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A THEOREM ON STRONG CONVERGENCE IN BANACH SPACES WITH APPLICATIONS TO FIXED POINTS OF NONEXPANSIVE AND PSEUDOCONTRACTIVE MAPPINGS

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Summary. The authors extend a theorem on strong convergence which originally has been given by M. G. Crandall and A. Pazy for Hilbert spaces (1969) to Banach spaces possessing a weakly continuous duality mapping. As a consequence they are able to prove several fixed point theorems of Leray/Schauder and Borsuk type for nonexpansive and (continuous) pseudocontractive mappings which are defined on arbitrary closed neighborhoods of the origin of the underlying space. Within this frame a simple example is presented showing that the well-known Krasnoselskii fixed point theorem for compact vector fields of antipodal type need not be true even in a Hilbert space for affine isometries being defined on the unit ball. As another application a couple of fixed point theorems is given for pseudocontractive mappings which are defined on closed starshaped subsets of Banach spaces of that type we mentioned above. In this area some corresponding results which are due to J. Reinermann and R. Schöneberg (1976) are improved. Finally the authors prove both a fixed point theorem of minimum-principle type for pseudocontractive mappings and a fixed point theorem of Ritt type (*Ann. Math*, 22(1922), 157-160) for nonexpansive mappings.

A convergence theorem due to M. G. Crandall and A. Pazy [3] implies several fixed point theorems for continuous pseudocontractive and especially for nonexpansive mappings in Hilbert space (see [10], [11], [12], [13], [14]). In the present paper we establish a variant of that theorem which guarantees that most of these results are valid for a more general class of spaces.

Definition 1. (i) $\mu:\mathsf{R}^+\to\mathsf{R}^+$ *is said to be a gauge function* $:\Leftrightarrow \mu$ *is continuous and strictly monotone,* $\mu(0)=0$, $\lim\limits_{t\to\infty}\mu(t)=\infty$.

(ii) *Let* $(E, \|\cdot\|)$ *be a real normed space,* $\mu:\mathsf{R}^+\to\mathsf{R}^+$ *be a gauge function and* $J:E\to E^*$. J *is called a duality mapping with respect to* $\mu:\Leftrightarrow \forall_{x\in E}\, J(x)(x)=\|x\|\cdot\mu(\|x\|)\wedge\|J(x)\|=\mu(\|x\|)$.

(iii) $((E, \|\cdot\|), \mu, J)$ *satisfies* $(*):\Leftrightarrow (E, \|\cdot\|)$ *is a reflexive real normed space,* $\mu:\mathsf{R}^+\to\mathsf{R}^+$ *is a gauge function and* $J:E\to E^*$ *is a weakly sequentially continuous duality mapping with respect to* μ[1].

[1] This implies that E^* is strictly-convex (see [6]) and consequently J is unique.

Remark 2. (i) Let $(E, \|\cdot\|)$ be a real normed space, let $\mu: \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function. Then the Hahn-Banach theorem implies the existence of a duality mapping $J: E \to E^*$ with respect to μ.

(ii) Let $(E, (\cdot\,,\cdot))$ be a real Hilbert space. Define $J: E \to E^*$ by $J(x)\,(y) := (y, x)$ and $\|\cdot\|: E \to \mathbb{R}$ by $\|x\| := (x, x)^{1/2}$. Then $((E, \|\cdot\|), Id_{\mathbb{R}^+}, J)$ satisfies $(*)$.

(iii) Let $p, q \in (1, \infty)$, $1/p + 1/q = 1$. Then we identify l_p^*, l_q in the usual manner. Define $J: l_p \to l_q$ by
$J((x_j)_{j\in\mathbb{N}}) := (|x_j|^{p-1} \cdot \operatorname{sign} x_j)_{j\in\mathbb{N}}$ and $\mu: \mathbb{R}^+ \to \mathbb{R}$ by $\mu(t) := t^{p-1}$. Then $((l_p, \|\cdot\|), \mu, J)$ satisfies $(*)$. (See [2]).

