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AN APPLICATION OF A BERNSTEIN-TYPE INEQUALITY FOR FRACTIONAL DERIVATIVES TO SOME PROBLEMS OF MODULAR SPACES

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Summary. Applying an inequality of Bernstein type for fractional derivatives in Orlicz metric generated by a convex function φ, there are established relations between modulars $\varrho(a, x) = \int_a^b \varphi(|D^a x(t)|)\,dt + |x(0)|$, $0 < a \leq 1$, in a class X of functions x defined by (1). These relations are given by inequality (2), which is applied to investigate modular spaces $\overline{X}$ and $\underline{X}$ defined in 2.1 and X_ϱ defined in 2.5.

1.1. Let X be the class of complex-valued functions x defined in the interval (a, b) ($(a, b) = (-\infty, \infty)$ or $0 < b-a < \infty$), which are of the form

$$x(t) = \int_{-R}^{R} e^{itu}\, d\omega(u), \tag{1}$$

where $R > 0$ and ω is a function of bounded variation in $[-R, R]$, R and ω dependent on x, and which are $(b-a)$-periodic in case of $b-a < \infty$. It is easily observed that X is a vector space. If $x \in X$ is of the form (1), then the derivative $D^a x$ of x with $0 \leq a \leq 1$ may be defined as

$$D^a x(t) = \int_{-R}^{R} i^a u^a e^{itu}\, d\omega(u)$$

(see [2], p. 662). It is obvious that $D^a x$ are continuous functions in $(-\infty, \infty)$, $D^0 x(t) = x(t)$ and $D^1 x(t) = x'(t)$ (the derivative in the usual sense). Moreover, the operator D^a is linear in X.

1.2. Now, let φ be a convex function defined for $u \geq 0$, $\varphi(u) = 0$, iff $u = 0$. We define the following functionals for $x \in X$:

$$\varrho(a, x) = \int_a^b \varphi(|D^a x(t)|)\, dt + |x(0)| \text{ for } 0 < a \leq 1.$$

This is a family of convex modulars depending on the parameter $a \in (0, 1\rangle$ (in the sense of [4]), i. e.

(1a) $\varrho(a, 0) = 0$,

(1b) if $\varrho(a, x) = 0$ for all $0 < a \leq 1$, then $x = 0$, (2') $\varrho(a, e^{ic}x)=\varrho(a, x)$ for all real c,

(3') $\varrho(a, \lambda_1 x+\lambda_2 y)\leq\lambda_1\varrho(a, x)+\lambda_2\varrho(a, y)$ for $\lambda_1, \lambda_2\geq 0$, $\lambda_1+\lambda_2=1$,

(4') $\varrho(a, x)$ is a Lebesgue measurable function of $a\in(0, 1\rangle$ for every $x\in X$.

1.3. We shall now investigate the relations between $\varrho(a, x)$ for different values of a. For this purpose we estimate the function $g(t)=\int_{-R}^{R}\mu(u)\gamma(u)e^{itu}\,d\omega(u)$ by means of the function $f(t)=\int_{-R}^{R}\gamma(u)e^{itu}\,d\omega(u)$ for $\gamma(u)=i^{a}u^{a}$, $\mu(u)=i^{\beta-a}u^{\beta-a}$, $0<a<\beta\leq 1$. In [3] there were investigated the connections between the norms of the functions f and g in Orlicz spaces $L^{\varphi}(-\infty, \infty)$ and $L^{\varphi}(a, b)$ (the second one in $(b-a)$-periodic case). Here, we shall consider connections between the respective modulars. Taking $p(\nu)=\nu\pi/R$ for $\nu=0, \pm 1, \pm 2,\ldots$ and $\mu(u)\quad e^{(iku)}=\sum_{\nu=-\infty}^{\infty}c_\nu e^{(ip(\nu)u)}$ for $u\in[-R, R]$ and some $k\geq 0$, where $\sum_{-\infty}^{\infty}|c_\nu|<\infty$, we have

$$g(t)=\sum_{\nu=-\infty}^{\infty}c_\nu f[p(\nu)-k+t] \text{ for } -\infty<t<\infty \text{ (see [3])}.$$

Moreover, taking $x(t)$ of the form (1), we have $D^{a}x(t)=f(t)$ and $D^{\beta}x(t)=g(t)$.

