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INTERPOLATION OF MONOTONE AND CONVEX FUNCTIONS

M. G. Nikolčeva

Summary. The problem of interpolation of monotone, convex and r-times convex functions is considered. Estimations for the degree of the interpolation polynomial in the upper cases, depending of the distance between the interpolating data and convexity of the interpolating function, are given.

I. Interpolation of Monotone Functions. Let $(x_i, y_i), i=0,\dots,m$, $x_i=i/m$ $y_i<y_{i+1}$, $y_0=0$, $y_m=1$ be given points. W. Wolibner [1] and S. W. Young [2] have proved that there exists an algebraic polynomial p, such that $p(x_i)=y_i$, $i=0,\dots,m$ and p is monotone in $[0,1]$, but they have said nothing about the degree of p.

The following results for the degree of p are known:

1. Let us denote $\Delta y_i=y_{i+1}-y_i$, $\Delta y_i>0$, $x_i=i/m$, $i=0,\dots,m$, $A=\max\{\Delta y_i : 0\leqq i\leqq m-1\}$, $B=\min\{\Delta y_i : 0\leqq i\leqq m-1\}$ and let H_n be the set of all algebraic polynomials of degree $\leqq n$.

Theorem 1 [3]. *For every integer k, $k\geqq 1$, if n is such that $2kn^{-1}\ln n<1/m$ and $\ln n>(2k-1)^{-1}\ln(e^4(A+B)m/B)$, then there exists a monotone in $[0, 1]$ algebraic polynomial $P_{n,k}\in H_{2n+1}$, such that $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$.*

Corollary 1 [3]. *Let* $\min\{\Delta y_i : 0\leqq i\leqq m-1\}\geqq cm^{-s}$, s-arbitrary, $m>s>1$, $m>ce^4$. If $n\geqq[15\, sm\ln m]+1$ then there exists a monotone algebraic polynomial $P_n\in H_{2n+1}$, such that $P_n(x_i)=y_i$, $i=0,\dots,m$.*

2. Let $\Delta=\Delta(y)=\min\{\Delta y_i : 0\leqq i\leqq m-1\}$, $\Delta y_i>0$, $\alpha=\alpha(x)=\min\{\Delta x_i : 0\leqq i\leqq m-1\}$ $\Delta x_i=x_{i+1}-x_i$, $M=M(x;y)=\max\{\Delta y_i/\Delta x_i : 0\leqq i\leqq m-1\}$ and $N(x;y)$ is the degree of the monotone interpolation polynomial p.

Theorem 2 [4]. *There exist constants A_r, $r=0, 1, 2,\dots$, such that* $N(x; y)\leqq\inf_r A_r(M/\alpha^r\Delta)^{(r+1)-1}$; $A_0\leqq A_1\leqq\dots$

Now we compare the estimations in 1. and 2. The estimation in 1. has been made for equidistant data, but it is not an essential assumption. Further we prove the following theorem:

Let the points (x_i, y_i), $i=0,\dots,m$, $x_0=0$, $x_m=1$, $\Delta x_i>0$, $y_0=0$, $y_m=1$, $\Delta y_i>0$ be given and $A=\max\{\Delta y_i : 0\leqq i\leqq m-1\}$, $B=\min\{\Delta y_i : 0\leqq i\leqq m-1\}$, $D=\min\{\Delta x_i : 0\leqq i\leqq m-1\}$.

Theorem 3. For every integer k, $k\geqq 0$, if n is such that $2kn^{-1}\ln n<D$ and $\ln n>(2k-1)^{-1}\ln(e^4(A+B)m/B)$, then there exists a monotone in $[0,1]$ algebraic polynomial $P_{n,k}\in H_{2n+1}$, such that $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$.

* Here and in what follows $C, C_0, C_1,\dots$ denote positive constants.