Definition 3. *Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq X \subset E$, $f: X \to E$.*

(i) *f is said to be nonexpansive* $:\Leftrightarrow \underset{x, y\in X}{\forall} \|f(x) - f(y)\| \leq \|x - y\|$

(ii) *f is said to be pseudocontractive*$:\Leftrightarrow$

$$\underset{x, y\in X}{\forall}\ \underset{r\in\mathbb{R}^+}{\forall} \|x - y\| \leq \|(1+r)(x-y) - r\cdot(f(x) - f(y))\|.$$

Remark 4. Let $(E, \|\cdot\|)$ be a real normed space, $\emptyset \neq X \subset E$, $f: X \to E$.

(i) If f is nonexpansive, then f is pseudocontractive.

(ii) If there is a uniquely determined duality mapping $J: E \to E^*$ with respect to some gauge function then we have: f pseudocontractive $\Leftrightarrow$

$$\underset{x, y\in X}{\forall} J(x-y)(f(x) - f(y)) \leq J(x-y)\ (x-y) \text{ (see [7]).}$$

The announced convergence theorem is

Lemma 5 [10]. *Let $(E, \|\cdot\|)$ be a real normed space admitting a weakly sequentially continuous duality mapping $J: E \to E^*$ with respect to some gauge function μ, let $(x_n) \in E^{\mathbb{Z}^+}$, $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that*

(i) $\lim (x_n) = x_0$ *(weakly)*, (ii) $\lim (r_n) = 0$,

(iii) $\underset{n, m\in\mathbb{N}}{\forall} J(x_n - x_m)\,(r_n x_n - r_m x_m) \leq 0.$

Then $\lim (x_n) = x_0$ *(strongly)*.

Proof: We have for $n \in \mathbb{N}$ $\lim\limits_m (J(x_n - x_m)) = J(x_n - x_0)$ (weakly) $\lim\limits_m (-r_n x_n + r_m x_m) = -r_n x_n$ (strongly). This together with (iii) implies: $J(x_n - x_0)(-r_n x_n) = \lim\limits_m (J(x_n - x_m)(-r_n x_n + r_m x_m)) \geq 0$, hence $J(x_n - x_0)(-x_0) = J(x_n - x_0)(x_n - x_0) + J(x_n - x_0)(-x_n) \geq J(x_n - x_0)(x_n - x_0) = \|x_n - x_0\| \cdot \mu(\|x_n - x_0\|)$. Because of $\lim (J(x_n - x_0)(-x_0)) = 0$ we get $\lim (\|x_n - x_0\|) = 0$.

As an evident consequence of lemma 5 we get

Lemma 6. *Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, $(x_n) \in E^{\mathbb{N}}$, $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that (x_n) is bounded,* $\lim (r_n) = 0$ *and* $\underset{n, m\in\mathbb{N}}{\forall} J(x_n - x_m)\,(r_n x_n - r_m x_m) \leq 0.$

Then there is a subsequence (y_n) of (x_n) and $y \in E$ such that $\lim (y_n) = y$ *(strongly)*.

Lemma 6 implies the following fixed point theorem for continuous pseudocontractive mappings:

Lemma 7 (See [14]). *Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $\emptyset \neq X \subset E$ be closed and $f: X \to E$ be continuous and pseudocontractive, let $(x_n) \in X^{\mathbb{N}}$, $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ such that*

(i) *(x_n) is bounded*, (ii) $\lim (\lambda_n) = 1$,

(iii) $\underset{n\in\mathbb{N}}{\forall} x_n = \lambda_n f(x_n).$

Then f has a fixed point.

Proof: We define $(r_n) \in (0, \infty)^{\mathbb{N}}$ by $r_n := 1/\lambda_n - 1$. As f is pseudocontractive, we get for $n, m \in \mathbb{N}$:
$-J(x_n - x_m)(r_n x_n - r_m x_m) = J(x_n - x_m)(x_n - x_m - f(x_n) + f(x_m)) \geq 0$ (see remark 4 (ii)). Lemma 6 guarantees $y \in E$ and a subsequence (y_n) of (x_n) such that $\lim(y_n) = y$ (strongly).

Then $y \in X$ and because of $\lim(y_n - f(y_n)) = 0$ and continuity of f we get: $f(y) = y$.

The following theorems are applications of lemma 7. For Hilbert spaces and Lipschitzian pseudocontractive mappings the theorems 8 and 9 are proved in [14].