1.4. Lemma 1. *Applying the notation of 1, 3, we have*

$$\int_a^b\varphi(|g(t)|)\,dt\leq\int_a^b\varphi\Big(|f(t)|\sum_{-\infty}^{\infty}|c_\nu|\Big)\,dt.$$

Proof. Denoting $g_n(t)=\sum_{\nu=-n}^{n}c_\nu f[p(\nu)-k+t]$ and applying Fatou lemma, we have

$$\int_a^b\varphi(|g(t)|)\,dt\leq\varliminf_{n\to\infty}\int_a^b\varphi(|g_n(t)|)\,dt.$$

Now, applying Jensen's inequality for convex functions, we obtain

$$\int_a^b\varphi(|g_n(t)|)\,dt\leq\int_a^b\varphi\Big(|f(t)|\sum_{\nu=-n}^{n}|c_\nu|\Big)\,dt\leq\int_a^b\varphi\Big(|f(t)|\sum_{\nu=-\infty}^{\infty}|c_\nu|\Big)\,dt.$$

Collecting the above two inequalities we get the desired result.

1.5. Theorem 1. *If* $0<a<\beta\leq 1$ *and* $x\in X$ *is of the form* (1) *then*

$$\varrho(\beta, x)\leq\varrho(a, 7R^{\beta-a}x/(\beta-a))+(1-7R^{\beta-a}/(\beta-a))|x(0)|. \tag{2}$$

Proof. Applying the above lemma and the inequality $\sum_{\nu=-\infty}^{\infty}|c_\nu|\leq 7R^{\beta-a}/(\beta-a)$ (see [2] or [3], Theorem 3), we obtain

$$\varrho(\beta, x)\leq\int_a^b\varphi\Big(|f(t)|\sum_{\nu=-\infty}^{\infty}|c_\nu|\Big)\,dt+|x(0)|\leq\int_a^b\varphi((\beta-a)^{-1}7R^{\beta-a}|f(t)|)\,dt+|x(0)|$$
$$=\varrho(a, 7R^{\beta-a}x/(\beta-a))+(1-7R^{\beta-a}/(\beta-a))|x(0)|.$$

2.1. Now, we shall investigate the modular space X_{ϱ_a}, defined by means of the convex pseudomodular $\varrho_a(x)=\varrho(a, x)$, $0<a\leq 1$, in the following manner:

$$X_{\varrho\alpha}=\{x:x\in X \text{ and } \varrho_\alpha(\lambda x)\to 0 \text{ as } \lambda\to 0\}.$$

Evidently, we have $X_{\varrho_\alpha}=\{x:x\in X \text{ and } D^\alpha x\in L^\varphi(a,\ b)\}$. It is easily seen that

(3) $$X_{\varrho_\alpha}\subset X_{\varrho_\beta} \text{ for } 0<\alpha\leq\beta\leq 1;$$

this follows from the inequality [2]. Let us remark that if $0<b-a<\infty$, i. e. X consists of $(b-a)$-periodic functions of the form (1), then $X_{\varrho_\alpha}=X$ for all $\alpha\in(0,\ 1)$, because then $D^\alpha x$ are continuous in $[a,\ b]$ and so $D^\alpha x\in L^\varphi(a,\ b)$. Hence in the subsequent investigation of relations between spaces X_{ϱ_α} with various α, we limit ourselves to the case $(a,\ b)=(-\infty,\ \infty)$. Let us denote

$$\overline{X}=\bigcup_{0<\alpha<1}X_{\varrho_\alpha} \quad\text{and}\quad \underline{X}=\bigcap_{0<\alpha<1}X_{\varrho_\alpha}.$$

By (3), we may write

$$\overline{X}=\bigcup_{n=1}^{\infty}X_{\varrho_{1-1/2^n}} \quad\text{and}\quad \underline{X}=\bigcap_{n=1}^{\infty}X_{\varrho_{1/2^n}}.$$

First, we prove the following proposition concerning $\overline{X}$:

2.2. Let $\overline{W}_\varphi^1$ be the space of functions x in $(-\infty,\ \infty)$ possessing the derivative $x'(t)$ for all $t\in(-\infty,\ \infty)$ and such that $x'\in L^\varphi(-\infty,\ \infty)$. Then $\overline{X}\subset\overline{W}_\varphi^1\cap X$.