Now we shall formulate Theorem 1 [3] using the notations of Theorem 2 [4]:

Theorem 1 [3]. *For every integer k, $k\geq 1$, if n is such that $2kn^{-1}\ln n<\alpha$ and $\ln n>(2k-1)^{-1}\ln e^4(M/\Delta+1/\alpha)$, then there exists a monotone in $[0,1]$ algebraic polynomial $P_{n,k}\in H_{2n+1}$ such that $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$.*

Obviously, if $\Delta=\Delta(y)=\min\{\Delta y_i:0\leq i\leq m-1\}$ is $0(m^{-s})$, $s>1$, then the estimation in Theorem 1 [3] is better than the estimation in Theorem 2 [4] and the order of the estimation in 1. is the best possible, in this case.

Let us consider the function

$$\sigma(x)=\begin{cases}0, & \text{for } 0\leq x<1/2,\\ 1, & \text{for } 1/2\leq x\leq 1\end{cases}$$

and the points $x_i=i/m$, $i=0,\dots,m$, $m>e^4$. Let k be such that $|x_k-1/2|\leq 1/m$ and let the points y_i, $i=0,\dots,m$, be such that $y_0=0$, $y_{i+1}=y_i+m^{-2}$, for $i\neq k$, $k+1$, $y_k=1/m$, $y_{k+1}=1-1/m$, $y_m=1$. Hence $\min\{\Delta y_i:0\leq i\leq m-1\}\geq m^{-2}$.

Let us assume that there exists a monotone in $[0,1]$ algebraic polynomial P_k of degree k, such that $P_k(x_i)=y_i$, $i=0,\dots,m$. In this case we have $r(\sigma,P_k)\leq 1/m$, where $r(\sigma,P_k)$ is the Hausdorff distance [5], because σ and P_k are located in the strip $1/m$ wide. On the other hand, for the best Hausdorff approximation of $\sigma-E_k(\sigma)_r$ with polynomials of degree k, the following asymptotic estimation holds: $E_k(\sigma)_r\sim k^{-1}\ln k$ [6]. Consequently $k^{-1}\ln k\leq 2/m$ for sufficiently large k. Then $k>2^{-1}m\ln k>2^{-1}m\ln m$ for sufficiently large k.

If $s=1$, then the estimation in 2. is better than the estimation in 1.

In the proofs in this work we shall use:

Lemma 1 [3]: *For every k, $1\leq k\leq n/2\ln n$, there exists an algebraic polynomial $A_{n,k}$ of degree $2n$, such that:*

1. $A_{n,k}(x)\geq 0$, $x\in[-1,1]$;
2. $A_{n,k}\leq 2e^4n^{-2k+1}$, $\lambda_{k,n}\leq|x|\leq 1$, *where* $\lambda_{k,n}=kn^{-1}\ln n$;
3. $\int_{-\lambda_{k,n}}^{\lambda_{k,n}}A_{n,k}(x)\,dx\geq 1$.

Lemma 2 [3]: *If $|\varepsilon_{ij}|<\varepsilon$, $i,j=1\dots,m$, and $M>m\varepsilon(A+B)/B$, $A=\max\{b_i:1\leq i\leq m\}$, $B=\min\{b_i:1\leq i\leq m\}$, $b_i>0$, $i=1,\dots,m$, then the system*

$$\begin{pmatrix}M+\varepsilon_{11} & \varepsilon_{12} & \dots & \varepsilon_{1m}\\ \varepsilon_{21} & M+\varepsilon_{22} & \dots & \varepsilon_{2m}\\ \vdots & & & \\ \varepsilon_{m1} & \varepsilon_{m2} & \dots & M+\varepsilon_{mm}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_m\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{pmatrix}$$

has unique positive solution.

Proof of Theorem 3: Let $\varepsilon>0$, ε — arbitrary and $d=m\varepsilon(A+B)/B$. Let us consider the polynomial $P_{n,k}(x)=\Sigma_{i=1}^{m}a_i\int_0^x dA_{n,k}(t-\Delta x_i/2)\,dt$, where $A_{n,k}$ is the polynomial from Lemma 1 [3]. Apparently $P_{n,k}\in H_{2n+1}$. We show that the system (1) $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$ (with respect to a_i) has positive solution. From the assumptions of the theorem it follows that $\lambda_{n,k}\leq D/2$. If we denote $\int_{-D/2}^{D/2}dA_{n,k}(x)dx=M>d$, then (1) can be written in the following way