We use the following lemma to prove theorem 8.

Lemma 8a. (R. Schöneberg). *Let $(E, \|\cdot\|)$ be a Banach space, U be an open neighbourhood of the origin and $f: \bar{U} \to E$ be continuous and strongly pseudocontractive such that f maps bounded sets into bounded sets and*

$$\text{(LS)} \quad \forall_{x \in \partial U} \ \forall_{t \in \mathbb{R}} \ f(x) = tx \Rightarrow t \leq 1.$$

Then f has a unique fixed point.

Proof: Without loss of generality we assume $f(x) \neq x$ for $x \in \partial U$ and define for $t \in [0, 1]$ $T_t : \bar{U} \to E$ by $T_t(x) := x - tf(x)$.

By assumption there is $k < 1$ and a duality mapping $J: E \to E^*$ with respect to $Id_{\mathbb{R}^+}$ such that

(+) $J(x-y)(T_t(x) - T_t(y)) \geq (1-tk)\|x-y\|^2$ for $x, y \in \bar{U}$, $t \in [0, 1]$. Without loss of generality we may assume that U is bounded (for $T_t(x) = 0$ and (+) imply $\|x\| \leq (1-tk)^{-1}\|T_t(0)\| \leq (1-k)^{-1}\|f(0)\|$). So choose $M > 0$ such that $\|f(x)\| < M$ for $x \in \bar{U}$. Then we have

(++) $\|T_t(x) - T_s(x)\| \leq M|t-s|$ for $t, s \in [0, 1]$, $x \in \bar{U}$.

Let $W = \{t \in [0,1] / T_t(x) = 0 \text{ for some } x \in \bar{U}\}$. Since W is closed (by (+) and (++)) and $0 \in W$ it is enough to show that W is open with respect to $[0, 1]$. Let $t_0 \in W$ and $x_0 \in \bar{U}$ such that $T_{t_0}(x_0) = 0$. Observe that (+) and (++) imply $\varepsilon := \inf\{\|T_t(x)\| / x \in \partial U, t \in [0, 1]\} > 0$ and choose $\delta > 0$ such that $\|T_t(x_0)\| < \varepsilon$ for $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$. Then we have for $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$ and $x \in \partial U$:

$$\|T_t(x_0)\| < \|T_t(x)\|.$$

Theorem 1 of [8] implies:

$0 \in \overline{T_t[\bar{U}]}$, and as $T_t[\bar{U}]$ is closed (because of (+)) we get $0 \in T_t[\bar{U}]$, that means $t \in W$.

Theorem 8. *Let $((E, \|\cdot\|), \mu, J)$ satisfy (*), let $X \subset E$ be a closed neighbourhood of the origin and $f: X \to E$ be continuous and pseudocontractive such that $f[X]$ is bounded and*

$$\text{(LS)} \quad \forall_{x \in \partial X} \ \forall_{\lambda \in \mathbb{R}} \ f(x) = \lambda x \Rightarrow \lambda \leq 1.$$

Then f has a fixed point.

Proof: Choose $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$ $\lambda_n \cdot f$ is continuous and strongly pseudocontractive with

$$\forall_{x \in \partial X} \ \forall_{\lambda \in \mathbb{R}} \ (\lambda_n f)(x) = \lambda x \Rightarrow \lambda \leq 1.$$

By lemma 8a there is a sequence $(x_n)\in X^{\mathbb{N}}$ such that $x_n=\lambda_n f(x_n)$ for $n\in\mathbb{N}$. According to lemma 7 we are done.

Theorem 9. *Let $((E,\|\cdot\|),\mu,J)$ satisfy (*), let $X\subset E$ be a closed and symmetric neighbourhood of the origin and $f:X\to E$ be continuous and pseudocontractive such that $f[X]$ is bounded and $\forall_{x\in\partial X} f(-x)=-f(x)$. Then f has a fixed point.*

Proof: For $x\in\partial X$ we have $J(2x)(2f(x))=J(x-(-x))(f(x)-f(-x))\leq J(-(-x))(x-(-x))=J(2x)(2x)$. Thus f satisfies condition (LS) of theorem 8.