Indeed, we have evidently $\overline{X}\subset X$. Supposing $x\in\overline{X}$, we have $D^\alpha x\in L^\varphi(-\infty,\ \infty)$ for $0<\alpha<1$, and so

$$\int_{-\infty}^{\infty}\varphi\left(\lambda\, 7(R^{1-\alpha}/(1-\alpha))\,|D^\alpha x(t)|\right)dt\to 0 \quad\text{as } \lambda\to 0,\ 0<\alpha<1.$$

Applying (2) with $\beta=1$, we get $\varrho(1,\lambda x)\to 0$ as $\lambda\to 0$, i. e. $x'\in L^\varphi(-\infty,\infty)$. This proves that $x\in\overline{W}_\varphi^1$.

2.3. Now, we proceed to investigation of $\underline{X}$. First, we establish the following

Lemma 2. *The sequence of convex pseudomodulars* $\varrho_{1/2^n}$ *in* X *satisfies the following conditions*:

(1°a) $\varrho_{1/2^n}(0)=0$,

(1°b) $\varrho_{1/2^n}(x)=0$ *for* $n=1,\ 2,\dots$ *implies* $x=0$,

(2°) $\varrho_{1/2^n}(e^{ic}x)=\varrho_{1/2^n}(x)$ *for all real* c,

(3°) $\varrho_{1/2^n}(\lambda_1x+\lambda_2y)\leq\lambda_1\varrho_{1/2^n}(x)+\lambda_2\varrho_{1/2^n}(y)$ *for* $\lambda_1,\ \lambda_2\geq 0,\ \lambda_1+\lambda_2=1$.

Proof. (1°a), (2°) and (3°) follow from (1a), (2′) and (3′) of 1.2, respectively. In order to prove (1°b), let us suppose that $\varrho_{1/2^n}(x)=0$ for $n=1,2,\dots$ Applying (2) with $\alpha=1/2^n$, $\beta=1$, we obtain

(4) $$\int_{-\infty}^{\infty}\varphi(|x'(t)|)\,dt\leq\varrho\left(2^{-n},\ 7R^{1-2^{-n}}x/(1-2^{-n})\right),$$

because from $\varrho_{1/2^n}(x)=0$ follows $x(0)=0$. Moreover, $\varrho_{1/2^n}(x)=0$ implies $D^{2-n}x(t)=0$ for all t and $n=1, 2, \dots$ Hence the right-hand side of inequality (4) is equal to 0. Consequently, we have $x'(t)=0$, and from $x(0)=0$ we conclude that $x(t)=0$ for all t.

2.4. Theorem 2. *Let us denote*

$$\underline{\varrho}(x)=\sum_{n=1}^{\infty} 2^{-n}\varrho_{1/2^n}(x)/(1+\varrho_{1/2^n}(x)) \quad \text{for } x\in X$$

and

$$X_{\underline{\varrho}}=\{x : x\in X \text{ and } \underline{\varrho}(\lambda x)\to 0 \text{ as } \lambda\to 0\}.$$

Then $\underline{\varrho}$ *is a modular in* X *and* $X_{\underline{\varrho}}=X$.

Proof. From Lemma 2 it follows that $\varrho_{1/2^n}$ satisfy all the assumptions of [1], concerning a sequence of pseudomodulars on a vector space X. Thus, by [1], $\underline{\varrho}$ is a modular in X and $X_{\underline{\varrho}}$ is the respective countably modulared space, and $X_{\underline{\varrho}}$ consists exactly of elements $x\in X$ such that $\varrho_{1/2^n}(\lambda x)\to 0$ as $\lambda\to 0$ for $n=1, 2, \dots$ Thus, we have $X_{\underline{\varrho}}=X$.

2.5. Now, we are going to show that the countably modulared space $X_{\underline{\varrho}}$ may be defined also by means of integration with respect to the ordinary derivative. By 1.2,

$$\varrho(x)=\int_0^1 (1+\varrho(\alpha, x))^{-1}\varrho(\alpha, x)\,d\alpha \tag{5}$$

is also a modular in X (see [4]); the respective modular space is

$$X_\varrho=\{x : x\in X \text{ and } \varrho(\lambda x)\to 0 \text{ as } \lambda\to 0\}.$$

First, we show some auxiliary inequalities.