$$\begin{pmatrix} M & \delta_{12} & \dots & \delta_{1m} \\ M+\delta_{21} & M+\delta_{22} & \dots & \delta_{2m} \\ \vdots & \vdots & & \\ M+\delta_{m1} & M+\delta_{m2} & \dots & M+\delta_{mm} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix},$$

where $0 \leq \delta_{ij} \leq de^4 n^{-2k+1}$, $\delta_{i+1,j} \geq \delta_{ij}$, $\delta_{i-1,j} = \delta_{ij}$. The system (1) is equivalent to

$$\begin{pmatrix} M & \varepsilon_{12} & \dots & \varepsilon_{1m} \\ \varepsilon_{21} & M+\varepsilon_{22} & \dots & \varepsilon_{2m} \\ \vdots & & & \\ \varepsilon_{m1} & \varepsilon_{m2} & \dots & M+\varepsilon_{mm} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \Delta y_0 \\ \Delta y_1 \\ \vdots \\ \Delta y_{m-1} \end{pmatrix}, \tag{2}$$

where $\varepsilon_{ij} = \delta_{ij} - \delta_{i-1,j} \geq 0$ and $\varepsilon_{ij} \leq de^4 n^{-2k+1}$, $\Delta y_i > 0$, $i = 0, \dots, m$, $j = 1, \dots, m$. But $d = m\varepsilon(A+B)/B$. Hence $M > m\varepsilon(A+B)/B$. On the other hand, $0 \leq \varepsilon_{ij} < \varepsilon$ if $de^4 n^{-2k+1} < \varepsilon$ or $n^{-2k+1} me^4 (A+B)/B < 1$. Consequently, if $\ln n > (2k-1)^{-1} \times \ln e^4 (A+B) m/B$, $k \geq 1$, k — integer, from Lemma 2 [3] it follows that the system (2) has unique positive solution. Then the monotone in [0, 1] algebraic polynomial

$$P_{n,k}(x) = \sum_{i=1}^{m} a_i \int_0^x dA_{n,k}(t - \Delta x_i/2)\, dt$$

satisfies the assumptions of the theorem, because $P_{n,k}(x_i) = y_i$, $i = 0, \dots, m$. Theorem 3 is proved.

Corollary 2: *Let* $\min\{\Delta y_i : 0 \leq i \leq m-1\} \geq cm^{-s}$, *$s$ — arbitrary*, $m > s > 1$, $m > ce^4$. *If* $n \geq [e^4 D^{-1} s \ln m] + 1$, *then there exists an algebraic polynomial* $P_n \in H_{2n+1}$ *such that* $P_n(x_i) = y_i$, $i = 0, \dots, m$.

Proof: If $\min\{\Delta y_i : 0 \leq i \leq m-1\} \geq cm^{-s}$, then $(A+B)/B \leq 2cm^s$ and the assumptions of Theorem 3 are transformed in

$$2n^{-2k+1} m^{s+1} ce^4 < 1 \tag{3}$$

and

$$2n^{-1} k \ln n < D. \tag{4}$$

Let $k = (s+3)/2$. Then from (3) we obtain

$$n^{-s-2}\, 2m^{s+1} ce^4 < 1 \tag{5}$$

and

$$(s+3)\, n^{-1} \ln n < D. \tag{6}$$

But, by the assumption, $n > m > ce^4$. Consequently (5) is fulfilled. On the other hand, $s > 1$, hence $s+3 < 4s$ and from (6) we obtain

$$4s \ln n/n < D. \tag{7}$$

Let $n > C_1 Ds \ln m$. Then (7) can be written as follows

$$4\,(c_1 D^2 \ln m)^{-1} \ln (C_1 Ds \ln m) < 1. \tag{8}$$

But $4\,(C_1 D^2 \ln m)^{-1} \ln (C_1 Ds \ln m) \leq 4\,(C_1 D^2 \ln m)^{-1} (\ln C_1 Ds + \ln m)$

$$= 4\,\{(C_1 D^2 \ln m)^{-1} \ln C_1 DS + 1/C_1 D^2\} \leq 4\,\{(5C_1 D^2)^{-1} \ln C_1 D + 2/C_1 D^2\},$$

because $m>s$, $m>e^4$ by the assumption. Hence (8) is fulfilled if

$$(9) \qquad 4\{(5C_1D^2)^{-1}\ln C_1D+2/C_1D^2\}<1.$$

Let $C_1=e^4/D^2$. Then (9) is fulfilled. Consequently, if $\min\{\Delta y_i:0\leq i\leq m-1\}\geq cm^{-s}$, $m>ce^4$, $m>s>1$ and $n>e^4D^{-1}s\ln m$, then there exists a monotone in $[0, 1]$ algebraic polynomial, such that $P_n(x_i)=y_i$, $i=0,\dots,m$. Corollary 2 is proved.