Lemma 10. *Let E be a topological linear space and $X\subset E$ be star-shaped with respect to the origin. Assume $f:\overline{X}\to E$ such that*

(R) $\forall_{x\in\partial X}\ \exists_{\lambda>0}\ \forall_{t\in(0,\lambda]}\ (1+t)x-tf(x)\notin\overline{X}$. *Then* $\forall_{x\in\partial X}\ \forall_{\lambda\in\mathbb{R}}\ f(x)=\lambda x\Rightarrow\lambda\leq 1$.

Proof: Let $x\in\partial X$, $\lambda\in\mathbb{R}$ and $f(x)=\lambda x$. Suppose $\lambda>1$. Choose $\tilde{\lambda}>0$ such that $(1+t)x-tf(x)\notin\overline{X}$ for $t\in(0,\tilde{\lambda}]$ and choose $t\in(0,\tilde{\lambda}]$ such that $(\lambda-1)t\in(0,1]$. Then we have $(1+t)x-tf(x)=(1-(\lambda-1)t)x\in\overline{X}$ since $\overline{X}$ is star-shaped with respect to the origin, too. This contradicts (R), thus $\lambda\leq 1$.

Observing lemma 10 and theorem 8 we obtain

Theorem 11. *Let $((E,\|\cdot\|),\mu,J)$ satisfy (*). Suppose $X\subset E$ is closed and star-shaped with respect to $0\in\operatorname{int}(X)$ and $f:X\to E$ is continuous and pseudocontractive such that $f[X]$ is bounded and*

(R) $\forall_{x\in\partial X}\ \exists_{\lambda>0}\ \forall_{t\in(0,\lambda]}\ (1+t)x-tf(x)\notin\overline{X}$. *Then f has a fixed point.*

Remark 12. Lemma 10 shows that H. Rothe's fixed point theorem for compact maps in [15] is only a special case of the general Leray-Schauder fixed point theorem for compact maps.

Theorem 13. *Let $((E,\|\cdot\|),\mu,J)$ satisfy (*). Suppose $X\subset E$ is a closed bounded and symmetric neighbourhood of the origin, $f:X\to E$ is continuous and pseudocontractive such that $f[X]$ is bounded and*

$$\text{(A)}\quad \exists_{\varepsilon>0}\ \forall_{x\in\partial X}\ \|f(x)+f(-x)\|^2-\|2x-f(x)+f(-x)\|^2$$

$$\leq 4(1-\varepsilon)\|x-f(x)\|\cdot\|x+f(-x)\| \quad \textit{and}$$

$$\text{(B)}\quad \inf\{\|x-f(x)\|/x\in\partial X\}>0.$$

Then f has a fixed point.

Proof: Let $\varepsilon>0$ be chosen according to (A). Let $M>0$ such that $\|f(x)\|<M$, $\|x\|<M$ for $x\in X$, $r:=\inf\{\|x-f(x)\|:x\in\partial X\}$, let $(\lambda_n)\in(0,1)^{\mathbb{N}}$ such that $\lim(\lambda_n)=1$ and $(1-\lambda_n)\cdot 9M^2<\varepsilon\cdot r^2$ for $n\in\mathbb{N}$. Then we have for $x\in\partial X$, $n\in\mathbb{N}$:

$$\frac{1}{4}\|\lambda_n f(x)+\lambda_n f(-x)\|^2-\frac{1}{4}\|2x-\lambda_n f(x)+\lambda_n f(-x)\|^2$$

$$\leq\frac{1}{4}\|f(x)+f(-x)\|^2-\frac{1}{4}\|2x-f(x)+f(-x)\|^2+(1-\lambda_n)4M^2$$

$$\leq(1-\varepsilon)\|x-f(x)\|\cdot\|x+f(-x)\|+(1-\lambda_n)\cdot 4M^2$$
$$\leq\|x-f(x)\|\cdot\|x+f(-x)\|-\varepsilon\cdot r^2+(1-\lambda_n)\cdot 4M^2$$
$$\leq(\|x-\lambda_n f(x)\|+(1-\lambda_n)M)(\|x+\lambda_n f(-x)\|+(1-\lambda_n)M)-\varepsilon\cdot r^2+(1-\lambda_n)\cdot 4M^2$$
$$<\|x-\lambda_n f(x)\|\cdot\|x+\lambda_n f(-x)\|, \text{ hence}$$

$x-\lambda_n f(x) \neq \mu(-x-\lambda_n f(-x))$ for $n \in \mathbb{N}$, $x \in \partial X$, $\mu \in (0, 1]$.