Lemma 3. *Let* $x\in X$ *be of the form* (1) *with* $R\geq 1$, *and let* $0<\alpha<1$, $0<\varepsilon<2$. *Then the following inequalities hold*:

$$3\cdot 2^{-n-2}\frac{\varrho(2^{-n+1}, 2^{-n-2}R^{-2^{-n}}x/7)}{1+\varrho(2^{-n+1}, 2^{-n-2}R^{-2^{-n}}x/7)}\leq\int_{2^{-n}}^{2^{-n+1}}\frac{\varrho(\alpha, x)}{1+\varrho(\alpha, x)}\,d\alpha \tag{6}$$

$$\leq\varepsilon 2^{-n-2}+2^{-n}\varrho(2^{-n}, 7\cdot 2^{n+3}\varepsilon^{-1}R^{2-n}x)/(1+\varrho(2^{-n}, 7\cdot 2^{n+3}\varepsilon^{-1}R^{2-n}x)).$$

Proof. First, we prove the left one of the inequalities (6). Supposing $0<\alpha<2^{-n+1}$ and applying (2) with $\beta=2^{-n+1}$ and with $(2^{-n+1}-\alpha)R^{-2^{-n+1+\alpha}}x/7$ in place of x, we obtain

$$\varrho(\alpha, x)\geq\varrho(2^{-n+1}, (2^{-n+1}-\alpha)x/(7R^{2-n+1}-\alpha))$$
$$+(1-(2^{-n+1}-\alpha)/(7R^{2-n+1}-\alpha))\,|x(0)|.$$

But the second term on the right-hand side of this inequality is nonnegative for $R\geq 1$. Hence

$$\varrho(\alpha, x)\geq\varrho(2^{-n+1}, (2^{-n+1}-\alpha)x/(7R^{2-n+1}-\alpha)).$$

Consequently, we have

$$\int_{2^{-n}}^{2^{-n+1}} (1+\varrho(a, x))^{-1}\varrho(a, x)\,da \geqq \int_{2^{-n}}^{7.2^{-n-2}} (1+\varrho(a, x))^{-1}\varrho(a, x)\,da$$

$$\geqq \int_{2^{-n}}^{7.2^{-n-2}} \varrho(2^{-n+1}, (2^{-n+1}-a)x/(7\cdot 2^{-n+1}-a))$$
$$\times[1+\varrho(2^{-n+1}, (2^{-n+1}-a)x/(7\cdot 2^{-n+1}-a))]^{-1}\,da$$

$$\geqq 3\cdot 2^{-n-2}\varrho(2^{-n+1}, \tfrac{1}{7}2^{-n-2}R^{-2^{-n}}x)[1+\varrho(2^{-n+1}, \tfrac{1}{7}2^{-n-2}R^{-2^{-n}}x)]^{-1}.$$

Now, we prove the right one of the inequalities (6). Applying (2) with a in place of β and with 2^{-n} in place of a, we obtain for $2^{-n}<a<1$:

$$\varrho(a, x)\leqq\varrho(2^{-n}, 7R^{a-2^{-n}}x/(a-2^{-n}))+(1-7R^{a-2^{-n}}/(a-2^{-n}))\,|\,x(0)\,|.$$

However, the second term on the right-hand side of this inequality is non-positive for $R\geqq 1$. Hence

$$\varrho(a, x)\leqq\varrho(2^{-n}, 7R^{a-2^{-n}}x/(a-2^{-n})).$$

Thus, taking $0<\varepsilon<2$, we obtain

$$\int_{2^{-n}}^{2^{-n+1}} (1+\varrho(a, x))^{-1}\varrho(a, x)\,da\leqq\varepsilon 2^{-n-3}+\int_{2^{-n}+\varepsilon.2^{-n-3}}^{2^{-n+1}} (1+\varrho(a, x))^{-1}\varrho(a, x)\,da$$

$$\leqq\varepsilon 2^{-n-3}+\int_{2^{-n}+\varepsilon.2^{-n-3}}^{2^{-n+1}} \varrho(2^{-n}, 7R^{a-2^{-n}}x/(a-2^{-n}))[1+\varrho(2^{-n}, 7R^{a-2^{-n}}x$$

$$:(a-2^{-n}))]^{-1}\,da\leqq\varepsilon 2^{-n-3}+2^{-n}(1-\varepsilon/8)\varrho(2^{-n}, 7\cdot 2^{n+3}\varepsilon^{-1}R^{2^{-n}}x)$$
$$\times[1+\varrho(2^{-n}, 7\cdot 2^{n+3}\varepsilon^{-1}R^{2^{-n}}x)]^{-1}.$$

Now, we are able to prove the following

Theorem 3. *If ϱ is defined by* (5), *then* $X=X_\varrho$.

