0)=u(1)=0$, the solution of which at the point $x=1-h$ is $u_\varepsilon(1-h)=1-h-(e^{-\varrho}-e^{-1/\varepsilon})/(1-e^{-1/\varepsilon})$.

Hence, for ϱ fixed and $h\to 0$, we have

$$u_\varepsilon(1-h)\to 1-e^{-\varrho}. \tag{9}$$

Applying (P^h_ε) to this special problem and using the hypotheses of the theorem, we obtain the difference scheme

$$-\varepsilon D_+D_-u^h(x_i)/\theta(\varrho)+D_0u^h(x_i)=1,\ u^h(0)=u^h(1)=0$$

the solution of which at the point $x_{N-1}=1-h$ is

$$u^h_\varepsilon(1-h)=1-h-\left(1-\left(\frac{1+\varrho\theta(\varrho)/2}{1-\varrho\theta(\varrho)/2}\right)^{N-1}\right)\Big/\left(1-\left(\frac{1+\varrho\theta(\varrho)/2}{1-\varrho\theta(\varrho)/2}\right)^{N}\right)$$

provided that $\varrho\theta(\varrho)/2\neq 1$. Hence, for ϱ fixed and $h\to 0$ we have

$$
(10) \qquad u^h_\varepsilon(1-h) \to \begin{cases} 1-\left(\dfrac{1-\varrho\theta(\varrho)/2}{1+\varrho\theta(\varrho)/2}\right) \text{ if } \left|\dfrac{1-\varrho\theta(\varrho)/2}{1+\varrho\theta(\varrho)/2}\right|<1 \\ 0 \text{ or } \infty \text{ otherwise} \end{cases}
$$

provided that $\varrho\theta(\varrho)/2 \neq 1$. In the case $\varrho\theta(\varrho)/2=1$ the difference scheme reduces to $(u_i-u_{i-1})/h=1$ and only one boundary condition can be satisfied, thus convergence cannot occur.

To obtain convergence for fixed ϱ and $h\to 0$ we must therefore have

$$
(11) \qquad \left|\frac{1-\varrho\theta(\varrho)/2}{1+\varrho\theta(\varrho)/2}\right|<1.
$$

Comparing (9) and (10) we see that we must also have $(1-\varrho\theta(\varrho)/2)/(1+\varrho\theta(\varrho)/2)=e^{-\varrho}$, which is equivalent to $\theta(\varrho)=(2/\varrho)\tanh \varrho/2$. But with this value of $\theta(\varrho)$ condition (11) is also fulfilled, and the proof is complete.

Corollary. *The difference schemes* (P^h_ε) *corresponding to the operators* L^h_ε *in* (3), (5) *and* (6) *do not converge uniformly in* ε, *for any* $p>0$; *those corresponding to* (4) *and* (7) *converge uniformly in* ε *only if the parameters are chosen so that the operators coincide with Ilin's operator* (8).

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