A lemma similar to lemma 8a guarantees that there is a sequence $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$. Hence f has a fixed point by lemma 7.

Remark 14. (i) In the case of a Hilbert space $(E, (\cdot, \cdot))$ condition (A) of theorem 13 is equivalent to

$$\underset{\varepsilon>0}{\exists}\ \underset{x \in \partial X}{\forall} \left(\frac{x-f(x)}{\|x-f(x)\|}, \quad \frac{-x-f(-x)}{\|-x-f(-x)\|} \right) \leq 1-\varepsilon.$$

(ii) For nonexpansive mappings we get the following

Theorem: *Let* $(E, \|\cdot\|)$ *be a uniformly convex space. Suppose* $X \subset E$ *is a closed bounded convex symmetric neighbourhood of the origin and let* $f: X \to E$ *be nonexpansive such that* (A) *of theorem* 13 *is fulfilled. Then* f *has a fixed point.*

The proof is based upon the fact that $Id_X - f$ is demi-closed.

(iii) Even in Hilbert space the condition (A) of theorem 13 cannot be weakened to

$$(+) \underset{x \in \partial X}{\forall} \left(\frac{x-f(x)}{\|x-f(x)\|}, \quad \frac{-x-f(-x)}{\|-x-f(-x)\|} \right) < 1 \text{ (Compare (i))}$$

sa we show by the following example:

Let $X = \{x \mid x \in l_2 \wedge \|x\| \leq 1\}$. X is a closed bounded canvex symmetric neighbourhood of the origin. Define $f: X \to l_2$ by

$f((x_n)_{n \in \mathbb{N}}) := \exp((2\pi i/(n+1)!)x_n + (-1)^n(1-\exp(2\pi i/(n+1)!)))_{n \in \mathbb{N}}$.

Then f is an isometry without fixed points since we get for $x = (x_n) \in X$:

$x - f(x) = ((1-\exp(2\pi i/(n+1)!))(x_n - (-1)^n))_{n \in \mathbb{N}}$;

$x \in X$, $\mu \in (0, 1]$ with $x - f(x) = \mu(-x - f(-x))$ imply

$(1+\mu)(1-\exp(2\pi i/(n+1)!)) \cdot x_n = (1-\mu)(1-\exp(2\pi i/(n+1)!))(-1)^n$,

i. e. $x_n = (-1)^n(1-\mu)/(1+\mu)$ for $n \in \mathbb{N}$, and because of $x \in l_2: \mu = 1$, thus $x = 0$. Hence condition (+) is fulfilled. (Condition (B) of theorem 13 is fulfilled, too, for otherwise f would have a fixed point since $Id_X - f$ is demi-closed.) Note that according to a theorem of M. A. Krasnoselskii (+) implies the existence of fixed points if f is assumed to be a compact map.

Theorem 15 (see [8]). *Let* $((E, \|\cdot\|), \mu, J)$ *satisfy* $(*)$, *let* $X \subset E$ *be closed and bounded with int* $(X) \neq \emptyset$. *Suppose* $f: X \to E$ *is continuous and pseudocontractive such that* $\underset{z \in X}{\exists}\ \underset{x \in \partial X}{\forall} \|z - f(z)\| < \|x - f(x)\|$ *("minimum principle"). Then* f *has a fixed point.*