II. Interpolation of Convex Functions. Let the points (x_i, y_i), $i=0,\dots,m$, $x_i=i/m$, $y_0=0$, $y_i>0$, $\Delta^2 y_i=y_{i+2}-2y_{i+1}+y_i$, $\Delta^2 y_i>0$ be given. The problem is to find a convex algebraic polynomial p, such that $p(x_i)=y_i$, $i=0,\dots,m$.

Lemma 3: *For every k, $1\leq k\leq n/2\ln n$, there exists a convex algebraic polynomial $B_{n,k}$ with the properties:*

1. $B_{n,k}(x)>0$, $x\in[-1,1]$, $B_{n,k}\in H_{2n+2}$
2. $B_{n,k}(x)\leq C_2e^4n^{-2k+1}$ *for* $-1\leq x\leq-\lambda_{k,n}$, *where* $\lambda_{k,n}=n^{-1}k\ln n$
3. $B_{n,k}(0)\leq C_2e^4n^{-2k+1}$
4. $B_{n,k}(x)-x\leq C_2e^4n^{-2k+1}$ *for* $\lambda_{k,n}\leq x\leq 1$.

Proof: Let us consider the polynomial $D_{n,k}(x)=\int_{-1}^{x}A_{n,k}(t)dt$, where $A_{n,k}$ is the polynomial from Lemma 1 [3]. $D_{n,k}\in H_{2n+1}$ and has the following properties:

1. $D_{n,k}(x)\geq 0$, $x\in[-1,1]$;
2. $D_{n,k}(x)\leq 2e^4n^{-2k+1}$ for $-1\leq x\leq-\lambda_{k,n}$;
3. $|D_{n,k}(x)-M_{n,k}|<2e^4n^{-2k+1}$ for $\lambda_{k,n}\leq x\leq 1$, where $M_{n,k}=\text{const}$, $M_{n,k}\geq 1$;
4. $D_{n,k}(x)$ is monotone for $|x|\leq\lambda_{k,n}$.

Let us denote $G_{n,k}(x)=(2D_{n,k}(x)-1)/M_{n,k}$, $L_{n,k}(x)=(G_{n,k}(x)-G_{n,k}(-x))/2$, $N_{n,k}(x)=\int_0^x L_{n,k}(t)\,dt$ and $B_{n,k}(x)=(N_{n,k}(x)+x)/2+C_0e^4n^{-2k+1}$, $x\in[-1,1]$. The polynomial $B_{n,k}\in H_{2n+2}$ and has the desired properties. Lemma 3 is proved.

Let the points (x_i, y_i), $i=0,\dots,m$, $x_i=i/m$, $\Delta^2y_i>0$, $A=\max\{\Delta^2y_i;0\leq i\leq m-2\}$ $B=\min\{\Delta^2y_i:0\leq i\leq m-2\}$ be given.

Theorem 4: *For every integer k, $k\geq 1$, if n is such that $n^{-1}2k\ln n<1/m$ and $\ln n>(2k-1)^{-1}\ln C_2e^4(A+B)m^2/B$, then there exists a convex in $[0, 1]$ algebraic polynomial $P_{n,k}\in H_{2n+2}$ such that $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$.*