Proof. First, we remark that without loss of generality we may assume $R\geqq 1$ in formula (1).

Let $x\in X_\varrho$, then $\varrho(\lambda x)\to 0$ as $\lambda\to 0$ and applying the left one of inequalities (6) with λx in place of x we obtain

$$3\cdot 2^{-n-3}\frac{\varrho(2^{-n+1}, 2^{-n-2}R^{-2^{-n}}\lambda x/7)}{1+\varrho(2^{-n+1}, 2^{-n-2}R^{-2^{-n}}\lambda x/7)}\leqq\int_{2^{-n}}^{2^{-n+1}}\varrho(a, \lambda x)(1+\varrho(a, \lambda x))^{-1}\,da$$

$$\leqq\varrho(\lambda x)\to 0$$

as $\lambda\to 0$. Hence, writing $\eta_n(\lambda)=2^{-n-3}R^{-2^{-n-1}}\lambda/7$, we have $\varrho_{1/2^n}(\eta_n(\lambda)x)\to 0$ as $\lambda\to 0$. Let us choose an arbitrary $0<\varepsilon<2$ and let us take N so large that $\sum_{n=N+1}^{\infty} 2^{-n}<\varepsilon/2$. Taking $\eta(\lambda)=\max\{\eta_n(\lambda):\ 1\leqq n\leqq N\}$, we have $\eta(\lambda)\to 0$ as $\lambda\to 0$; moreover, $\varrho_{1/2^n}(\eta(\lambda)x)\to 0$ as $\lambda\to 0$. Choosing $\lambda_0>0$ in such a way that $\varrho_{1/2^n}(\eta(\lambda)x)<\varepsilon/(2-\varepsilon)$ for $0\leqq\lambda\leqq\lambda_0$ and $n=1, 2, \ldots, N$, we thus have

$\varrho(\eta x)<\varepsilon$ for $0\leqq\eta\leqq\eta(\lambda_0)$. This shows that $x\in X_\varrho$; by Theorem 2, we obtain $x\in \bar{X}$.

Now, let us suppose that $x\in \bar{X}$, i. e. by Theorem 2, $x\in X_\varrho$. Hence $\varrho(\eta x)\to 0$, as $\eta\to 0$, and so $\varrho_{1/2^n}(\eta x)\to 0$ as $\eta\to 0$ for $n=1, 2, \ldots$ Let $0<\varepsilon<2$. Applying the right one of inequalites (6) with ηx in place of x, we obtain

$$\varrho(\eta x)=\sum_{n=1}^{\infty}\int_{2^{-n}}^{2^{-n+1}}(1+\varrho(a, \eta x))^{-1}\varrho(a, \eta x))^{-1}\,da$$

$$\leq(3\varepsilon/4+\sum_{n=1}^{N}2^{-n}\varrho_{1/2^n}(7\cdot 2^{n+3}\varepsilon^{-1}R^{2^{-n}}\eta x)[1+\varrho_{1/2^n}(7\cdot 2^{n+3}\varepsilon^{-1}R^{2^{-n}}\eta x)]^{-1}$$

for any $\eta>0$, where N is chosen in the same manner as in the first part of the proof. Since $\varrho_{1/2^n}(7\cdot 2^{n+3}\varepsilon^{-1}R^{2^{-n}}\eta x)\to 0$ as $\eta\to 0$, we may choose an $\eta_0>0$ such that

$$\varrho_{1/2^n}(7\,.\,2^{n+3}\varepsilon^{-1}R^{2^{-n}}\eta x)<\varepsilon/(4-\varepsilon) \text{ for } 0\leqq\eta\leqq\eta_0$$

and $n=1, 2, \ldots, N$. Consequently, $\varrho(\eta x)<\varepsilon$ for $0\leqq\eta\leqq\eta_0$, i. e. $x\in X_\varrho$.

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