Proof: Theorem 1 of [8] implies $\inf\{\|x - f(x)\| / x \in X\} = 0$. Without loss of generality we may assume that $a = \inf\{\|x - f(x)\| / x \in \partial X\} > 0$ and that there exists $z \in X$ such that $\|z - f(z)\| < a$. Moreover, we may assume $z = 0$. Choose $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that $\lim(r_n) = 0$ and $r_n \|x\| + \|f(0)\| < a$ for $n \in \mathbb{N}$ and $x \in X$. Define $T_n: X \to E$ by $T_n = (1 + r_n) Id_X - f$, let $n \in \mathbb{N}$. Then we have for $x, y \in X$: $\mu(\|x-y\|)\|T_n(x) - T_n(y)\| \geq J(x-y)(T_n(x) - T_n(y)) \geq J(x-y)(r_n x - r_n y) = r_n \mu(\|x-y\|)\|x-y\|$, hence $(+)\ \|T_n(x) - T_n(y)\| \geq r_n \|x-y\|$, and for $x \in \partial X$: $\|T_n(0)\| = \|f(0)\| < \|x - f(x)\| - r_n \|x\| \leq \|T_n(x)\|$. Theorem 1 of [8] implies: $0 \in \overline{T_n[X]}$, and because of $(+): 0 \in T_n[X]$. That means: there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = (1/(1+r_n)) f(x_n)$ for $n \in \mathbb{N}$. Lemma 7 completes the proof.

Remark 16. From theorem 1 of [8] we learn that theorem 15 remains true if the assumption "$((E, \|\cdot\|), \mu, J)$ satisfies $(*)$" is replaced by "$(E, \|\cdot\|)$ is a Banach space and X has the fixed point property with respect to nonexpansive self-mappings".

Lemma 17. *Let $(E, \|\cdot\|)$ be a normed space. Suppose $X \subset E$ is closed and star-shaped with respect to the origin, $\lambda \in (0, 1)$ and $f: X \to E$ such that*

$$\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1-t)x + tf(x), X) = 0.$$

Then $\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1-t)x + t \cdot \lambda f(x), X) = 0.$

Theorem 18. *Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $X \subset E$ is closed, bounded and star-shaped and $f: X \to E$ is continuous and pseudocontractive such that*

$$\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1-t)x + tf(x), X) = 0.$$

Then f has a fixed point.

Proof: Define $\tilde{J}: E \to E^*$ by $J(0) := 0$, $\tilde{J}(x) := \frac{\|x\|}{\mu(\|x\|)} J(x)$ for $x \in E \setminus \{0\}$. $\tilde{J}$ is the (uniquely determined) duality mapping with respect to $Id_{\mathbb{R}^+}$. Without loss of generality we assume X to be star-shaped with respect to the origin. Choose $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ such that $\lim(\lambda_n) = 1$. Then we have for $n \in \mathbb{N}$:

(i) $\lambda_n f$ is continuous

(ii) $\tilde{J}(x-y)(\lambda_n f(x) - \lambda_n f(y)) \leq \lambda_n \|x-y\|^2$

(iii) $\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1-t)x + t \cdot \lambda_n f(x), X) = 0$ (Lemma 17).

A theorem of R. H. Martin [9] and K. Deimling [4] implies the existence of $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and lemma 7 completes the proof.

Remark 19. For Lipschitzian pseudocontractive mappings in Hilbert spaces theorem 18 was proved by D. Göhde [5]. If X is assumed to be convex it was shown in [14] that the assumption "f be Lipschitzian" can be dropped.

Theorem 20. *Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ is closed and bounded and $f: X \to E$ is nonexpansive such that $[0, 1](f(\partial X] - \{z\}) \subset X - \{z\}$ for some $z \in E$. Then f has a fixed point.*

Proof: Without loss of generality $[0, 1] f[\partial X] \subset X$. Let $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$, $x \in \partial X$ we have: $\lambda_n f$ is a Banach-contraction and $(\lambda_n f)(x) \in [0, 1] f[\partial X] \subset X$, thus $(\lambda_n f)[\partial X] \subset X$. According to a theorem due to N. A. Assad [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and by lemma 7 we obtain the conclusion.

L e m m a 21. *Let* $(E, \|\cdot\|)$ *be a normed space,* $J: E \to E^*$ *be a duality mapping with respect to some gauge function* $\mu: \mathbb{R}^+ \to \mathbb{R}$. *Suppose* $x, z \in E$, $M>0$, $\|x\| \geqq 3M$, $\|z\|<M$. *Then* (i) $0<\mu(2M) \leqq \|J(x-z)\| \leqq \mu(\|x\|+M)$ (ii) $J(x-z)(x) \geqq \mu(2M) \cdot M$.