Proof: Let $\varepsilon>0$ be arbitrary and $c>\varepsilon m^2(A+B)/B$. Let us consider the polynomial $P_{n,k}(x)=\sum_{i=1}^{m}ca_iB_{n,k}(x-(i-1)/m)$, where $B_{n,k}$ is the polynomial from Lemma 3. Obviously $P_{n,k}\in H_{2n+2}$. We show that the system (10) $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$ (with respect to a_i) has unique positive solution. We can write the system (10) as follows:

$$(11)\quad \begin{pmatrix} cm^{-1}+\delta_{11} & \delta_{12} & \dots\ \delta_{1m} \\ 2cm^{-1}+\delta_{21} & cm^{-1}+\delta_{22} & \dots\ \delta_{2m} \\ \vdots & & \\ (m-1)cm^{-1}+\delta_{m1} & (m-2)cm^{-1}+\delta_{m2} & \dots\ cm^{-1}+\delta_{mm} \end{pmatrix} \begin{pmatrix} a_1\\ a_2\\ \vdots\\ a_m\end{pmatrix} = \begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_m\end{pmatrix},$$

where $0\leq\delta_{ij}\leq c_3e^4n^{-2k+1}$, $\delta_{ij}\geq\delta_{i-1,j}$, $C_3=CC_2$. The system (11) is equivalent to

$$(12)\quad \begin{pmatrix} cm^{-1}+\delta'_{11} & \delta'_{12} & \dots & \delta'_{1m} \\ cm^{-1}+\delta'_{21} & cm^{-1}+\delta'_{22} & \dots & \delta'_{2m} \\ \vdots & & & \\ cm^{-1}+\delta'_{m1} & cm^{-1}+\delta'_{m2} & \dots & cm^{-1}+\delta'_{mm} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \Delta y_0 \\ \Delta y_1 \\ \vdots \\ \Delta y_{m-1} \end{pmatrix},$$

where $\delta'_{ij}=\delta_{ij}-\delta_{i-1,j}$, $0\leq\delta'_{ij}\leq c_3 e^4 n^{-2k+1}$, $\delta'_{ij}\geq\delta'_{i-1,j}$, $\Delta y_0>0$.
But (12) is equivalent to

$$(13)\quad \begin{pmatrix} cm^{-1}+\varepsilon_{11} & \varepsilon_{12} & \dots & \varepsilon_{1m} \\ \varepsilon_{21} & cm^{-1}+\varepsilon_{22} & \dots & \varepsilon_{2m} \\ \vdots & & & \\ \varepsilon_{m1} & \varepsilon_{m2} & \dots & cm^{-1}+\varepsilon_{mm} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \Delta y_0 \\ \Delta^2 y_1 \\ \vdots \\ \Delta^2 y_{m-2} \end{pmatrix},$$

where $\varepsilon_{ij}=\delta'_{ij}-\delta'_{i-1,j}$, $0\leq\varepsilon_{ij}\leq C_3 e^4 n^{-2k+1}$, $\Delta^2 y_i>0$, $cm^{-1}>m\varepsilon(A+B)/B$. On the other hand, $0\leq\varepsilon_{ij}<\varepsilon$ if $C_3 e^4 n^{-2k+1}<\varepsilon$ or $C_2 e^4 m^2 n^{-2k+1}(A+B)/B<1$ or $\ln n>(2k-1)^{-1}\ln C_2 e^4(A+B)m^2/B$, $k\geq 1$. Consequently, if $\ln n>(2k-1)^{-1}\times\ln C_2 e^4(A+B)m^2/B$, $k\geq 1$, k — integer, then from Lemma 1 [3] it follows that the system (13) — respectively (10) — has unique positive solution and the convex polynomial $P_{n,k}(x)=\sum_{i=1}^{m} Ca_i B_{n,k}(x-(i-1)/m)$ satisfies $P_{n,k}(x_i)=y_i$, $i=0,\dots,m$. Theorem 4 is proved.

Corollary 3: *Let* $\min\{\Delta^2 y_i : 0\leq i\leq m-2\}\geq cm^{-s}$, s — *arbitrary*, $m>s>1$, $m>CC_2e^4$. *If* $n>[e^5 sm\ln m]+1$, *then there exists a convex algebraic polynomial* P_n, *such that* $P_n(x_i)=y_i$, $i=0,\dots,m$.

