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ON IMBEDDING THEOREMS

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Summary. In this paper three theorems are proved which give necessary and sufficient conditions, applying the moduli of continuity of a nonicreasing function $f(x)(\in L^p(0,\infty))$ assuring that this function $f(x)$ should belong to another class $L^\nu(0,\infty)\cdot\varrho(L(0,\infty))$, where $\nu, p\geq 1$, and the function $\varrho(u)$ is a more general function than $|\log u|$.

Several authors, among others G. H. Hardy, J. E. Littlewood, S. M. Nikolskiĭ, P. L. Uljanov, L. Leindler, E. A. Storoženko have proved theorems which give conditions, assuring that a function f of a space A should belong to another space $A_1(\subseteq A)$, in terms of the modulus of continuity of f. P. L. Uljanov in [4], for example, proved the following

Theorem A ([4], Theorem 3). *If* $1\leq p<\nu<\infty$, *and* $\omega(\delta)$ *is a given modulus of continuity,* $0\leq\beta<\infty$, *then a necessary and sufficient condition for* $H_p^{\omega(\delta)}\subset L^\nu(\ln^+ L)^\beta$ *is that* $\sum_{n=1}^{\infty} n^{(\nu/p)-2}\cdot\omega^\nu(1/n)\ln^\beta(n+1)<\infty$, *on the interval* $[0, 1]$. Where

$$H_p^{\omega(\delta)}=\{f: f\in L^p,\ \omega(p,\delta;f)=\sup_{0\leq h\leq\delta}[\int_0^{1-h}|f(x+h)-f(x)^p\,dx]^{1/p}=O(\omega(\delta))\}.$$

L. Leindler in [1] and [2] generalized this result in some direction, proving for example a theorem including the case $p=\nu$, too. The most important part of this theorem is the following statement:

Theorem B([1] Theorem 3.) *A necessary and sufficient condition for* $H_p^{\omega(\delta)}\subset L^p\wedge(L)$, *is that* $\sum_{k=1}^{\infty}(\lambda_k/k)\,\omega^p(1/k)<\infty$ *on the interval* $[0, 1]$, *where* $\{\lambda_k\}$ *is a nonnegative monotonic sequence, satisfying* $\lambda_{k^2}\leq c\cdot\lambda_k$ *for any* k, *and the function* $\Lambda(x)$ *is defined as follows*

$$\Lambda(x)=\begin{cases}\sum_{k=1}^{[x]}\lambda_k/k, & \text{if } 1\leq x,\\ 0, & \text{if } 0\leq x<1.\end{cases}$$

($[x]$ denotes the integer part of x).

E. A. Storoženko in [3] proved theorems similar to that of P. L. Uljanov and L. Leindler, using $\omega(\delta;f^*)$ instead of $\omega(\delta;f)$, where f^* is a nonincreasing function, equidistributed with $|f(x)|$, that is

$$\text{mes}\{x : x \in [0,\ 1],\ |f(x)| > y\} = \text{mes}\{z : z \in [0,\ 1],\ f^*(z) > y\}.$$

For example, she proved theorems which are equivalent to the following

Theorem C ([3] Theorem 1 and Theorem 2). *Let f be a function belonging to L^p. Then $f \in L^p \wedge(L)$ if and only if*

$$\sum_{k=1}^{\infty} (\lambda_k/k)\, \omega^p (1/k), f^* < \infty$$

on the interval $[0, 1]$, and if $\nu > p$, then $f(x) \in L^\nu \varrho(L)$ if and only if

$$\sum_{k=1}^{\infty} \varrho(k) \,.\, k^{(\nu/p)-2}\, \omega^\nu (1/k,\ f^*) < \infty, \text{ on } [0,\ 1],$$

where the function $\Lambda(x)$ has the same meaning as in Theorem B, and $\varrho(x)$ is a nondecreasing function with $0 \leq \varrho(x^2) \leq c \cdot \varrho(x)$.

We remark that all of the above-mentioned results are valid on the interval $[0,\ 1]$.

P. L. Uljanov in [5] proved that the Theorem A is valid on the interval $(0,\ \infty)$, too. Of course, in this case the definition of $\omega(p, \delta, f)$ is the following:

$$\omega(p,\ \delta,\ f) = \sup\{(\int_0^\infty |f(x+h) - f(x)|^p\, dx)^{1/p} : 0 \leq h \leq \delta\}.$$

We proved theorems of Storoženko's type, concerning the interval $(0,\ \infty)$ instead of $[0,\ 1]$, which contain the case $\nu \leq p$, too, and are valid for more general functions $\Lambda(x)$ and $\varrho(x)$ than before.

Before formulating our theorems we give some definitions.

If $f(x) \in L^p(0,\ \infty)$, then

$$\omega(p;\ \delta;\ f) \equiv \omega(\delta;\ f) := \sup\ \{(\int_0^\infty |f(x+h) - f(x)|^p dx)^{1/p} : 0 \leq h \leq \delta\},$$

$$\widehat{\omega}(p;\ \delta;\ f) \equiv \widehat{\omega}(\delta;\ f) := \{\int_{1/\delta}^\infty |f(x)|^p\, dx\}^{1/p},$$

$$\widetilde{\omega}(p;\ \delta;\ f) \equiv \widetilde{\omega}(\delta;\ f) := \omega(\delta;\ f) + \widehat{\omega}(\delta;\ f).$$

The definition of the function $\Lambda(x)$ is the following:

$$\Lambda(x) = \begin{cases} \sum_{k=1}^{[x]} \lambda_k/k, & \text{if } 1 \leq x < \infty, \\ \sum_{k=1}^{[1/x]} \lambda_k/k, & \text{if } 0 < x < 1, \\ 0, & \text{if } x = 0, \end{cases}$$

where the sequence $\{\lambda_k\}$ means the same as before (see Theorem B).

And the function $\varrho(x)$ is nondecreasing on $(1;\ \infty)$ and nonincreasing on $(0;\ 1)$ with the following property: $0 \leq \varrho(x^2) \leq c \cdot \varrho(x)$ for any $x \in (0;\ \infty)$. We proved the following theorems:

Theorem 1. *If $f(x) \in L^p(0,\ \infty)$ and $\nu > p$, then $f(x) \in L^\nu \varrho(L)$ if and only if*

$$\sum_{n=1}^{\infty} \varrho(n) \cdot n^{(\nu/p)-2}\, \omega^\nu (1/n;\ f^*) < \infty.$$

Theorem 2. *If* $f(x) \in L^p(0, \infty)$ *and* $\nu < p$, *then* $f(x) \in L^\nu \varrho(L)$ *if and only if*

$$\sum_{n=1}^{\infty} \varrho(n)\, n^{-\nu/p}\, \widehat{\omega}^\nu(1/n;\, f^*) < \infty.$$

Theorem 3. *If* $f(x) \in L^p(0, \infty)$, *then* $f(x) \in L^p \wedge (L)$ *if and only if*

$$\sum_{n=1}^{\infty} (\lambda_n/n)\, \widetilde{\omega}^p(1/n;\, f^*) < \infty.$$

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