P r o o f: (i) $\|J(x-z)\|=\mu(\|x-z\|) \geqq \mu(\|x\|-\|z\|) \geqq \mu(3M-M)$,

$$\|J(x-z)\| \geqq \mu(\|x\|+\|z\|) \geqq \mu(\|x\|+M)$$

(ii) $J(x-z)(x)=J(x-z)(x-z)+J(x-z)(z)$

$$\geqq J(x-z)\|(\|x-z\|-\|z\|) \geqq \mu(2M) \cdot M.$$

L e m m a 22. *Let* $(E, \|\cdot\|)$ *be a normed space,* $M, r>0$, $x \in E$, $\|x\| \geqq 3M$, $\emptyset \neq S \subset E$ *and suppose* $\|z\|<M$ *for* $z \in S$. *Then* $\inf\{\|(1+r)x-z\|: z \in S\}>\inf\{\|x-z\|: z \in S\}$.

P r o o f: Let $J: E \to E^*$ be a duality mapping with respect to $Id_{\mathbb{R}^+}$. Then we have for $z \in S: \|J(x-z)\|\,\|(1+r)x-z\| \geqq J(x-z)(x-z)+J(x-z)(rx)=\|J(x-z)\|\,\|x-z\|+rJ(x-z)(x) \geqq \|J(x-z)\|\,\|x-z\|+r \cdot 2 \cdot M \cdot M$, and from

$$\|(1+r)x-z\| \geqq \|x-z\|+\frac{2rM^2}{\|J(x-z)\|} \geqq \|x-z\|+\frac{2rM^2}{\|x\|+M} \text{ and } \frac{2rM^2}{\|x\|+M}>0$$

the conclusion follows.

L e m m a 23. *Let* $(E, \|\cdot\|)$ *be a normed space,* $\emptyset \neq X \subset E$ *and* $f: X \to E$ *be nonexpansive. Suppose* $x_0 \in X$ *such that* $(f^n(x_0))_{n \in \mathbb{Z}^+}$ *is bounded. Finally let* $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ *and* $(\lambda_n) \in (0,1)^{\mathbb{N}}$ *such that* $x_n=\lambda_n f(x_n)$ *for* $n \in \mathbb{N}$. *Then* $(x_n)_{n \in \mathbb{N}}$ *is bounded.*

P r o o f: Define $S=\{f^n(x_0)/n \in \mathbb{Z}^+\}$. Obviously $S \neq \emptyset$. Choose $M>0$ such that $\|z\|<M$ for $z \in S$. We claim $\|x_n\|<3M$ for $n \in \mathbb{N}$. Otherwise we would have $\|x_n\| \geqq 3M$ for a suitable $n \in \mathbb{N}$, hence by lemma 22 $\inf\{\|f(x_n)-z\|: z \in S\}=\inf\{\|\frac{1}{\lambda_n} \cdot x_n-z\|: z \in S\}>\inf\{\|x_n-z\|: z \in S\}$ for this n. Choose $y \in S$ with $\|x_n-y\|<\inf\{\|f(x_n)-z\|: z \in S\}$. Observing $f(y) \in S$ we get $\|f(x_n)-f(y)\| \geqq \inf\{\|f(x_n)-z\|: z \in S\}>\|x_n-y\|$ and this is a contradiction to the nonexpansiveness of f.

T h e o r e m 24. *Let* $((E, \|\cdot\|), \mu, J)$ *satisfy* (*). *Suppose* $\emptyset \neq X \subset E$ *is closed and star-shaped and* $f: X \to E$ *is nonexpansive such that* $f[\partial X] \subset X$. *Then* f *has a fixed point if and only if there is* $x_0 \in X$ *such that* $(f^n(x_0))_{n \in \mathbb{Z}^+}$ *is bounded.*

P r o o f: "$\Longrightarrow$": trivial. "$\Longleftarrow$": Without loss of generality let X be star-shaped with respect to the origin. Then for $n \in \mathbb{N}$ $(1-1/(n+1))f$ is a Banach-contraction with $(1-1/(n+1))f[\partial X] \subset X$. By a theorem of N. A. A s s a d [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n=(1-1/(n+1))f(x_n)$ for $n \in \mathbb{N}$. The boundedness of (x_n) follows from lemma 23; lemma 7 completes the proof.