Proof: If $\min\{\Delta^2 y_i : 0\leq i\leq m-2\}\geq cm^{-s}$, then $(A+B)/B<2cm^s$ and the assumptions of Theorem 4 are transformed in

$$(14)\qquad n^{-1}2k\ln n<m^{-1}$$

and

$$(15)\qquad n^{-2k+1}2CC_2e^4m^{s+2}<1.$$

Let $k=(s+5)/2$. Then from (14) and (15) we obtain

$$(16)\qquad (s+5)n^{-1}\ln n<m^{-1}$$

and

$$(17)\qquad m^{-s-4}2m^{s+2}CC_2e^4<1.$$

Since $n>m>CC_2e^4$, (17) is fulfilled. But $s>1$. Hence $s+5<6s$ and from (16) we obtain

$$(18)\qquad n^{-1}6s\ln n<m^{-1}.$$

Let $n>C_4 sm\ln m$. Then from (18) we get

$$(19)\qquad 6(C_4\ln m)^{-1}\ln(C_4 sm\ln m)<1.$$

But $6(C_4\ln m)^{-1}\ln(C_4 sm\ln m)\leq 6(C_4\ln m)^{-1}\{\ln(C_4 s)+2\ln m\}$ $=6\{(C_4\ln m)^{-1}\ln(C_4 s)+2/C_4\}\leq 6\{(C_4\ln m)^{-1}\ln C_4+3/C_4\}\leq 6\{(\ln(C_4)/5C_4+3/C_4\}$, because $m>s$, $m>e^4$ by the assumption. Hence (19) is fulfilled, if

$$(20)\qquad 6\{(\ln C_4)/5C_4+3/C_4\}<1.$$

Let $C_4=e^5$. Then (20) is fulfilled. Consequently, if $\min\{\Delta^2 y_i : 0\leq i\leq m-2\} \geq cm^{-s}$, $m>CC_2e^4$, $m>s>1$ and $n\geq[e^5 sm\ln m]+1$, than there exists a convex algebraic polynomial, such that $P_n(x_i)=y_i$, $i=0,\ldots,m$. Corollary 3 is proved.

III. Interpolation of *r*-Times Convex Functions. Let the points (x_i, y_i), $i=0,\ldots,m$, $x_i=i/m$, $y_0=0$, $y_i>0$ be given and $\Delta^r y_i>0$, where $\Delta^r y_i=y_{i+r} -\binom{r}{1}y_{i+r-1}+\binom{r}{2}y_{i+r-2}-\cdots+(-1)^r f_i$

Using the method from I. and II. we can prove the following

Lemma 4: *For every* k, $1\leq k\leq n/2\ln n$, *there exists r-times convex algebraic polynomial* $B_{n,k,r}$, *such that:*

1. $B_{n,k,r}\in H_{2n+r}$, $B_{n,k,r}(x)>0$, $x\in[-1,1]$
2. $B_{n,k,r}(x)\leq C_r e^4 n^{-2k+1}$, *for* $-1\leq x\leq -\lambda_{k,n}$, *where* $\lambda_{k,n}=n^{-1}k\ln n$
3. $B_{n,k,r}(0)\leq C_r e^4 n^{-2k+1}$
4. $B_{n,k,r}(x)-x^r\leq C_r e^4 n^{-2k+1}$, *for* $\lambda_{k,n}\leq x\leq 1$.

Theorem 5: *For every integer* k, $k\geq 1$, *if* n *is such that* $n^{-1}2k\ln n <m^{-1}$ *and* $\ln n>(2k-1)^{-1}\ln C_r e^4(A+B)m^{r+1}/B$ *then there exists r-times convex in* $[0,1]$ *algebraic polynomial* $P_{n,k,r}\in H_{2n+r}$, *such that* $P_{n,k,r}(x_i)=y_i$, $i=0,\ldots,m$.

Corollary 4: *Let* $\min\{\Delta^r y_i : 0\leq i\leq m-r\}\geq cm^{-s}$, *s — arbitrary*, $m>CC_re^4$, $m>s>1$. If $n>[e^{3+r}sm\ln m]+1$, *then there exists a convex (r-times) algebraic polynomial, such that* $P_{n,r}(x_i)=y_i$, $i=0,\ldots,m$.

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Centre for Mathematics and Mechanics P. O. Box 373 1090 Sofia Bulgaria

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