R e m a r k 25 (i). Theorem 24 was originally proved for Hilbert spaces by J. R e i n e r m a n n and R. S c h ö n e b e r g [14].

(ii) In the case of a Hilbert space $(E, (\cdot,\cdot))$ and a convex X theorem 24 remains valid if the condition "$f[\partial X] \subset X$" is omitted.

(iii) Clearly the boundedness of X implies the boundedness of every defined Picard-sequence of f, but there are examples of unbounded star-shaped sets such that every nonexpansive self-mapping has a fixed point (thus every Picard-sequence is bounded).

Theorem 26. *Let* $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ *is closed and bounded and* $f: X \to E$ *is nonexpansive such that* $\|f[\partial X], E \setminus X\| > 0$. *Then* f *has a fixed point.*

Proof: Assume $0 \in X$ and choose $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ with $\lim (\lambda_n) = 1$. For $n \in \mathbb{N}$, $x \in \partial X$ and $y \in E \setminus X$ we have

$$\|\lambda_n f(x) - y\| \geq \lambda_n \|f(x) - y\| - (1 - \lambda_n)\|y\| \geq \lambda_n \|f[\partial X], E \setminus X\| - (1 - \lambda_n)\|y\|.$$

Choose $n_0 \in \mathbb{N}$ such that $\lambda_n \geq 1/2$ and $(1 - \lambda_n)$ diam $X < (1/4)\|f[\partial X], E \setminus X\|$ for $n \geqq n_0$. Then we have for $n \geq n_0$: $\|\lambda_n f[\partial X], E \setminus X\| = \inf\{\|\lambda_n f(x) - y\| / x \in \partial X \wedge y \in E \setminus X \wedge \|y\| < \text{diam } X\} \geq (1/4)\|f[\partial X], E \setminus X\| > 0$, hence $\lambda_n f[\partial X] \subset X$ and by a theorem of N. A. Assad [1] there is $x_n \in X$ such that $x_n = \lambda_n f(x_n)$.

Lemma 7 completes the proof.

Remark 27. (i) Obviously theorem 26 is not true for compact maps.

(ii) The condition "X is bounded" cannot be cancelled in theorem 26 as is shown by the example $(E, \|\cdot\|) := (\mathbb{R}, |\cdot|)$, $X := \mathbb{R}^+$, $f(x) := x + 1$.

Finally, we give a continuation theorem for nonexpansive mappings:

Theorem 28. *Let* $(E, \|\cdot\|)$ *be a Banach space for which the Browder-Göhde-Kirk fixed point theorem for nonexpansive mappings is valid. Assume* $\emptyset \neq X \subset E$ *is closed and bounded and* $f: X \to E$ *is nonexpansive such that*

$$\underset{g:\, \overline{\text{co}}\, X \to X}{\exists} g \text{ nonexpansive} \wedge g|_{\partial X} = f|_{\partial X}.$$

Then f *has a fixed point.*

Proof: Let $g: \overline{\text{co}}\, X \to X$ be nonexpansive with $f|_{\partial X} = g|_{\partial X}$ and define $h: \overline{\text{co}}\, X \to E$ to be

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in \overline{\text{co}}\, X \setminus X. \end{cases}$$

We claim that h is nonexpansive, too. To show this let $x \in \text{int}(X)$ and $y \in \overline{\text{co}}\, X \setminus X$ without loss of generality. There is $z \in \partial X$ and $t \in [0, 1]$ such that $z = tx + (1 - t)y$. Hence $\|h(x) - h(y)\| \leq \|h(x) - f(z)\| + \|f(z) - h(y)\| = \|f(x) - f(z)\| + \|g(z) - g(y)\| \leq \|x - z\| + \|z - y\| = \|x - y\|$.

Because of $\partial\, \overline{\text{co}}\, X \subset \overline{\text{co}}\, X \setminus \text{int}(X)$ and $h[\partial\, \overline{\text{co}}\, X] \subset X \subset \overline{\text{co}}\, X$ there is $\alpha \in (0, 1)$ such that $\alpha h + (1 - \alpha) Id$ is a nonexpansive self-mapping and thus $\alpha h + (1 - \alpha) Id$ has a fixed point by assumption. This point is a fixed point of h and of f, too.

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